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Quasigroup covers of division groupoids

† JAROSLAV J. JEŽEK, TOMÁŠ J. KEPKA, PETR C. NĚMEC

Abstract. Let G be a division groupoid that is not a quasigroup. For each regular cardinal $\alpha > |G|$ we construct a quasigroup Q on $G \times \alpha$ that is a quasigroup cover of G (i.e., G is a homomorphic image of Q and G is not an image of any quasigroup that is a proper factor of Q). We also show how to easily obtain quasigroup covers from free quasigroups.

Keywords: groupoid; division; quasigroup; cover

Classification: 20N05

Throughout the paper, a set with a binary operation will be called a *groupoid*, following the classical usage of B. A. Hausmann and O. Ore in [9]; groupoids are also known as *magmas*, *binars* or *multiplicative systems*. One rather familiar groupoid/magma is the multiplicative semigroup $\mathbb{N}(\cdot)$ of positive integers (recognised for several millenia). The multiplicative group $\mathbb{Q}^+(\cdot)$ of positive rationals is widely used as well. Positive integers form a subgroupoid/submagma of positive rationals and the latter serve as a group of fractions of the former ones.

It is a notoriously known fact that a division semigroup is a group, whence a quasigroup. This is not true for general groupoids anymore. E.g., setting $n * m = |n - m|$ for all $n, m \in \mathbb{N}_0$ (nonnegative integers), we get a division groupoid $\mathbb{N}_0(*)$ that is not a quasigroup. Homomorphic images of quasigroups are division groupoids, however not necessarily quasigroups. The investigation of homomorphic images (i.e., factors by stable equivalences) of quasigroups started already in the forties of the last century, see, e.g., [1], [2], [3], [4], [5], [6], [8], [11], and is continued in the present note. Stable equivalences that are not quasigroup congruences were constructed, e.g., in [3] and [6]. Our main aim is to show that every proper division groupoid possesses a quasigroup cover of “arbitrary size”.

This paper (after many years) finalizes a common research started several months before untimely death of our colleague J. J. Ježek who deceased in 2011, see also [10].

1. Basic notions

Let G be a groupoid. For every element $a \in G$ we define two mappings L_a and R_a (called the *left* and the *right translation* by a) of G into G by $L_a(x) = ax$ and $R_a(x) = xa$. The submonoid of the transformation monoid of G generated by all the mappings L_a and R_a (for $a \in G$) is called the *multiplication monoid* of G .

We shall say that G is a *left* (*right*, respectively) *division groupoid* if for any two elements $a, b \in G$ there is an element $x \in G$ with $a = bx$ ($a = xb$, respectively). Clearly, G is a left (*right*, respectively) *division groupoid* if and only if all left (*right*, respectively) translations are surjective. Similarly, we shall say that G is a *left* (*right*, respectively) *cancellation groupoid* if all left (*right*, respectively) translations are injective. If G is both left and right division (*cancellation*, respectively) groupoid then it is said to be a *division* (*cancellation*, respectively) groupoid.

A relation r on a groupoid G is said to be *left stable* if $(x, y) \in r$ implies $(ax, ay) \in r$ for any $a \in G$. Right stable relations are defined dually. A relation is said to be *stable* if it is both left and right stable. A congruence on G turns out to be precisely a stable equivalent relation.

A congruence r of a groupoid G is said to be *left* (*right*, respectively) *cancellative* if for all $a, x, y \in G$, $(ax, ay) \in r$ ($(xa, ya) \in r$, respectively) implies $(x, y) \in r$. A congruence which is both left and right cancellative is said to be *cancellative*.

A subset S of a groupoid G is said to be *left* (*right*, respectively) *closed* (in G) if $x \in S$ whenever $x \in G$ and $ax \in S$ ($xa \in S$, respectively) for some $a \in S$. A subset is said to be *closed* if it is both left and right closed.

An element $x \in G$ is said to be *idempotent* if $xx = x$. The set of idempotent elements of a groupoid G will be denoted by $\text{Id}(G)$.

By a *quasigroup* we mean a groupoid which is both division and cancellation groupoid (i.e., all left and right translations are bijective). Evidently, every finite division (*cancellation*) groupoid is a quasigroup.

A homomorphic image of a quasigroup is not necessarily a quasigroup, however it is a division groupoid. We will see that, conversely, every division groupoid is a homomorphic image of a quasigroup.

By a *quasigroup cover* of a division groupoid G we mean a pair consisting of a quasigroup Q and a homomorphism h of Q onto G such that there is no quasigroup congruence (i.e., no cancellative congruence) of Q contained in the kernel of h , except the identity.

By a *coreflection* of a division groupoid G in the class of quasigroups we mean a quasigroup Q together with a homomorphism h of Q to G such that whenever

Q' is a quasigroup and h' is a homomorphism of Q' into G then there is a unique homomorphism g of Q' into Q with $h' = hg$.

In this paper, we will prove that if G is a division groupoid such that G is not a quasigroup then G has arbitrarily large quasigroup covers and G has no coreflection in the class of quasigroups.

2. Cancellative congruences of (division) groupoids

Lemma 2.1. *Let r, s be two congruences of a groupoid G ; let A be a block of r and B a block of s .*

- (i) *If G is a left division groupoid, s is right cancellative and $A \subseteq B$ then $r \subseteq s$.*
- (ii) *If G is a division groupoid, s is right cancellative and $B \subseteq A$ then $s \subseteq r$.*

PROOF: (i) Let $(x, y) \in r$. Take an element $a \in A$. There is an element $u \in G$ such that $a = xu$. We have $(xu, yu) \in r$, $xu \in A$, $yu \in A$, so that xu and yu belong to B , and hence $(xu, yu) \in s$. Since s is right cancellative, this implies $(x, y) \in s$.

(ii) Let $(x, y) \in s$. Take an element $b \in B$. There are elements $u, c \in G$ such that $x = bu$ and $y = cu$. Since $(bu, cu) \in s$ and s is right cancellative, $(b, c) \in s$. But then $c \in B$, $b, c \in A$, $(b, c) \in r$, $(bu, cu) \in r$ and $(x, y) \in r$. $\square$

Corollary 2.2. *Let r, s be two congruences of a groupoid G . Then $r = s$, provided that r and s have a common block and, moreover, at least one of the following three conditions is satisfied:*

- (i) *G is a left division groupoid and both r and s are right cancellative.*
- (ii) *G is a right division groupoid and both r and s are left cancellative.*
- (iii) *G is a division groupoid and at least one of r, s is either left or right cancellative.*

Lemma 2.3. *Let A be a block of a congruence r of a groupoid G .*

- (i) *If r is right cancellative and $x, y \in G$ are elements such that $xA \cap yA \neq \emptyset$ then $(x, y) \in r$.*
- (ii) *If G is a left division groupoid, r is left cancellative and $(x, y) \in r$ then $xA = yA$.*

PROOF: (i) We have $xa = yb$ for some elements $a, b \in A$. Now $(a, b) \in r$ implies $(xa, xb) \in r$, so that $(yb, xb) \in r$, and hence $(y, x) \in r$.

(ii) Let ya , with $a \in A$, be an arbitrary element of yA . There is an element $b \in G$ with $ya = xb$. Since $(ya, yb) \in r$, we have $(a, b) \in r$, and hence $b \in A$. So, $ya = xb \in xA$ and we get $yA \subseteq xA$. Quite similarly, $xA \subseteq yA$. $\square$

Corollary 2.4. *Let A be a block of a cancellative congruence r of a left division groupoid G . The following are equivalent for $x, y \in G$:*

- (i) $(x, y) \in r$;
- (ii) $xA \cap yA \neq \emptyset$;
- (iii) $xA = yA$.

Corollary 2.5. *Let r be a cancellative congruence of a division groupoid G . The following are equivalent for $x, y \in G$:*

- (i) $(x, y) \in r$;
- (ii) $xA \cap yA \neq \emptyset$ for at least one block A of r ;
- (iii) $Ax \cap Ay \neq \emptyset$ for at least one block A of r ;
- (iv) $xA = yA$ for every block A of r ;
- (v) $Ax = Ay$ for every block A of r .

Proposition 2.6. *Let A be a block of a cancellative congruence r of a division groupoid G . Then any block B of r can be expressed as $B = xA$ for some $x \in G$, and also as $B = Ay$ for some $y \in G$.*

PROOF: According to Lemma 2.3, the family $\{xA : x \in G\}$ is a partition of G . Denote by s the corresponding equivalence relation. If $(uv) \in r$ and $a \in A$ then there are elements $w, b \in G$ with $u = wa$ and $v = wb$, and we have $(a, b) \in r$, $b \in A$, $u, v \in wA$, and hence $(u, v) \in s$. This shows that $r \subseteq s$. On the other hand, each of the sets xA , for $x \in G$, is contained in a block of r . Consequently, $s \subseteq r$, and we have shown that $r = s$. Now $B = Ay$ is a matter of duality. $\square$

Proposition 2.7. *Let r be a cancellative congruence of a division groupoid G and let A, B be two blocks of r . Then $\text{card}(A) = \text{card}(B)$ and*

$$\text{card}(G) = \text{card}(A) \cdot \text{card}\left(\frac{G}{r}\right).$$

PROOF: By Proposition 2.6, $B = xA$ and $A = yB$ for some $x, y \in G$. The rest is clear. $\square$

Proposition 2.8. *Let r be a congruence of a right cancellation left division groupoid G and let A, B be two blocks of r . Then $\text{card}(A) = \text{card}(B)$ and $\text{card}(G) = \text{card}(A) \cdot \text{card}(G/r)$.*

PROOF: Take an element $a \in A$. We have $ab \in B$ for some element $b \in G$, and it follows that $R_b(A) \subseteq B$. Since R_b is injective, we have $\text{card}(A) \leq \text{card}(B)$. The rest is clear. $\square$

Lemma 2.9. *Let r be a congruence of a right cancellation left division groupoid. If at least one of the blocks of r is finite then r is right cancellative.*

PROOF: Let $x, y, z \in G$ be such that $(yx, zx) \in r$. Denote by A and B the blocks of r containing the elements y and yx , respectively. Then $zx \in B$ and $Ax \subseteq B$. Since G is a right cancellation groupoid, we have $\text{card}(Ax) = \text{card}(A)$. On the other hand, according to Proposition 2.8, A and B are both finite and $\text{card}(A) = \text{card}(B)$. Consequently, $Ax = B$ and $zx = ax$ for some $a \in A$. But then $z = a$ and $(y, z) \in r$. $\square$

Corollary 2.10. *Let r be a congruence of a quasigroup Q such that at least one of the blocks of r is finite. Then r is cancellative.*

Proposition 2.11. *Let G be a left division groupoid. The join of any nonempty collection S of left cancellative congruences of G in the congruence lattice of G is a left cancellative congruence.*

PROOF: Denote by r the join of S , so that r is the transitive closure of the union of S . Let $a, b, c \in G$ and $(ab, ac) \in r$. There is a finite sequence $d_0, \dots, d_n$ of elements of G such that $d_0 = ab$, $d_n = ac$ and, for every $i = 1, \dots, n$, the pair (d_{i-1}, d_i) belongs to a congruence $s_i \in S$. Since G is a left division groupoid, for every $i = 0, \dots, n$ there is an element $e_i \in G$ with $d_i = ae_i$; of course, we can take $e_0 = b$ and $e_n = c$. Since the congruences s_i are left cancellative, we have $(e_{i-1}, e_i) \in s_i$. Hence $(e_0, e_n) \in r$, i.e., $(b, c) \in R$. $\square$

Proposition 2.12. *Let G be a division groupoid. The join of any nonempty collection of cancellative congruences of G in the congruence lattice of G is a cancellative congruence.*

PROOF: It is similar to the proof of Proposition 2.11. $\square$

Proposition 2.13. *Let r, s be two cancellative congruences of a division groupoid G . Then $r \circ s = s \circ r$ is a cancellative congruence of G .*

PROOF: Let $(x, z) \in r \circ s$, so that $(x, y) \in r$ and $(y, z) \in s$ for an element $y \in G$. There are elements $a, b, c \in G$ such that $x = xa$, $y = ba$ and $z = bc$. We have $(xa, ba) \in r$, $(ba, bc) \in s$, and hence $(x, b) \in r$ and $(a, c) \in s$. Then also $(xa, xc) \in s$, $(xc, bc) \in r$, $(x, xc) \in s$ and $(xc, z) \in r$. It follows that $r \circ s \subseteq s \circ r$. The converse inclusion can be proved similarly. Now it is easy to see that this is a cancellative congruence of G . $\square$

Lemma 2.14. *Let r be a congruence of a left division groupoid G and let H be a subgroupoid of G containing at least one block of r . Then H is a union of blocks of r , provided that either H is a right division groupoid and r is left cancellative or H is a closed subgroupoid of G .*

PROOF: Let A be a block of r contained in H . Let $x \in H$, $y \in G$ and $(x, y) \in r$. We have to show that $y \in H$.

Assume first that H is a right division groupoid and r is left cancellative. There are elements $a \in A$, $b \in H$ and $c \in G$ such that $x = ba$ and $y = bc$. Then $(ba, bc) \in r$, $(a, c) \in r$, $c \in A \subseteq H$ and $y = bc \in H$.

Next assume that H is a closed subgroupoid of G . We have $xa \in A$ for some $a \in G$. Then $(xa, ya) \in r$ and thus $ya \in A$. Since $x, xa, ya \in H$ and H is closed, we get $y \in H$. $\square$

Lemma 2.15. *Let H be a subgroupoid of a groupoid G such that H is a block of a left (or right) cancellative congruence r of G . Then H is a left (or right, respectively) closed subgroupoid of G .*

PROOF: Consider, e.g., the left case. Let $ax \in H$, where $a \in H$ and $x \in G$. We have $aa \in H$, $(aa, ax) \in r$, $(a, x) \in r$, and hence $x \in H$. $\square$

Lemma 2.16. *Let r, s be two cancellative congruences of a division groupoid G and let $t = r \circ s$. Let A be a block of r and B that of s such that A is a subgroupoid of G and $A \cap B \neq \emptyset$. The closed subset of G generated by $A \cup B$ is a block of the cancellative congruence t ; it is a subgroupoid of G .*

PROOF: According to Proposition 2.13, $t = r \circ s = s \circ r$ is a cancellative congruence. Since both r and s are contained in t and $A \cap B$ is nonempty, there is a block D of t such that $A \cup B \subseteq D$. If $a \in A$ then $(a, aa) \in r \subseteq t$ and it follows that D is a subgroupoid of G . By Lemma 2.15, D is a closed subgroupoid of G . Denote by C the closed subset of G generated by $A \cup B$. We get $C \subseteq D$. In order to prove the converse inclusion, let $x \in A$ and $y \in D$. Then $(x, y) \in t$ and there is an element $z \in G$ with $(x, z) \in r$ and $(z, y) \in s$. Then also $z \in A$ and we have $uy \in B$ for some $u \in G$. Consequently, $(uz, uy) \in s$, $uz \in B$, and hence $u \in C$, since C is right closed. But C is also left closed and the elements u, uy belong to C , so we get $y \in C$. This proves $D \subseteq C$. $\square$

Proposition 2.17. *Let r be a congruence of a groupoid G . Then r is a subgroupoid of $G \times G$. This subgroupoid is left (or right) closed in $G \times G$ if and only if r is a left (or right, respectively) cancellative congruence of G .*

PROOF: It is easy. $\square$

3. Cancellative cores of congruences

Definition 3.1. Let G be a groupoid. We denote by $\mathbf{c}_l(G)$ (or $\mathbf{c}_r(G)$) the smallest left (or right, respectively) cancellative congruence of G and by $\mathbf{c}(G)$ the smallest cancellative congruence of G . Further, we denote by $\mathbf{a}_l(G)$ (or $\mathbf{a}_r(G)$, respectively) the congruence of G generated by $\bigcup_{a \in G} \ker L_a$ (or $\bigcup_{a \in G} \ker R_a$, respectively) and by $\mathbf{a}(G)$ the congruence generated by $\mathbf{a}_l(G) \cup \mathbf{a}_r(G)$.

Clearly, $\mathbf{a}_l(G) \subseteq \mathbf{c}_l(G)$, $\mathbf{a}_r(G) \subseteq \mathbf{c}_r(G)$ and $\mathbf{a}(G) \subseteq \mathbf{c}(G)$. If G is a left cancellation groupoid then $\mathbf{a}_l(G) = \mathbf{c}_l(G) = \text{id}_G$. If G is a cancellation groupoid then $\mathbf{a}(G) = \mathbf{c}(G) = \text{id}_G$.

Definition 3.2. Let r be a congruence of a groupoid G . If it exists, the largest left (or right, respectively) cancellative congruence of G contained in r is called the *left* (or *right*, respectively) *cancellative core* of r and it is denoted by $\text{core}_{l,G}(r)$ (or $\text{core}_{r,G}$, respectively). Similarly, the largest cancellative congruence of G contained in r , if it exists, is called the *cancellative core* of r and it is denoted by $\text{core}_G(r)$.

Proposition 3.3. Let r be a congruence of a division (left division, respectively) groupoid G . Then $\text{core}_G(r)$ (or $\text{core}_{l,G}(r)$, respectively) exists if and only if $\mathbf{c}(G) \subseteq r$ (or $\mathbf{c}_l(G) \subseteq r$, respectively).

PROOF: Suppose that $\mathbf{c}(G) \subseteq r$. Then the set S of cancellative congruences contained in r is nonempty and, according to Proposition 2.12, $\text{core}_G(r)$ is just the congruence generated by the union of S . The rest is clear. $\square$

Definition 3.4. For any binary relation r on a groupoid G we define two relations $\alpha_G(r)$ and $\beta_G(r)$ as follows: $(a, b) \in \alpha_G(r)$ if and only if $(y, z) \in r$ whenever $a = xy$ and $b = xz$ for some $x, y, z \in G$.

Dually, $(a, b) \in \beta_G(r)$ if and only if $(y, z) \in r$ whenever $a = yx$ and $b = zx$ for some $x, y, z \in G$.

Lemma 3.5. Let r be a binary relation on a groupoid G . Put $s = \alpha_G(r)$.

- (i) If r is symmetric then s is symmetric.
- (ii) If G is a left division groupoid and r is transitive then s is transitive.
- (iii) If G is a left cancellation groupoid and r is reflexive then s is reflexive.
- (iv) If G is a left division groupoid and r is left stable then $s \subseteq r$.
- (v) If t is a left cancellative congruence of G such that $t \subseteq r$ then $t \subseteq s$.

PROOF: It is easy. $\square$

Lemma 3.6. Let r be a congruence of a left quasigroup G . Then $\alpha_G(r)$ is an equivalence and $\text{core}_{l,G}(r) \subseteq \alpha_G(r) \subseteq r$.

PROOF: Combine Proposition 3.3 and Lemma 3.5. $\square$

Lemma 3.7. Let r be a binary relation on a groupoid G . Put $s = \alpha_G(r) \cap \beta_G(r)$.

- (i) If r is symmetric then s is symmetric.
- (ii) If G is a division groupoid and r is transitive then s is transitive.
- (iii) If G is a cancellation groupoid and r is reflexive then s is reflexive.

- (iv) If G is a division groupoid and r is stable then $s \subseteq r$.
- (v) If t is a cancellative congruence of G such that $t \subseteq r$ then $t \subseteq s$.

PROOF: Combine Lemma 3.5 and its dual. $\square$

Lemma 3.8. Let r be a congruence of a quasigroup Q . Then $\text{core}_Q(r) \subseteq \alpha_Q(r) \cap \beta_Q(r) \subseteq r$ and $\alpha_Q(r) \cap \beta_Q(r)$ is an equivalence.

PROOF: Combine Proposition 3.3 and Lemma 3.7. $\square$

Definition 3.9. For any binary relation r on a groupoid G we define two relations $\gamma_G(r)$ and $\delta_G(r)$ as follows: $(a, b) \in \gamma_G(r)$ if and only if $(f(a), f(b)) \in \alpha_G(r)$ for every f from the multiplication monoid of G .

Dually, $(a, b) \in \delta_G(r)$ if and only if $(f(a), f(b)) \in \beta_G(r)$ for every f from the multiplication monoid of G .

Lemma 3.10. Let r be a binary relation on a groupoid G . Put $s = \gamma_G(r)$.

- (i) Then s is stable and $s \subseteq \alpha_G(r)$.
- (ii) If r is symmetric then s is symmetric.
- (iii) If G is a left division groupoid and r is transitive then s is transitive.
- (iv) If G is a left cancellation groupoid and r is reflexive then s is reflexive.
- (v) If G is a left division groupoid and r is left stable then $s \subseteq r$.
- (vi) If t is a left cancellative congruence of G such that $t \subseteq r$ then $t \subseteq s$.

PROOF: Use Lemma 3.5. $\square$

Lemma 3.11. Let r be a congruence of a left quasigroup G . Then $\text{core}_{l,G}(r) \subseteq \gamma_G(r) \subseteq \alpha_G(r) \subseteq r$ and $\gamma_G(r)$ is a congruence of G .

PROOF: Combine Proposition 3.3 and Lemma 3.10. $\square$

Lemma 3.12. Let r be a binary relation on a groupoid G . Put $s = \gamma_G(r) \cap \delta_G(r)$.

- (i) Then s is stable and $s \subseteq \alpha_G(r) \cap \beta_G(r)$.
- (ii) If r is symmetric then s is symmetric.
- (iii) If G is a division groupoid and r is transitive then s is transitive.
- (iv) If G is a cancellation groupoid and r is reflexive then s is reflexive.
- (v) If G is a division groupoid and r is left stable then $s \subseteq r$.
- (vi) If t is a cancellative congruence of G such that $t \subseteq r$ then $t \subseteq s$.

PROOF: Combine Lemma 3.10 and its dual. $\square$

Lemma 3.13. Let r be a congruence of a quasigroup Q . Then

$$\text{core}_Q(r) \subseteq \gamma_Q(r) \cap \delta_Q(r) \subseteq \alpha_Q(r) \cap \beta_Q(r) \subseteq r$$

and $\gamma_Q(r) \cap \delta_Q(r)$ is a congruence of Q .

PROOF: Combine Proposition 3.3 and Lemma 3.12. $\square$

Lemma 3.14. *Let r be a congruence of a left division groupoid G . If $\mathbf{c}_l(G) \subseteq r$ then*

$$\mathbf{a}_l(G) \subseteq \mathbf{c}_l(G) \subseteq \text{core}_{l,G}(r) \subseteq \gamma_G(r) \subseteq \alpha_G(r) \subseteq r$$

and $\gamma_G(r)$ is a congruence of G .

PROOF: Since $\mathbf{a}_l(G) \subseteq r$, one can easily see that $\alpha_G(r)$ is reflexive. The rest follows from Proposition 3.3 and Lemmas 3.5 and 3.10. $\square$

Proposition 3.15. *Let r be a congruence of a left division groupoid G such that $\mathbf{c}_l(G) \subseteq r$. Define a chain $r_0 \supseteq r_1 \supseteq r_2 \supseteq \dots$ of congruences of G by $r_0 = r$ and $r_{i+1} = \gamma_G(r_i)$. Then $\text{core}_{l,G}(r) = \bigcap_{i \geq 0} r_i$.*

PROOF: Put $s = \bigcap_{i \geq 0} r_i$. It follows from Lemma 3.14 that s is a congruence of G and $\mathbf{c}_l(G) \subseteq \text{core}_{l,G}(r) \subseteq s \subseteq r$. In order to prove $s = \text{core}_{l,G}(r)$, it remains to show that s is left cancellative. Let $(xy, xz) \in s$, so that $(xy, xz) \in r_{i+1} = \gamma_G(r_i)$ for all i . We have $(xy, xz) \in \alpha_G(r_i)$, and hence $(y, z) \in r_i$. Since this is true for all i , we get $(y, z) \in s$. $\square$

Lemma 3.16. *Let r be a congruence of a division groupoid G . If $\mathbf{c}(G) \subseteq r$ then*

$$\mathbf{a}(G) \subseteq \mathbf{c}(G) \subseteq \text{core}_G(r) \subseteq \gamma_G(r) \cap \delta_G(r) \subseteq \alpha_G(r) \cap \beta_G(r) \subseteq r$$

and $\gamma_G(r) \cap \delta_G(r)$ is a congruence of G .

PROOF: Use Lemma 3.14 and its dual. $\square$

Proposition 3.17. *Let r be a congruence of a division groupoid G such that $\mathbf{c}(G) \subseteq r$. Define a chain $r_0 \supseteq r_1 \supseteq r_2 \supseteq \dots$ of congruences of G by $r_0 = r$ and $r_{i+1} = \gamma_G(r_i) \cap \delta_G(r_i)$. Then $\text{core}_G(r) = \bigcap_{i \geq 0} r_i$.*

PROOF: Similar to the proof of Proposition 3.15. $\square$

4. Quasigroup covers and pseudo-coreflections

Definition 4.1. Let V be a variety of groupoids and G be a division groupoid from V . By a V -*quasigroup cover* of G we mean a surjective homomorphism $\pi: Q \rightarrow G$ such that Q is a quasigroup belonging to V and $\text{core}_Q(\ker(\pi)) = \text{id}_Q$. The cover is said to be *minimal* if $P = Q$ whenever P is a subquasigroup of Q such that $\pi(P) = G$.

By a *pseudo-coreflection* of G in the class of quasigroups from V we mean a homomorphism $\sigma: Q \rightarrow G$ such that Q is a quasigroup from V and whenever $\sigma': Q' \rightarrow G$ is a homomorphism of a quasigroup Q' from V into G then there is at least one homomorphism $\varrho: Q' \rightarrow Q$ such that $\sigma' = \sigma\varrho$.

By a *coreflection* of G in the class of quasigroups from V we mean a homomorphism $\sigma: Q \rightarrow G$ such that Q is a quasigroup from V and whenever $\sigma': Q' \rightarrow G$ is a homomorphism of a quasigroup Q' from V into G then there is a unique homomorphism $\varrho: Q' \rightarrow Q$ such that $\sigma' = \sigma\varrho$.

Lemma 4.2. *Let $\pi: Q \rightarrow G$ be a surjective homomorphism of a quasigroup Q onto a division groupoid G . Put $s = \text{core}_Q(\ker(\pi))$ and denote by σ the natural projection of Q onto Q/s . There is a unique homomorphism $\varrho: Q/s \rightarrow G$ with $\pi = \varrho\sigma$ and $\varrho: Q/s \rightarrow G$ is a quasigroup cover of G .*

PROOF: The core exists by Proposition 3.3 and the rest is clear. $\square$

Proposition 4.3. *Let G be a division groupoid and let there exist a surjective homomorphism $\pi: Q \rightarrow G$, where Q is a quasigroup. Then G has a quasigroup cover $\pi_1: Q_1 \rightarrow G$ such that $\text{card}(Q_1) = \text{card}(Q)$ and Q_1 is a homomorphic image of a subquasigroup of Q .*

PROOF: If G is finite then G is a quasigroup and we can put $Q_1 = G$. Let G be infinite. There is a subset S of Q such that $\text{card}(S) = \text{card}(G)$ and $\pi(S) = G$. Denote by P the subquasigroup of Q generated by S . According to Lemma 4.2, there is a congruence s of P such that $\varrho: P/s \rightarrow G$ is a quasigroup cover of G . We have $\text{card}(P/s) = \text{card}(G)$. $\square$

Proposition 4.4. *If $\pi: Q \rightarrow G$ is a minimal quasigroup cover of a division groupoid G then $\text{card}(Q) = \text{card}(G)$.*

PROOF: It is easy. $\square$

Proposition 4.5. *Let V be a variety of groupoids and let $G \in V$ be a division groupoid having a pseudo-coreflection $\sigma: Q \rightarrow G$ in the class of quasigroups from V . Let $\pi: P \rightarrow G$ be a V -quasigroup cover of G . Then σ is surjective and there is an injective homomorphism $\varrho: P \rightarrow Q$ such that $\pi = \sigma\varrho$. Consequently, $\text{card}(P) \leq \text{card}(Q)$. If $\sigma: Q \rightarrow G$ is a minimal quasigroup cover of G then ϱ is an isomorphism of P onto Q .*

PROOF: Since σ is a pseudo-coreflection, there is a homomorphism $\varrho: P \rightarrow Q$ with $\pi = \sigma\varrho$. We have $\ker(\varrho) \subseteq \text{core}_P(\ker(\pi)) = \text{id}_P$, so that ϱ is injective. The rest is clear. $\square$

Proposition 4.6. *Let V be a variety of groupoids and let $G \in V$ be a division groupoid having a pseudo-coreflection $\sigma: Q \rightarrow G$ in the class of quasigroups from V . Put $s = \text{core}_Q(\ker(\sigma))$ and denote by $\pi: Q \rightarrow Q/s$ the natural projection. There is a homomorphism $\varrho: Q/s \rightarrow G$ with $\sigma = \varrho\pi$, and this homomorphism is also a pseudo-coreflection of G in the class of quasigroups from V . If σ is surjective then ϱ is a quasigroup cover of G . If σ is a coreflection then $s = \text{id}_Q$.*

PROOF: Since σ is a pseudo-coreflection, there is a homomorphism $\tau: Q/s \rightarrow Q$ such that $\varrho = \sigma\tau$. Clearly, τ is injective. We have $\sigma \text{id}_Q = \sigma = \varrho\pi = \sigma\tau\pi$. So, if σ is a coreflection then $\tau\pi = \text{id}_Q$, and hence $s = \text{id}_Q$. The rest is clear. $\square$

5. Quasigroup covers of large regular cardinalities

Let G be a division groupoid and let α be a regular cardinal number (i.e., no set of cardinality less than α is the union of fewer than α sets for cardinality less than α) such that $\alpha > \text{card}(G)$. In this section we are going to construct a quasigroup cover of G of cardinality α .

Construction 5.1. Put $Q = G \times \alpha$. An element $(a, i) \in Q$ will be denoted by a_i . We have $\text{card}(Q) = \alpha$.

A partial binary operation f on Q is said to be *small* if $\text{card}(f) < \alpha$. An element $a_i \in Q$ is said to be *f-unused* if $f(x, y) = z$ implies $a_i \notin \{x, y, z\}$. If f is small then for every $a \in G$ the set of f -unused elements in $\{a_i : i \in \alpha\}$ is of cardinality α .

Denote by Z the set of ordered quintuples $(a_i, b_j, a_k, b_m, \varepsilon)$, where $a, b \in G$, $i, j, k, m \in \alpha$ and $\varepsilon \in \{0, 1, 2, 3\}$.

Take a bijection B of α onto Z . We are going to define, by transfinite induction, a small partial binary operation f_I on Q for any $I \in \alpha$, in such a way that $I \leq J$ implies $f_I \subseteq f_J$. Let $I \in \alpha$ and suppose that f_J has been defined for all $J < I$. Denote by f'_I the union of all these f_J , $J < I$. Then f'_I is a partial binary operation on Q and f'_I is small (since α is regular). Let $B(I) = (a_i, b_j, a_k, b_m, \varepsilon)$. We distinguish four cases.

Case 1: Let $\varepsilon = 0$. If $f'_I(a_i, b_j)$ is defined, put $f_I = f'_I$. Otherwise, put $c = ab$ (the product in G), choose $p \in \alpha$ in such a way that c_p is f'_I -unused, and let f_I be the extension of f'_I by $f_I(a_i, b_j) = c_p$.

Case 2: Let $\varepsilon = 1$. If there is an element $x \in Q$ with $f'_I(a_i, x) = b_j$, put $f_I = f'_I$. Otherwise, choose $c \in G$ such that $ac = b$ (this is possible, since G is a division groupoid), choose $p \in \alpha$ in such a way that c_p is f'_I -unused, and let f_I be the extension of f'_I by $f_I(a_i, c_p) = b_j$.

Case 3: Let $\varepsilon = 2$. If there is an element $y \in Q$ with $f'_I(y, a_i) = b_j$, put $f_I = f'_I$. Otherwise, choose $c \in G$ so that $ca = b$, choose $p \in \alpha$ in such a way that c_p is f'_I -unused, and let f_I be the extension of f'_I by $f_I(c_p, a_i) = b_j$.

Case 4: Let $\varepsilon = 3$. Take elements $c, d, e \in G$ such that $ac = d$ and $de = b$ (the element c can be taken arbitrarily, and then the existence of e follows from the fact that G is a division groupoid). Choose $p, q, r, s \in \alpha$ in such a way that c_p, d_q, d_r, e_s are pairwise different and f'_I -unused, and let f_I be the extension of f'_I by $f_I(a_i, c_p) = d_q$, $f_I(d_q, e_s) = b_j$, $f_I(a_k, c_p) = d_r$, $f_I(d_r, e_s) = b_m$.

This completes the transfinite induction. Denote by f the union of all the partial operations f_I , $I \in \alpha$. Clearly, f is a binary operation on Q and it is not difficult to check that Q is a quasigroup with respect to f . The mapping $a_i \mapsto a$ is a homomorphism of Q onto G . Denote by κ its kernel, i.e., $(a_i, b_j) \in \kappa$ if and only if $a = b$.

Let θ be a groupoid congruence of Q such that $\theta \subseteq \kappa$ and $\theta \neq \text{id}_Q$. We are going to prove that $\theta = \kappa$. We have $(a_i, a_k) \in \theta$ for some $a \in G$ and some $i \neq j$. Consider any pair $(b_j, b_m) \in \kappa$ with $j \neq m$. There is an $I \in \alpha$ with $B(I) = (a_i, b_j, a_k, b_m, 3)$. By Case 4, there are elements c_p, d_q, d_r, e_s with $f(a_i, c_p) = d_q$, $f(a_k, c_p) = d_r$ (so that $(d_q, d_r) \in \theta$) and $f(d_q, e_s) = b_j$, $f(d_r, e_s) = b_m$, so that $(b_j, b_m) \in \theta$. Hence $\theta = \kappa$.

We have proved the following theorem.

Theorem 5.2. *Let G be a division groupoid and α be a regular cardinal number such that $\alpha > \text{card}(G)$. The mapping h , defined by $h(a_i) = a$, is a homomorphism of the (above constructed) quasigroup Q of cardinality α onto G . There is no groupoid congruence of Q contained in the kernel of h , other than the identity on Q and the kernel of h . In particular, if G is not a quasigroup then $h: Q \rightarrow G$ is a quasigroup cover of G .*

Theorem 5.3. *Let G be a division groupoid such that G is not a quasigroup. Then G has no pseudo-coreflection in the class of quasigroups.*

PROOF: Suppose that G has a pseudo-coreflection $\sigma: Q \rightarrow G$. Take an arbitrary regular cardinal number $\alpha > \text{card}(G)$. By Theorem 5.2, there is a quasigroup cover $h_\alpha: P_\alpha \rightarrow G$ such that $\text{card}(P_\alpha) = \alpha$. According to Proposition 4.5, $\alpha = \text{card}(P_\alpha) \leq \text{card}(Q)$. But there are arbitrarily large regular cardinal numbers, so we get a contradiction. $\square$

Theorem 5.4. *Let V be any of the following three varieties:*

- (i) *the variety of idempotent groupoids;*
- (ii) *the variety of commutative groupoids;*
- (iii) *the variety of idempotent commutative groupoids.*

Let $G \in V$ be a division groupoid such that G is not a quasigroup. Then for every regular cardinal number $\alpha > \text{card}(G)$ there is a V -quasigroup cover $h: Q \rightarrow G$ such that $\text{card}(Q) = \alpha$. The groupoid G has no pseudo-coreflection in the class of quasigroups from V .

PROOF: Construction 5.1 can be slightly modified in an obvious individual way to the needs of each of these three cases. Then we can proceed as in the proofs of Theorems 5.2 and 5.3. $\square$

6. A simple construction of a quasigroup cover

Let G be a division groupoid (which is not a quasigroup). Then G has also a quasigroup cover of cardinality $\text{card}(G)$. This could be proved by the techniques used in the proof of Construction 5.1 (cf. Proposition 4.3), at the cost of some technical difficulties. We will give here a different, more simple proof. We first need a construction of free quasigroups that can be traced back to T. Evans, see [7].

Construction 6.1. Let X be a nonempty set. Denote by T the absolutely free (term) algebra with three binary operations $\cdot, /, \backslash$ over X and let FQ_X be the subset of T consisting of the terms that contain no subterm of any of the forms $(uv)/v, u \backslash (uv), u(u \backslash v), (v/u)u, v/(u \backslash v), (v/u) \backslash v$ (for any terms u and v). Define three binary operations $\circ, \nearrow, \nwarrow$ on FQ_X as follows. If $uv \in FQ_X$, put $u \circ v = uv$; otherwise, if either $v = u \backslash w$ or $u = w/v$ for a term w then put $u \circ v = w$. If $u/v \in FQ_X$, put $u \nearrow v = u/v$; otherwise, if either $u = uv$ or $v = w \backslash u$ then put $u \nearrow v = w$. If $u \backslash v \in FQ_X$, put $u \nwarrow v = u \backslash v$; otherwise, if either $v = uw$ or $u = v/w$ then put $u \nwarrow v = w$. Then FQ_X is a free quasigroup over X with respect to these operations $\circ, \nearrow, \nwarrow$.

Construction 6.2. Now, let G be a division groupoid. For every $t \in FQ_G$ define $h(t) \in G$ by induction on the length of the term t as follows. If $t \in G$, let $h(t) = t$. If $t = uv$, let $h(t) = h(u)h(v)$. If $t = u/v$, let $h(t)$ be an arbitrarily chosen element $a \in G$ with $ah(v) = h(u)$ (its existence follows from the fact that G is a division groupoid). If $t = u \backslash v$, let $h(t)$ be an arbitrarily chosen element $a \in G$ with $h(u)a = h(v)$.

Let us prove that h is a homomorphism of the quasigroup FQ_G onto G . Let $u, v \in FQ_G$. We must prove $h(u)h(v) = h(u \circ v)$ in G . If $u \circ v = uv$, it follows from the definition. Let $v = u \backslash w$, so that $u \circ v = w$. Then $h(v)$ was defined in such a way that $h(u)h(v) = h(w) = h(u \circ v)$. In the remaining case $u = w/v$ the proof is similar.

Therefore for every division groupoid G we have constructed a quasigroup FQ_G and a homomorphism h of FQ_G onto G . Clearly, if G is infinite then $\text{card}(FQ_G) = \text{card}(G)$. This need not to be a quasigroup cover, but according to Lemma 4.2, we obtain a quasigroup cover of G if we take the factor of FQ_G through the congruence $\text{core}_{FQ_G}(\ker(h))$.

Observe that this construction of quasigroup cover is canonical in some sense (which we do not fully understand).

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