

Attila Nagy; Csaba Tóth

On special Rees matrix semigroups over semigroups

Commentationes Mathematicae Universitatis Carolinae, Vol. 64 (2023), No. 3, 279–288

Persistent URL: <http://dml.cz/dmlcz/152299>

Terms of use:

© Charles University in Prague, Faculty of Mathematics and Physics, 2023

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

On special Rees matrix semigroups over semigroups

ATTILA NAGY, CSABA TÓTH

Abstract. We study the right regular representation of special Rees matrix semigroups over semigroups, and discuss their embedding in idempotent-free left simple semigroups.

Keywords: semigroup; Rees matrix semigroup; representation of semigroups

Classification: 20M10, 20M30

1. Introduction

The concept of the Rees matrix semigroup is one of the tools that can be effectively applied in the study of the structure of semigroups. The Rees matrix construction was first used by D. Rees in [27] to give a description of completely simple (and completely 0-simple) semigroups in terms of their maximal subgroups. Completely simple semigroups are simple semigroups containing a primitive idempotent. By an other characterization, they are simple semigroups containing both a minimal left ideal and a minimal right ideal. By the Rees theorem, see, for example, [4, Theorem 3.5], a semigroup is completely simple if and only if it is isomorphic with a Rees matrix semigroup over a group. The group in the construction of a Rees matrix semigroup can be replaced with a semigroup, see, for example, [1], [8], [10], [12], [13], [26]. If S is a semigroup, I and Λ are nonempty sets, and P is a $\Lambda \times I$ matrix with entries $P(\lambda, i)$, then the set $\mathcal{M}(S; I, \Lambda; P)$ of all triples $(i, s, \lambda) \in I \times S \times \Lambda$ is a semigroup under the multiplication $(i, s, \lambda)(j, t, \mu) = (i, sP(\lambda, j)t, \mu)$, which is called a *Rees $I \times \Lambda$ matrix semigroup over the semigroup S with $\Lambda \times I$ sandwich matrix P* . Some studies have focused on Rees $I \times \Lambda$ matrix semigroups over semigroups satisfying $|I| = 1$. In this case the entries of P can be denoted by $P(\lambda)$, the elements of $\mathcal{M}(S; I, \Lambda; P)$ can be considered in the form (s, λ) , and the operation on $\mathcal{M}(S; I, \Lambda; P)$ is modified as follows: $(s, \lambda)(t, \mu) = (sP(\lambda)t, \mu)$. We denote this Rees matrix semigroups as $\mathcal{M}(S; \Lambda; P)$. By [4, Theorem 8.14], a simple semigroup containing a minimal left ideal (or a minimal right ideal) contains an

idempotent element if and only if it is completely simple. This result focuses attention on idempotent-free simple semigroups containing minimal left (but not minimal right) ideals, studied by M. Teissier in [28] and R. Croisot in [7]. Rees $I \times \Lambda$ matrix semigroups over semigroups satisfying $|I| = 1$ are suitable for constructing such semigroups. If S is an idempotent-free left simple semigroup and P is an arbitrary mapping of a nonempty set Λ into S , then the Rees matrix semigroup $\mathcal{M}(S; \Lambda; P)$ is an idempotent-free simple semigroup containing a minimal left ideal, see, for example, the dual of [5, Exercise 6 for Section 8.2]. The above role of Rees $I \times \Lambda$ matrix semigroups over semigroups satisfying $|I| = 1$ motivated us to examine them and investigate their usefulness in other areas of the theory of semigroups. In our present paper we focus on Rees matrix semigroups $\mathcal{M}(S; S/\theta_S; P)$, where θ_S is the kernel of the right regular representation of S and P is a choice function on the collection of all θ_S -classes of S (i.e., $P([s]_{\theta_S}) \in [s]_{\theta_S}$ for every θ_S -class $[s]_{\theta_S}$ of S). These Rees matrix semigroups seem interesting because they show how far the right regular representation of semigroups is from being faithful. In Section 2 we investigate the right regular representation of Rees matrix semigroups $\mathcal{M}(S; S/\theta_S; P)$. We show that the right regular representation of a Rees matrix semigroup $\mathcal{M}(S; S/\theta_S; P)$ is isomorphic to the Rees matrix semigroup $\mathcal{M}(S/\theta_S^*; S/\theta_S; P')$ of the same type, where θ_S^* is the colon congruence of θ_S and P' is the mapping of S/θ_S onto S/θ_S^* defined by the following way: for every $s \in S$, $P': [s]_{\theta_S} \mapsto [s]_{\theta_S^*}$. In Section 3 we discuss the embedding of Rees matrix semigroups $\mathcal{M}(S; S/\theta_S; P)$ in idempotent-free left simple semigroups. The modern semigroup theory tends to focus more on semigroups with at least one idempotent (especially finite semigroups) than on the more mysterious idempotent-free situation; hence whenever we can say something interesting about the idempotent-free case, it is worth considering. We show that if P is an arbitrary choice function on the collection of all θ_S -classes of a semigroup S , then the Rees matrix semigroup $\mathcal{M}(S; S/\theta_S; P)$ can be embedded in an idempotent-free left simple semigroup if and only if S is idempotent-free and the right colon congruence θ_S^* is left cancellative.

2. Right regular representation

A transformation ϱ of a semigroup S is called a *right translation* of S if $(xy)\varrho = x(y\varrho)$ is satisfied for every $x, y \in S$. For an arbitrary element a of a semigroup S , $\varrho_a: x \mapsto xa$, $x \in S$, is a right translation of S which is called an *inner right translation* of S corresponding to the element a . For an arbitrary semigroup S , the mapping $\Phi_S: a \mapsto \varrho_a$ is a homomorphism of S into the semigroup of all right translations of S . The homomorphism Φ_S is called the *right regular representation* of S . We mention that there is a distinction between the

notion of the right regular representation of a semigroup S and the representation induced in S by the right regular representation of S^1 , where S^1 denotes S if S has an identity element and otherwise denotes the semigroup $S \cup \{1\}$ which can be obtained from S by the adjunction of an identity element to S . This latter representation (called the extended right regular representation of S) is always faithful. Using the terminology of [2], the semigroup $\Phi_S(S)$ is also called the right regular representation of S . The right regular representation of semigroups plays an important role in the investigation of the structure of semigroups. We refer to papers [2], [3], [20], [21], [23], [22], [24] in which the structure of semigroups S are examined by the help of the right regular representation $\Phi_S(S)$ of S . For an arbitrary semigroup S , let θ_S denote the kernel of Φ_S . It is clear that $(a, b) \in \theta_S$ for elements $a, b \in S$ if and only if $xa = xb$ for all $x \in S$. In this section we focus on the right regular representation of Rees matrix semigroups $\mathcal{M}(S; S/\theta_S; P)$, where S is an arbitrary semigroup and P is an arbitrary choice function on the collection of all θ_S -classes of S .

First we prove a lemma, which will be used in the proof of Theorem 2.2.

Lemma 2.1. *For arbitrary congruences α, β on a semigroup S with $\alpha \subseteq \beta$ and an arbitrary choice function P on the collection of all α -classes of S , the mapping $\mathcal{F}_{\alpha, \beta}: (a, [s]_\alpha) \mapsto ([a]_\beta, [s]_\alpha)$ is a surjective homomorphism of the Rees matrix semigroup $\mathcal{M}(S; S/\alpha; P)$ onto the Rees matrix semigroup $\mathcal{M}(S/\beta; S/\alpha; P')$, where P' is the mapping of S/α onto S/β defined by $P'([s]_\alpha) = [s]_\beta$ for every $[s]_\alpha \in S/\alpha$.*

PROOF: It is clear that $\mathcal{F}_{\alpha, \beta}$ is surjective. We show that $\mathcal{F}_{\alpha, \beta}$ is a homomorphism. Let $(a, [s]_\alpha)$ and $(b, [t]_\alpha)$ be arbitrary elements of $\mathcal{M}(S; S/\alpha; P)$. Then

$$\begin{aligned} \mathcal{F}_{\alpha, \beta}((a, [s]_\alpha)(b, [t]_\alpha)) &= \mathcal{F}_{\alpha, \beta}((aP([s]_\alpha)b, [t]_\alpha)) = ([aP([s]_\alpha)b]_\beta, [t]_\alpha) \\ &= ([a]_\beta[P([s]_\alpha)]_\beta[b]_\beta, [t]_\alpha). \end{aligned}$$

Since P is a choice function on the collection of all α -classes of S , we have $P([s]_\alpha) \in [s]_\alpha$. Since $\alpha \subseteq \beta$, we have $[P([s]_\alpha)]_\beta = [s]_\beta$ for every $s \in S$. Thus

$$\begin{aligned} ([a]_\beta[P([s]_\alpha)]_\beta[b]_\beta, [t]_\alpha) &= ([a]_\beta[s]_\beta[b]_\beta, [t]_\alpha) = ([a]_\beta P'([s]_\alpha)[b]_\beta, [t]_\alpha) \\ &= ([a]_\beta, [s]_\alpha)([b]_\beta, [t]_\alpha) = \mathcal{F}_{\alpha, \beta}((a, [s]_\alpha))\mathcal{F}_{\alpha, \beta}((b, [t]_\alpha)). \end{aligned}$$

Hence $\mathcal{F}_{\alpha, \beta}$ is a homomorphism. $\square$

If ϱ is a congruence on a semigroup S , then the congruence

$$\varrho^* = \{(a, b) \in S \times S : (\forall s \in S)(sa, sb) \in \varrho\}$$

on S (defined in [19]) will be called the *right colon congruence of ϱ* . This name comes from ring theory: if ϱ_J denotes the congruence on a ring R defined by an ideal J of R , then the congruence on R defined by the right colon ideal of J is the right colon congruence of ϱ_J . It is clear that $\varrho \subseteq \varrho^*$ for an arbitrary congruence ϱ on a semigroup S , and hence we can consider the Rees matrix semigroup $\mathcal{M}(S/\varrho^*; S/\varrho; P')$, where P' is the mapping of S/ϱ onto S/ϱ^* defined by the following way: for every $a \in S$, $P': [a]_\varrho \mapsto [a]_{\varrho^*}$.

Theorem 2.2. *For an arbitrary semigroup S and an arbitrary choice function P on the collection of all θ_S -classes of S , the right regular representation of the Rees matrix semigroup $\mathcal{M}(S; S/\theta_S; P)$ is isomorphic to the Rees matrix semigroup $\mathcal{M}(S/\theta_S^*; S/\theta_S; P')$, where θ_S^* is the right colon congruence of θ_S and P' is the mapping of S/θ_S onto S/θ_S^* defined by the following way: for every $s \in S$, $P': [s]_{\theta_S} \mapsto [s]_{\theta_S^*}$.*

PROOF: By Lemma 2.1, it is sufficient to show that $\theta_{\mathcal{M}(S; S/\theta_S; P)}$ equals the kernel of $\mathcal{F}_{\theta_S, \theta_S^*}$. For elements $(a, [s]_{\theta_S})$ and $(b, [t]_{\theta_S})$ of the Rees matrix semigroup $\mathcal{M}(S; S/\theta_S; P)$, we have $((a, [s]_{\theta_S}), (b, [t]_{\theta_S})) \in \ker \mathcal{F}_{\theta_S, \theta_S^*}$ if and only if $([a]_{\theta_S^*}, [s]_{\theta_S}) = ([b]_{\theta_S^*}, [t]_{\theta_S})$, that is,

$$(1) \quad (\forall x, y \in S) \quad xya = xyb \quad \text{and} \quad [s]_{\theta_S} = [t]_{\theta_S}.$$

We show that (1) and $((a, [s]_{\theta_S}), (b, [t]_{\theta_S})) \in \theta_{\mathcal{M}(S; S/\theta_S; P)}$ are equivalent. Since P is a choice function on the collection of all θ_S -classes of S , $(P([y]_{\theta_S}), y) \in \theta_S$ for every $y \in S$, and hence $xP([y]_{\theta_S}) = xy$ for every $x, y \in S$. This fact will be used in the proof several times.

Assume (1). Then, for every $(x, [y]_{\theta_S}) \in \mathcal{M}(S; S/\theta_S; P)$, we have

$$\begin{aligned} (x, [y]_{\theta_S})(a, [s]_{\theta_S}) &= (xP([y]_{\theta_S})a, [s]_{\theta_S}) = (xya, [s]_{\theta_S}) = (xyb, [s]_{\theta_S}) \\ &= (xP([y]_{\theta_S})b, [s]_{\theta_S}) = (x, [y]_{\theta_S})(b, [t]_{\theta_S}), \end{aligned}$$

and hence

$$((a, [s]_{\theta_S}), (b, [t]_{\theta_S})) \in \theta_{\mathcal{M}(S; S/\theta_S; P)}.$$

Conversely, assume that $((a, [s]_{\theta_S}), (b, [t]_{\theta_S})) \in \theta_{\mathcal{M}(S; S/\theta_S; P)}$. Then, for every $x, y \in S$,

$$(x, [y]_{\theta_S})(a, [s]_{\theta_S}) = (x, [y]_{\theta_S})(b, [t]_{\theta_S}),$$

and hence

$$\begin{aligned} (xya, [s]_{\theta_S}) &= (xP([y]_{\theta_S})a, [s]_{\theta_S}) = (x, [y]_{\theta_S})(a, [s]_{\theta_S}) = (x, [y]_{\theta_S})(b, [t]_{\theta_S}) \\ &= (xP([y]_{\theta_S})b, [t]_{\theta_S}) = (xyb, [t]_{\theta_S}), \end{aligned}$$

which implies (1). Consequently $\theta_{\mathcal{M}(S; S/\theta_S; P)}$ equals the kernel of the homomorphism $\mathcal{F}_{\theta_S, \theta_S^*}$. $\square$

By the proof of Theorem 2.2, if β is a congruence on a semigroup S such that $\beta = \theta_S^*$ and P is a choice function on the collection of all θ_S -classes of S , then $\theta_{\mathcal{M}(S;S/\theta_S;P)}$ equals the kernel of $\mathcal{F}_{\theta_S,\beta}$. The next proposition shows that the converse assertion is also true.

Proposition 2.3. *Let P be a choice function on the collection of all θ_S -classes of a semigroup S . If β is a congruence on S such that $\theta_S \subseteq \beta$ and $\theta_{\mathcal{M}(S;S/\theta_S;P)}$ equals the kernel of $\mathcal{F}_{\theta_S,\beta}$, then $\beta = \theta_S^*$.*

PROOF: Let β be a congruence on a semigroup S with conditions that $\theta_S \subseteq \beta$ and $\theta_{\mathcal{M}(S;S/\theta_S;P)}$ equals the kernel of $\mathcal{F}_{\theta_S,\beta}$, where P is a choice function on the collection of all θ_S -classes of S . For elements $a, b \in S$, we have $(a, b) \in \beta$ if and only if for arbitrary $[s]_{\theta_S} \in S/\theta_S$, $\mathcal{F}_{\theta_S,\beta}((a, [s]_{\theta_S})) = \mathcal{F}_{\theta_S,\beta}((b, [s]_{\theta_S}))$. Since the kernel of $\mathcal{F}_{\theta_S,\beta}$ equals $\theta_{\mathcal{M}(S;S/\theta_S;P)}$, this last condition is equivalent to the condition that for every $x, y \in S$, $(x, [y]_{\theta_S})(a, [s]_{\theta_S}) = (x, [y]_{\theta_S})(b, [s]_{\theta_S})$. Since P is a choice function on the collection of all θ_S -classes of S , $P([y]_{\theta_S}) \in [y]_{\theta_S}$ and hence $xP([y]_{\theta_S}) = xy$. Thus the equation $(x, [y]_{\theta_S})(a, [s]_{\theta_S}) = (x, [y]_{\theta_S})(b, [s]_{\theta_S})$ has the form $(xya, [s]_{\theta_S}) = (xyb, [s]_{\theta_S})$. Thus $(a, b) \in \beta$ if and only if, for every $x, y \in S$, we have $xya = xyb$, that is, $(a, b) \in \theta^*$. Hence $\beta = \theta^*$. $\square$

By the result of Theorem 2.2, a triple $S, S/\theta_S, S/\theta_S^*$ of semigroups has the property: there are mappings $S \xleftarrow{P} S/\theta_S \xrightarrow{P'} S/\theta_S^*$ such that the right regular representation of the Rees matrix semigroup $\mathcal{M}(S;S/\theta_S;P)$ is isomorphic to the Rees matrix semigroup $\mathcal{M}(S/\theta_S^*;S/\theta_S;P')$. For brevity, this property will be called right regularity.

Definition 2.4. We say that a triple A, B, C of semigroups is right regular, if there are mappings $A \xleftarrow{P} B \xrightarrow{P'} C$ such that the right regular representation of the Rees matrix semigroup $\mathcal{M}(A;B;P)$ is isomorphic to the Rees matrix semigroup $\mathcal{M}(C;B;P')$, that is, $\mathcal{M}(A;B;P)/\theta_{\mathcal{M}(A;B;P)} \cong \mathcal{M}(C;B;P')$.

By [9, Theorem 5.6 on page 24], if ϱ is a congruence on a semigroup S , then ϱ^*/ϱ is a congruence on the factor semigroup S/ϱ , and $(S/\varrho)/(\varrho^*/\varrho) \cong S/\varrho^*$. Recall that ϱ^*/ϱ is defined by the following way: for elements $a, b \in S$, $([a]_{\varrho}, [b]_{\varrho}) \in \varrho^*/\varrho$ if and only if $(a, b) \in \varrho^*$. It is a matter of checking to see that $\varrho^*/\varrho = \theta_{S/\varrho}$. Thus $(S/\varrho)/\theta_{S/\varrho} \cong S/\varrho^*$. This implies the following proposition.

Proposition 2.5. *For an arbitrary congruence ϱ on an arbitrary semigroup S , the triple $S/\varrho, S/\varrho^*, S/(\varrho^*)^*$ of factor semigroups is right regular.*

By Theorem 2.2, if A, B, C are semigroups with $A/\theta_A \cong B$ and $B/\theta_B \cong C$, then the triple A, B, C is right regular. The following example shows that conditions $A/\theta_A \cong B$, $B/\theta_B \cong C$ are not necessary for a triple A, B, C of semigroups to be right regular.

Example 2.6. Let A be a left zero semigroup, B be a right zero semigroup with $|B| \geq 2$, and $C = \{e\}$ be a one-element semigroup. Recall that a semigroup is a left (or right) zero semigroup if it satisfies the identity $ab = a$ ($ab = b$, respectively). It is easy to see that θ_A is the universal relation on A , and θ_B is the identity relation on B . Then $A/\theta \not\cong B$ and $B/\theta \not\cong C$. We show that the triple A, B, C is right regular. Let $P: B \rightarrow A$ be an arbitrary mapping. For every elements (a_1, b_1) and (a_2, b_2) of $\mathcal{M}(A; B; P)$, $(a_1, b_1)(a_2, b_2) = (a_1 P(b_1) a_2, b_2) = (a_1, b_2)$. Thus $\mathcal{M}(A; B; P)$ is isomorphic to the direct product of the left zero semigroup A and the right zero semigroup B . Then the right regular representation of $\mathcal{M}(A; B; P)$ is isomorphic to the semigroup B . Let P' denote the (only possible) mapping of B onto the one-element semigroup $C = \{e\}$. In the semigroup $\mathcal{M}(C; B; P')$, we have $(e, b_1)(e, b_2) = (e P'(b_1) e, b_2) = (e, b_2)$ for every $b_1, b_2 \in B$ from which it follows that $b \mapsto (e, b)$ is an isomorphism of B onto $\mathcal{M}(C; B; P')$. Thus $B \cong \mathcal{M}(C; B; P')$. Consequently the right regular representation of the Rees matrix semigroup $\mathcal{M}(A; B; P)$ is isomorphic to the Rees matrix semigroup $\mathcal{M}(C; B; P')$. Consequently the triple A, B, C is right regular.

Further results on right regular triples of semigroups can be found in [29], in which the connection between the structure of semigroups belonging to a right regular triples of semigroups are examined, and examples of right regular triples of semigroups are presented.

3. Embedding theorems

Let $\mathcal{C}$ be a class of semigroups. A congruence ρ on a semigroup S is called a $\mathcal{C}$ -congruence if the factor semigroup S/ρ belongs to $\mathcal{C}$. According to this definition, a congruence ρ on a semigroup S is called a *left cancellative congruence* if the factor semigroup S/ρ is left cancellative, that is, $(xa, xb) \in \rho$ implies $(a, b) \in \rho$ for every $x, a, b \in S$. In [6, Theorem 1], P. M. Cohn gave necessary and sufficient conditions for a semigroup to be embedded in a left simple semigroup. The conditions differ essentially according to whether or not the semigroup contains an idempotent element. He proved that a semigroup S is embedded in an idempotent-free left simple semigroup if and only if S is idempotent-free and satisfies the condition: for all $a, b, x, y \in S$, $xa = xb$ implies $ya = yb$. Using the terminology of [21], a semigroup S satisfying this last condition is called a *left equalizer simple semigroup*. In other words, a semigroup S is left equalizer simple if for arbitrary elements $a, b \in S$, the assumption that $xa = xb$ is satisfied for some $x \in S$ implies that $ya = yb$ is satisfied for all $y \in S$. By [21, Theorem 2.1], a semigroup S is left equalizer simple if and only if θ_S is a left cancellative congruence (i.e., the right regular representation of S is left

cancellative). Thus a semigroup S is embedded in an idempotent-free left simple semigroup if and only if S is idempotent-free and θ_S is a left cancellative congruence. The next theorem gives necessary and sufficient conditions for a Rees matrix semigroup $\mathcal{M}(S; S/\theta_S; P)$ to be embedded in an idempotent-free left simple semigroup, where P is a choice function on the collection of all θ_S -classes of S . The right colon congruence of θ_S plays an important role.

Theorem 3.1. *Let S be a semigroup and P be a choice function on the collection of all θ_S -classes of S . Then the Rees matrix semigroup $\mathcal{M}(S; S/\theta_S; P)$ can be embedded in an idempotent-free left simple semigroup if and only if S is idempotent-free and the right colon congruence θ_S^* is left cancellative.*

PROOF: By [22, Corollary 2.4], a semigroup S is idempotent-free if and only if the right regular representation of S (i.e., the factor semigroup S/θ_S) is idempotent-free. This fact will be used in the proof several times.

Assume that S is an idempotent-free semigroup such that the right colon congruence θ_S^* is left cancellative. Then the factor semigroup S/θ_S^* is left cancellative. Since $(S/\theta_S)/\theta_{S/\theta_S} \cong S/\theta_S^*$, it follows from [22, Corollary 2.4] that S/θ_S^* is idempotent-free. Thus S/θ_S^* is left cancellative and idempotent-free. We show that the Rees matrix semigroup $\mathcal{M}(S/\theta_S^*; S/\theta_S; P')$ is also left cancellative and idempotent-free, where $P': S/\theta_S \mapsto S/\theta_S^*$ is defined by $P'([s]_{\theta_S}) = [s]_{\theta_S^*}$. Let $([x]_{\theta_S^*}, [y]_{\theta_S}), ([a]_{\theta_S^*}, [s]_{\theta_S}), ([b]_{\theta_S^*}, [t]_{\theta_S})$ be arbitrary elements of the Rees matrix semigroup $\mathcal{M}(S/\theta_S^*; S/\theta_S; P')$. Assume

$$([x]_{\theta_S^*}, [y]_{\theta_S})([a]_{\theta_S^*}, [s]_{\theta_S}) = ([x]_{\theta_S^*}, [y]_{\theta_S})([b]_{\theta_S^*}, [t]_{\theta_S}).$$

Then

$$\begin{aligned} ([xya]_{\theta_S^*}, [s]_{\theta_S}) &= ([x]_{\theta_S^*}[y]_{\theta_S^*}[a]_{\theta_S^*}, [s]_{\theta_S}) = ([x]_{\theta_S^*}P'([y]_{\theta_S})[a]_{\theta_S^*}, [s]_{\theta_S}) \\ &= ([x]_{\theta_S^*}, [y]_{\theta_S})([a]_{\theta_S^*}, [s]_{\theta_S}) = ([x]_{\theta_S^*}, [y]_{\theta_S})([b]_{\theta_S^*}, [t]_{\theta_S}) \\ &= ([x]_{\theta_S^*}P'([y]_{\theta_S})[b]_{\theta_S^*}, [t]_{\theta_S}) = ([x]_{\theta_S^*}[y]_{\theta_S^*}[b]_{\theta_S^*}, [t]_{\theta_S}) \\ &= ([xyb]_{\theta_S^*}, [t]_{\theta_S}), \end{aligned}$$

and hence $(xya, xyb) \in \theta_S^*$ and $[s]_{\theta_S} = [t]_{\theta_S}$. Since θ_S^* is a left cancellative congruence on S , we have $(a, b) \in \theta_S^*$. Thus $([a]_{\theta_S^*}, [s]_{\theta_S}) = ([b]_{\theta_S^*}, [t]_{\theta_S})$, and hence the Rees matrix semigroup $\mathcal{M}(S/\theta_S^*; S/\theta_S; P')$ is left cancellative. Suppose, indirectly, that the Rees matrix semigroup $\mathcal{M}(S/\theta_S^*; S/\theta_S; P')$ contains an idempotent element $([e]_{\theta_S^*}, [f]_{\theta_S})$. Then

$$([e]_{\theta_S^*}, [f]_{\theta_S}) = ([e]_{\theta_S^*}, [f]_{\theta_S})([e]_{\theta_S^*}, [f]_{\theta_S}) = ([e]_{\theta_S^*}[f]_{\theta_S^*}[e]_{\theta_S^*}, [f]_{\theta_S}),$$

and hence $[e]_{\theta_S^*}$ is a regular element of the semigroup S/θ_S^* . Thus S/θ_S^* contains an idempotent element. This is a contradiction. Consequently the Rees matrix

semigroup $\mathcal{M}(S/\theta_S^*; S/\theta_S; P')$ is an idempotent-free left cancellative semigroup. By Theorem 2.2, the right regular representation of the Rees matrix semigroup $\mathcal{M}(S; S/\theta_S; P)$ is isomorphic to the Rees matrix semigroup $\mathcal{M}(S/\theta_S^*; S/\theta_S; P')$. Thus $\theta_{\mathcal{M}(S; S/\theta_S; P)}$ is a left cancellative congruence, and hence $\mathcal{M}(S; S/\theta_S; P)$ is left equalizer simple by [21, Theorem 2.1]. Since $\mathcal{M}(S/\theta_S^*; S/\theta_S; P')$ is idempotent-free, it also follows from [22, Corollary 2.4] that $\mathcal{M}(S; S/\theta_S; P)$ is idempotent-free. Thus $\mathcal{M}(S; S/\theta_S; P)$ is an idempotent-free left equalizer simple semigroup. By [6, Theorem 1], $\mathcal{M}(S; S/\theta_S; P)$ can be embedded in an idempotent-free left simple semigroup.

Conversely, assume that the Rees matrix semigroup $\mathcal{M}(S; S/\theta_S; P)$ can be embedded in an idempotent-free left simple semigroup. Then $\mathcal{M}(S; S/\theta_S; P)$ is an idempotent-free left equalizer simple semigroup by [6, Theorem 1]. Since the right regular representation of the Rees matrix semigroup $\mathcal{M}(S; S/\theta_S; P)$ is isomorphic to the Rees matrix semigroup $\mathcal{M}(S/\theta_S^*; S/\theta_S; P')$, [22, Corollary 2.4] and [21, Theorem 2.1] imply that $\mathcal{M}(S/\theta_S^*; S/\theta_S; P')$ is an idempotent-free left cancellative semigroup. If e is an idempotent element of S , then

$$([e]_{\theta_S^*}, [e]_{\theta_S})([e]_{\theta_S^*}, [e]_{\theta_S}) = ([e]_{\theta_S^*})^3, [e]_{\theta_S} = ([e^3]_{\theta_S^*}, [e]_{\theta_S}) = ([e]_{\theta_S^*}, [e]_{\theta_S}),$$

that is, $([e]_{\theta_S^*}, [e]_{\theta_S})$ is an idempotent element of $\mathcal{M}(S/\theta_S^*; S/\theta_S; P')$. This implies that S is idempotent-free, because $\mathcal{M}(S/\theta_S^*; S/\theta_S; P')$ is idempotent-free. It remains to show that θ_S^* is a left cancellative congruence on S . Assume $(xa, xb) \in \theta_S^*$ for elements $x, a, b \in S$. Then, for arbitrary $y, s \in S$, we have

$$\begin{aligned} ([y]_{\theta_S^*}, [x]_{\theta_S})([a]_{\theta_S^*}, [s]_{\theta_S}) &= ([y]_{\theta_S^*} P'([x]_{\theta_S})[a]_{\theta_S^*}, [s]_{\theta_S}) = ([yxa]_{\theta_S^*}, [s]_{\theta_S}) \\ &= ([yxb]_{\theta_S^*}, [s]_{\theta_S}) = ([y]_{\theta_S^*} P'([x]_{\theta_S})[b]_{\theta_S^*}, [s]_{\theta_S}) \\ &= ([y]_{\theta_S^*}, [x]_{\theta_S})([b]_{\theta_S^*}, [s]_{\theta_S}), \end{aligned}$$

from which we get $([a]_{\theta_S^*}, [s]_{\theta_S}) = ([b]_{\theta_S^*}, [s]_{\theta_S})$, because $\mathcal{M}(S/\theta_S^*; S/\theta_S; P')$ is left cancellative. Thus $(a, b) \in \theta_S^*$. Consequently θ_S^* is a left cancellative congruence. $\square$

A semigroup satisfying the identity $axy = ayx$ is called a *right commutative semigroup*. Right commutative semigroups are investigated in many papers, see, for example, [11], [14], [15], [17], [18], [22], and in books [16], [25]. It is clear that every left zero semigroup is right commutative. A semigroup is called a *left abelian group* if it is a direct product of a left zero semigroup and an abelian group. It is known, see the dual of [25, III. 5. 7. Exercise 6]) that a semigroup is embeddable into a left abelian group if and only if it is right commutative and right cancellative. The next lemma is on the congruence θ_S^* , where S is a right commutative right cancellative semigroup.

Lemma 3.2. *If S is a right commutative right cancellative semigroup, then the congruence θ_S^* is left cancellative.*

PROOF: Let S be a right commutative right cancellative semigroup. Assume $(xa, xb) \in \theta_S^*$ for elements $x, a, b \in S$. Then, for every $s, t \in S$, $stxa = stxb$, and hence $stax = stbx$ by right commutativity of S . Since S is right cancellative, $sta = stb$ (for every $s, t \in S$), and hence $(a, b) \in \theta_S^*$. Consequently θ_S^* is a left cancellative congruence on S . $\square$

Corollary 3.3. *Let S be an idempotent-free right commutative right cancellative semigroup and P be a choice function on the collection of all θ_S -classes of S . Then the Rees matrix semigroup $\mathcal{M}(S; S/\theta_S; P)$ is embedded in an idempotent-free left simple semigroup.*

PROOF: By Lemma 3.2 and Theorem 3.1, it is obvious. $\square$

Acknowledgement. The authors are grateful to the referee for useful comments and suggestions.

REFERENCES

- [1] Ayik H., Ruškuc N., *Generators and relations of Rees matrix semigroups*, Proc. Edinburgh Math. Soc. (2) **42** (1999), no. 3, 481–495.
- [2] Chrislock J. L., *Semigroups whose regular representation is a group*, Proc. Japan Acad. **40** (1964), no. 10, 799–800.
- [3] Chrislock J. L., *Semigroups whose regular representation is a right group*, Amer. Math. Monthly **74** (1967), no. 9, 1097–1100.
- [4] Clifford A. H., Preston G. B., *The Algebraic Theory of Semigroups. Vol. I*, Math. Surveys, 7, American Mathematical Society, Providence, 1961.
- [5] Clifford A. H., Preston G. B., *The Algebraic Theory of Semigroups. Vol. II*, Math. Surveys, 7, American Mathematical Society, Providence, 1967.
- [6] Cohn P. M., *Embeddings in semigroups with one-sided division*, J. London Math. Soc. **31** (1956), no. 2, 169–181.
- [7] Croisot R., *Demi-groupes simple inversifs à gauche*, C. R. Acad. Sci. Paris **239** (1954), 845–847 (French).
- [8] Descalço L., Ruškuc N., *On automatic Rees matrix semigroups*, Comm. Algebra **30** (2002), no. 3, 1207–1226.
- [9] Howie J. M., *An Introduction to Semigroup Theory*, L. M. S. Monographs, 7, Academic Press, London, 1976.
- [10] Kambites M., *The loop problem for Rees matrix semigroups*, Semigroup Forum **76** (2008), no. 2, 204–216.
- [11] Kehayopulu N., Tsingelis M., *Ordered semigroups which are both right commutative and right cancellative*, Semigroup Forum **84** (2012), no. 3, 562–568.

- [12] Lawson M. V., *Rees matrix semigroups*, Proc. Edinburgh Math. Soc. (2) **33** (1990), no. 1, 23–37.
- [13] McAlister D. B., *Rees matrix covers for locally inverse semigroups*, Trans. Amer. Math. Soc. **227** (1983), no. 2, 727–738.
- [14] Nagy A., *Subdirectly irreducible right commutative semigroups*, Semigroup Forum **46** (1993), no. 2, 187–198.
- [15] Nagy A., *Right commutative Δ -semigroups*, Acta Sci. Math. (Szeged) **66** (2000), no. 1–2, 33–46.
- [16] Nagy A., *Special Classes of Semigroups*, Adv. Math. (Dordr.), 1, Kluwer Academic Publishers, Dordrecht, 2001.
- [17] Nagy A., *A supplement to my paper “Right commutative Δ -semigroups”*, Acta Sci. Math. (Szeged) **71** (2005), no. 1–2, 35–36.
- [18] Nagy A., *Medial permutable semigroups of the first kind*, Semigroup Forum **76** (2008), no. 2, 297–308.
- [19] Nagy A., *Left reductive congruences on semigroups*, Semigroup Forum **87** (2013), no. 1, 129–148.
- [20] Nagy A., *Remarks on the paper “M. Kolibiar, On a construction of semigroups”*, Period. Math. Hungar. **71** (2015), no. 2, 261–264.
- [21] Nagy A., *Left equalizer simple semigroups*, Acta Math. Hungar. **148** (2016), no. 2, 300–311.
- [22] Nagy A., *A construction of left equalizer simple medial semigroups*, Period. Math. Hungar. **86** (2023), no. 1, 37–42.
- [23] Nagy A., Nagy O., *A construction of semigroups whose elements are middle units*, International Journal of Algebra **14** (2020), no. 3, 163–169.
- [24] Nagy A., Tóth C., *On the probability that two elements of a finite semigroup have the same right matrix*, Comment. Math. Univ. Carolin. **63** (2022), no. 1, 21–31.
- [25] Petrich M., *Lectures in Semigroups*, Math. Lehrbücher Monogr. I. Abt. Math. Lehrbücher, 29, Akademie-Verlag, Berlin, 1977.
- [26] Petrich M., *Embedding semigroups into idempotent generated ones*, Monatsh. Math. **141** (2004), no. 4, 315–322.
- [27] Rees D., *On semi-groups*, Proc. Cambridge Philos. Soc. **36** (1940), no. 4, 387–400.
- [28] Teissier M., *Sur les demi-groupes ne contenant pas d’élément idempotent*, C. R. Acad. Sci. Paris **237** (1953), 1375–1377 (French).
- [29] Tóth C., *Right regular triples of semigroups*, available at arXiv:2211.06600v1 [math.GR] (2022), 13 pages.

A. Nagy, C. Tóth:

DEPARTMENT OF ALGEBRA, BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS,
MŰEGYETEM RKP. 3, BUDAPEST, 1111, HUNGARY

E-mail: nagyat@math.bme.hu

E-mail: tcsaba94@gmail.com

(Received November 15, 2022, revised May 30, 2023)