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Complex interpolation of function spaces with general weights

DOUADI DRIHEM

Abstract. We present the complex interpolation of Besov and Triebel–Lizorkin spaces with generalized smoothness. In some particular cases these function spaces are just weighted Besov and Triebel–Lizorkin spaces. As a corollary of our results, we obtain the complex interpolation between the weighted Triebel–Lizorkin spaces $\dot{F}_{p_0, q_0}^{s_0}(\omega_0)$ and $\dot{F}_{\infty, q_1}^{s_1}(\omega_1)$ with suitable assumptions on the parameters s_0, s_1, p_0, q_0 and q_1 , and the pair of weights (ω_0, ω_1) .

Keywords: Besov space; Triebel–Lizorkin space; complex interpolation; Muckenhoupt class

Classification: 42B25, 42B35, 26B35, 46E35

1. Introduction

The theory of complex interpolation had a remarkable development due to its usefulness in applications in mathematical analysis. For general literature on complex interpolation we refer to [2], [36] and references therein. Let us recall briefly the results of complex interpolation of some known function spaces. For Lebesgue spaces we have

$$[L_{p_0}(\mathbb{R}^n), L_{p_1}(\mathbb{R}^n)]_\theta = L_p(\mathbb{R}^n),$$

see [2, Theorem 5.1.1], where

$$(1.1) \quad 0 < \theta < 1, \quad 1 \leq p_0, p_1 < \infty, \quad \frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Let $L_p(\mathbb{R}^n, \omega)$ denote the weighted Lebesgue space with weight ω , see below. The weighted version is given as follows:

$$(1.2) \quad [L_{p_0}(\mathbb{R}^n, \omega_0), L_{p_1}(\mathbb{R}^n, \omega_1)]_\theta = L_p(\mathbb{R}^n, \omega_0^\theta \omega_1^{1-\theta}),$$

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see [2, Theorem 5.5.3] and [36, Theorem 1.18.5] with the same assumptions (1.1). Clearly all the above results are given for Banach case, but in [34, Lemma 3.4] the authors gave a generalization of (1.2) to the case $0 < p_0, p_1 < \infty$. For Sobolev spaces

$$[W_{p_0}^{m_0}(\mathbb{R}^n), W_{p_1}^{m_1}(\mathbb{R}^n)]_\theta = W_p^m(\mathbb{R}^n),$$

with the same assumptions (1.1), $1 < p_0, p_1 < \infty$, $m_0, m_1 \in \mathbb{N}$ and $m = (1 - \theta)m_0 + \theta m_1$, see [36, Remark 2.4.2/2]. The extension of the above results to generalized scale of function spaces are given in [2] and [36]. For convenience of the reader, we recall some known interpolation results on Besov and Triebel–Lizorkin spaces. Such function spaces cover many well-known classical function spaces such as Hölder–Zygmund spaces, Hardy spaces and Sobolev spaces. For more details one can refer to Triebel’s books [37] and [38]. Let

$$(1.3) \quad 0 < \theta < 1, \quad 0 < p_0, p_1, q_0, q_1 \leq \infty, \quad \frac{1}{p} := \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} := \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}.$$

Then we have

$$[B_{p_0, q_0}^{s_0}(\mathbb{R}^n), B_{p_1, q_1}^{s_1}(\mathbb{R}^n)]_\theta = B_{p, q}^s(\mathbb{R}^n),$$

provided that

$$(1.4) \quad s_0, s_1 \in \mathbb{R}, \quad s = (1 - \theta)s_0 + \theta s_1, \quad 1 < p_0, p_1 < \infty, \quad 1 \leq q_0 < \infty, \quad 1 \leq q_1 \leq \infty,$$

see [36, Theorem 2.4.1]. We mention that in some cases complex interpolation of pairs of Besov spaces does not result in a Besov space. More precisely,

$$(1.5) \quad [B_{p, \infty}^{s_0}(\mathbb{R}^n), B_{p, \infty}^{s_1}(\mathbb{R}^n)]_\theta = \dot{B}_{p, \infty}^s(\mathbb{R}^n),$$

see again [36, Theorem 2.4.1], with $s_0 \neq s_1 \in \mathbb{R}$, $s = (1 - \theta)s_0 + \theta s_1$, $1 < p < \infty$, where $\dot{B}_{p, \infty}^s(\mathbb{R}^n)$ is the closure of the set of test functions in $B_{p, \infty}^s(\mathbb{R}^n)$. The result (1.5) is in contradiction with [2, Theorem 6.4.5] in the case of $s_0 \neq s_1$ and $q_0 = q_1 = \infty$, because of $\dot{B}_{p, \infty}^s(\mathbb{R}^n)$ is smaller than $B_{p, \infty}^s(\mathbb{R}^n)$. For Triebel–Lizorkin spaces we have

$$[F_{p_0, q_0}^{s_0}(\mathbb{R}^n), F_{p_1, q_1}^{s_1}(\mathbb{R}^n)]_\theta = F_{p, q}^s(\mathbb{R}^n),$$

with the same assumptions (1.3) and (1.4) but with $1 < q_0 < \infty$, $1 < q_1 < \infty$, see [36, Theorem 2.4.2/1]. The quasi-Banach version of complex interpolation of Besov and Triebel–Lizorkin spaces can be found in [19] and [27]. For weighted Besov and Lizorkin–Triebel spaces M. Bownik in [6] and A. Wojciechowska in [43] studied the problem

$$(1.6) \quad [F_{p_0, q_0}^{s_0}(\mathbb{R}^n, \omega_0), F_{p_1, q_1}^{s_1}(\mathbb{R}^n, \omega_1)]_\theta.$$

But both authors only deal with the case $\omega_0 = \omega_1 = \omega$. The general problem, $\omega_0 \neq \omega_1$, was completed in [34] but with $p_0 < \infty$ and $p_1 < \infty$ and the weights ω_0, ω_1 being local Muckenhoupt weights in the sense of Rychkov, see [32].

In this direction we present the complex interpolation of function spaces of general weights, introduced in [11] and [12], that are based on Tyulenev class given in [40], [41] and [42]. As an application, we study (1.6), when $p_1 = \infty$ and $\omega_0 \neq \omega_1$.

These types of spaces of generalized smoothness have been introduced by several authors. We refer, for instance, to references [5], [9], [22], [23], [25]; see also [3], [4], [26], and [28].

Recently, O. Dominguez and S. Tikhonov in [10] gave a treatment of function spaces with logarithmic smoothness (Besov, Sobolev, Triebel–Lizorkin), including various new characterizations for Besov norms in terms of different, sharp estimates for Besov norms of derivatives and potential operators (Riesz and Bessel potentials) in terms of norms of functions themselves and sharp embeddings between the Besov spaces defined by differences and by Fourier-analytical decompositions as well as between Besov and Sobolev/Triebel–Lizorkin spaces.

The theory of these spaces had a remarkable development in part due to its usefulness in applications. For instance, they appear in the study of trace spaces on fractals, see [14] and [15], for which D. Edmunds and H. Triebel introduced the spaces $B_{p,q}^{s,\Psi}$, where Ψ is a so-called admissible function, typically of log-type near 0. For a complete treatment of these spaces we refer the reader to the work [30]. More general function spaces of generalized smoothness can be found in [16], and references therein.

The paper is organized as follows. First we recall some basic facts on the Muckenhoupt classes and the weighted class of Tyulenev. Also we give some key technical lemmas needed in the proofs of the main statements. In Section 3, we present some properties of $\dot{B}_{p,q}(\mathbb{R}^n, \{t_k\})$ and $\dot{F}_{p,q}(\mathbb{R}^n, \{t_k\})$ spaces. In Section 4 we shall apply a method which has been used in [34] where we calculate the Calderón products of associated sequence spaces. From an abstract theory on the relation between the complex interpolation and the Calderón product of Banach lattices obtained by A.P. Calderón in [8], we deduce the complex interpolation theorems of these sequence spaces. Finally, the desired complex interpolation theorem is lifted by the φ -transform characterization in the sense of Frazier and Jawerth.

We will adopt the following convention throughout this paper. As usual, we denote by $\mathbb{R}^n$ the n -dimensional real Euclidean space, while $\mathbb{N}$ is the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter $\mathbb{Z}$ stands for the set of all

integer numbers. The expression $f \lesssim g$ means that $f \leq cg$ for some independent constant c (and nonnegative functions f and g), and $f \approx g$ means $f \lesssim g \lesssim f$.

By $\text{supp} f$ we denote the support of the function f , i.e., the closure of its nonzero set. If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of E and χ_E denotes its characteristic function. By c we denote generic positive constants, which may have different values at different occurrences.

A weight is a nonnegative locally integrable function on $\mathbb{R}^n$ that takes values in $(0, \infty)$ almost everywhere. For measurable set $E \subset \mathbb{R}^n$ and a weight γ , $\gamma(E)$ denotes

$$\int_E \gamma(x) dx.$$

Given a measurable set $E \subset \mathbb{R}^n$ and $0 < p \leq \infty$, we denote by $L_p(E)$ the space of all functions $f: E \rightarrow \mathbb{C}$ equipped with the quasi-norm

$$\|f\|_{L_p(E)} := \left(\int_E |f(x)|^p dx \right)^{1/p} < \infty,$$

with $0 < p < \infty$ and

$$\|f\|_{L_\infty(E)} := \text{ess-sup}_{x \in E} |f(x)| < \infty.$$

For a function f in $L_1^{\text{loc}}(\mathbb{R}^n)$, we set

$$M_A(f) := \frac{1}{|A|} \int_A |f(x)| dx$$

for any $A \subset \mathbb{R}^n$. Furthermore, we put

$$M_{A,p}(f) := \left(\frac{1}{|A|} \int_A |f(x)|^p dx \right)^{1/p},$$

with $0 < p < \infty$. Further, given a measurable set $E \subset \mathbb{R}^n$ and a weight γ , we denote the space of all functions $f: \mathbb{R}^n \rightarrow \mathbb{C}$ with finite quasi-norm

$$\|f\|_{L_p(\mathbb{R}^n, \gamma)} := \|f\gamma\|_{L_p(\mathbb{R}^n)}$$

by $L_p(\mathbb{R}^n, \gamma)$.

If $1 \leq p \leq \infty$ and $1/p + 1/p' = 1$, then p' is called the conjugate exponent of p .

The symbol $\mathcal{S}(\mathbb{R}^n)$ is used to denote the set of all Schwartz functions on $\mathbb{R}^n$. In what follows, Q will denote a cube in the space $\mathbb{R}^n$ with sides parallel to the coordinate axes and $l(Q)$ will denote the side length of the cube Q . For $v \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$, denote by $Q_{v,m}$ the dyadic cube,

$$Q_{v,m} := 2^{-v}([0, 1)^n + m).$$

For the collection of all such cubes we use $\mathcal{Q} := \{Q_{v,m} : v \in \mathbb{Z}, m \in \mathbb{Z}^n\}$.

2. Background tools

In this section we present some useful results.

2.1 Muckenhoupt weights. The purpose of this subsection is to review some known properties of Muckenhoupt class.

Definition 2.1. Let $1 < p < \infty$. We say that a weight γ belongs to the Muckenhoupt class $A_p(\mathbb{R}^n)$ if there exists a constant $C > 0$ such that for every cube Q the following inequality holds

$$(2.1) \quad M_Q(\gamma)M_{Q,p'/p}(\gamma^{-1}) \leq C.$$

The smallest constant C for which inequality (2.1) holds is denoted by $A_p(\gamma)$. As an example, we can take

$$\gamma(x) = |x|^\alpha, \quad \alpha \in \mathbb{R}.$$

Then $\gamma \in A_p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $-n < \alpha < n(p-1)$.

For $p = 1$, we rewrite the above definition in the following way.

Definition 2.2. We say that a weight γ belongs to the Muckenhoupt class $A_1(\mathbb{R}^n)$ if there exists a constant $C > 0$ such that for every cube Q and for a.e. $y \in Q$ the following inequality holds

$$(2.2) \quad M_Q(\gamma) \leq C\gamma(y).$$

The smallest constant C for which inequality (2.2) holds denoted by $A_1(\gamma)$. The above classes have been first studied by B. Muckenhoupt in [31] and used to characterize the boundedness of the Hardy–Littlewood maximal function on $L_p(\gamma)$, see the monographs [21] and [24] for a complete account on the theory of Muckenhoupt weights.

We recall a few basic properties of the class of $A_p(\mathbb{R}^n)$ weights, see [24, Chapter 7].

Lemma 2.3. Let $1 \leq p < \infty$.

- (i) If $\gamma \in A_p(\mathbb{R}^n)$, then for any $1 \leq p < q$, $\gamma \in A_q(\mathbb{R}^n)$.
- (ii) Let $1 < p < \infty$. Then $\gamma \in A_p(\mathbb{R}^n)$ if and only if $\gamma^{1-p'} \in A_{p'}(\mathbb{R}^n)$.
- (iii) Suppose that $\gamma \in A_p(\mathbb{R}^n)$ for some $1 < p < \infty$. Then there exists a $1 < p_1 < p < \infty$ such that $\gamma \in A_{p_1}(\mathbb{R}^n)$.

2.2 The weight class $\dot{X}_{\alpha,\sigma,p}$. Let $0 < p \leq \infty$. A weight sequence $\{t_k\}$ is called p -admissible if $t_k \in L_p^{\text{loc}}(\mathbb{R}^n)$ for all $k \in \mathbb{Z}$. We mention here that

$$\int_E t_k^p(x) dx < c(k, E)$$

for any $k \in \mathbb{Z}$ and any compact set $E \subset \mathbb{R}^n$. For a p -admissible weight sequence $\{t_k\}$ we set

$$t_{k,m,p} := \|t_k\| L_p(Q_{k,m}), \quad k \in \mathbb{Z}, \quad m \in \mathbb{Z}^n.$$

A. I. Tyulenev in [39] introduced the following new weighted class and used it to study Besov spaces of variable smoothness.

Definition 2.4. Let $\alpha_1, \alpha_2 \in \mathbb{R}$, $p, \sigma_1, \sigma_2 \in (0, \infty]$, $\alpha = (\alpha_1, \alpha_2)$ and let $\sigma = (\sigma_1, \sigma_2)$. We let $\dot{X}_{\alpha, \sigma, p} = \dot{X}_{\alpha, \sigma, p}(\mathbb{R}^n)$ denote the set of p -admissible weight sequences $\{t_k\}$ satisfying the following conditions. There exist numbers $C_1, C_2 > 0$ such that for any $k \leq j$ and every cube Q ,

$$(2.3) \quad M_{Q,p}(t_k) M_{Q,\sigma_1}(t_j^{-1}) \leq C_1 2^{\alpha_1(k-j)},$$

$$(2.4) \quad (M_{Q,p}(t_k))^{-1} M_{Q,\sigma_2}(t_j) \leq C_2 2^{\alpha_2(j-k)}.$$

The constants $C_1, C_2 > 0$ are independent of both the indexes k and j .

Remark 2.5.

- (i) We would like to mention that if $\{t_k\}$ satisfies inequality (2.3) with $\sigma_1 = r(p/r)'$ and $0 < r < p \leq \infty$, then $t_k^p \in A_{p/r}(\mathbb{R}^n)$ for any $k \in \mathbb{Z}$ with $0 < r < p < \infty$ and $t_k^{-r} \in A_1(\mathbb{R}^n)$ for any $k \in \mathbb{Z}$ with $p = \infty$.
- (ii) We say that $t_k \in A_p(\mathbb{R}^n)$, $k \in \mathbb{Z}$, $1 < p < \infty$, have the same Muckenhoupt constant if

$$A_p(t_k) = c, \quad k \in \mathbb{Z},$$

where c is independent of k .

- (iii) Definition 2.4 is different from the one used in [39, Definition 2.1] and Definition 2.7 in [40], because we used the boundedness of the maximal function on weighted Lebesgue spaces.

Example 2.6. Let $0 < r < p < \infty$, a weight $\omega^p \in A_{p/r}(\mathbb{R}^n)$ and $\{s_k\} = \{2^{ks} \omega^p\}_{k \in \mathbb{Z}}$, $s \in \mathbb{R}$. Clearly, $\{s_k\}_{k \in \mathbb{Z}}$ lies in $\dot{X}_{\alpha, \sigma, p}$ for $\alpha_1 = \alpha_2 = s$, $\sigma = (r(p/r)', p)$.

Remark 2.7. Let $0 < \theta \leq p \leq \infty$. Let $\alpha_1, \alpha_2 \in \mathbb{R}$, $\sigma_1, \sigma_2 \in (0, \infty]$, $\sigma_2 \geq p$, $\alpha = (\alpha_1, \alpha_2)$ and let $\sigma = (\sigma_1 = \theta(p/\theta)', \sigma_2)$. Let a p -admissible weight sequence $\{t_k\} \in \dot{X}_{\alpha, \sigma, p}$. Then

$$\alpha_2 \geq \alpha_1,$$

see [11].

As usual, we put

$$\mathcal{M}(f)(x) := \sup_Q \frac{1}{|Q|} \int_Q |f(y)| dy, \quad f \in L_1^{\text{loc}}(\mathbb{R}^n),$$

where the supremum is taken over all cubes with sides parallel to the axis and $x \in Q$. Also we set

$$\mathcal{M}_\sigma(f) := (\mathcal{M}(|f|^\sigma))^{1/\sigma}, \quad 0 < \sigma < \infty.$$

Recall the vector-valued maximal inequality of C. Fefferman and E. M. Stein in [17].

Theorem 2.8. *Let $0 < p < \infty$, $0 < q \leq \infty$ and $0 < \sigma < \min(p, q)$. Then*

$$(2.5) \quad \left\| \left(\sum_{k=-\infty}^{\infty} (\mathcal{M}_\sigma(f_k))^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)} \lesssim \left\| \left(\sum_{k=-\infty}^{\infty} |f_k|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)}$$

holds for all sequence of functions $\{f_k\} \in L_p(l_q)$.

We state one of the main tools of this paper, see [12].

Lemma 2.9. *Let $1 < \theta \leq p < \infty$ and $1 < q < \infty$. Let $\{t_k\}$ be a p -admissible weight sequence such that $t_k^p \in A_{p/\theta}(\mathbb{R}^n)$, $k \in \mathbb{Z}$. Assume that t_k^p , $k \in \mathbb{Z}$, have the same Muckenhoupt constant, $A_{p/\theta}(t_k^p) = c$, $k \in \mathbb{Z}$. Then*

$$\left\| \left(\sum_{k=-\infty}^{\infty} t_k^q (\mathcal{M}(f_k))^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)} \lesssim \left\| \left(\sum_{k=-\infty}^{\infty} t_k^q |f_k|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)}$$

holds for all sequences of functions $\{f_k\} \in L_p(l_q)$. In particular

$$\|\mathcal{M}(f_k)\|_{L_p(\mathbb{R}^n, t_k)} \leq c \|f_k\|_{L_p(\mathbb{R}^n, t_k)}$$

holds for all sequences $f_k \in L_p(\mathbb{R}^n, t_k)$, $k \in \mathbb{Z}$, where $c > 0$ is independent of k .

Remark 2.10.

- (i) We would like to mention that the result of this lemma is true if we assume that $t_k^p \in A_{p/\theta}(\mathbb{R}^n)$, $k \in \mathbb{Z}$, $1 < p < \infty$ with

$$A_{p/\theta}(t_k^p) \leq c, \quad k \in \mathbb{Z},$$

where $c > 0$ is independent of k .

- (ii) A proof of this result for $t_k^p = \omega$, $k \in \mathbb{Z}$, may be found in [1] and [29].
 (iii) In view of Lemma 2.3 (iii) we can assume that $t_k^p \in A_p(\mathbb{R}^n)$, $k \in \mathbb{Z}$, $1 < p < \infty$ with

$$A_p(t_k^p) \leq c, \quad k \in \mathbb{Z},$$

where $c > 0$ is independent of k .

3. Function spaces

In this section we present the Fourier analytical definition of Besov and Triebel–Lizorkin spaces of variable smoothness and recall their basic properties. Select a pair of Schwartz functions φ and ψ satisfying

$$(3.1) \quad \text{supp } \mathcal{F}\varphi, \mathcal{F}\psi \subset \left\{ \xi: \frac{1}{2} \leq |\xi| \leq 2 \right\},$$

$$(3.2) \quad |\mathcal{F}\varphi(\xi)|, |\mathcal{F}\psi(\xi)| \geq c \quad \text{if } \frac{3}{5} \leq |\xi| \leq \frac{5}{3}$$

and

$$(3.3) \quad \sum_{k=-\infty}^{\infty} \overline{\mathcal{F}\varphi(2^{-k}\xi)} \mathcal{F}\psi(2^{-k}\xi) = 1 \quad \text{if } \xi \neq 0,$$

where $c > 0$. Throughout the paper, for all $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, we put $\varphi_k(x) := 2^{kn}\varphi(2^k x)$ and $\tilde{\varphi}(x) := \overline{\varphi(-x)}$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a function satisfying (3.1) and (3.2). Recall that there exists a function $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (3.1)–(3.3), see [20, Lemma (6.9)]. We set

$$\mathcal{S}_{\infty}(\mathbb{R}^n) := \left\{ \varphi \in \mathcal{S}(\mathbb{R}^n): \int_{\mathbb{R}^n} x^{\beta} \varphi(x) dx = 0 \text{ for all multi-indices } \beta \in \mathbb{N}_0^n \right\}.$$

Let $\mathcal{S}'_{\infty}(\mathbb{R}^n)$ be the topological dual of $\mathcal{S}_{\infty}(\mathbb{R}^n)$, namely, the set of all continuous linear functionals on $\mathcal{S}_{\infty}(\mathbb{R}^n)$.

Now, we define the spaces under consideration.

Definition 3.1. Let $0 < p \leq \infty$ and $0 < q \leq \infty$. Let $\{t_k\}$ be a p -admissible weight sequence, and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (3.1) and (3.2).

- (i) The Besov space $\dot{B}_{p,q}(\mathbb{R}^n, \{t_k\})$ is the collection of all $f \in \mathcal{S}'_{\infty}(\mathbb{R}^n)$ such that

$$\|f| \dot{B}_{p,q}(\mathbb{R}^n, \{t_k\})\| := \left(\sum_{k=-\infty}^{\infty} \|t_k(\varphi_k * f)|_{L_p(\mathbb{R}^n)}\|^q \right)^{1/q} < \infty$$

with the usual modifications if $q = \infty$.

- (ii) Let $0 < p < \infty$. The Triebel–Lizorkin space $\dot{F}_{p,q}(\mathbb{R}^n, \{t_k\})$ is the collection of all $f \in \mathcal{S}'_{\infty}(\mathbb{R}^n)$ such that

$$\|f| \dot{F}_{p,q}(\mathbb{R}^n, \{t_k\})\| := \left\| \left(\sum_{k=-\infty}^{\infty} t_k^q |\varphi_k * f|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)} < \infty$$

with the usual modifications if $q = \infty$.

Remark 3.2. Some properties of these function spaces, such as the φ -transform characterization in the sense of Frazier and Jawerth, the smooth atomic and molecular decomposition and the characterization of $\dot{B}_{p,q}(\mathbb{R}^n, \{t_k\})$ spaces in terms of the difference relations are given in [11], [12] and [13].

Remark 3.3. We would like to mention that the elements of the above spaces are not distributions but equivalence classes of distributions. Instead of $\dot{B}_{p,q}(\mathbb{R}^n, \{t_k\})$ or $\dot{F}_{p,q}(\mathbb{R}^n, \{t_k\})$ we will use the notation $\dot{A}_{p,q}(\mathbb{R}^n, \{t_k\})$.

Using the system $\{\varphi_k\}_{k \in \mathbb{Z}}$ we can define the quasi-norms

$$\|f|\dot{B}_{p,q}^s(\mathbb{R}^n)\| := \left(\sum_{k=-\infty}^{\infty} 2^{ksq} \|\varphi_k * f\|_{L_p(\mathbb{R}^n)}^q \right)^{1/q}$$

and

$$\|f|\dot{F}_{p,q}^s(\mathbb{R}^n)\| := \left\| \left(\sum_{k=-\infty}^{\infty} 2^{ksq} |\varphi_k * f|^q \right)^{1/q} \middle| L_p(\mathbb{R}^n) \right\|$$

for constants $s \in \mathbb{R}$ and $0 < p, q \leq \infty$ with $0 < p < \infty$ in the $\dot{F}$ -case. The Besov space $\dot{B}_{p,q}^s(\mathbb{R}^n)$ consist of all distributions $f \in \mathcal{S}'_{\infty}(\mathbb{R}^n)$ for which

$$\|f|\dot{B}_{p,q}^s(\mathbb{R}^n)\| < \infty.$$

The Triebel–Lizorkin space $\dot{F}_{p,q}^s(\mathbb{R}^n)$ consist of all distributions $f \in \mathcal{S}'_{\infty}(\mathbb{R}^n)$ for which

$$\|f|\dot{F}_{p,q}^s(\mathbb{R}^n)\| < \infty.$$

Further details on the classical theory of these spaces can be found in [18], [19], [20], [37] and [38].

One recognizes immediately that if $\{t_k\} = \{2^{sk}\}$, $s \in \mathbb{R}$, then

$$\dot{B}_{p,q}(\mathbb{R}^n, \{2^{sk}\}) = \dot{B}_{p,q}^s(\mathbb{R}^n) \quad \text{and} \quad \dot{F}_{p,q}(\mathbb{R}^n, \{2^{sk}\}) = \dot{F}_{p,q}^s(\mathbb{R}^n).$$

Moreover, for $\{t_k\} = \{2^{sk}w\}$, $s \in \mathbb{R}$, with a weight w we re-obtain the weighted Besov and Triebel–Lizorkin spaces; we refer, in particular, to the papers [7], [32] and [33] for a comprehensive treatment of the weighted function spaces.

Let $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (3.1) through (3.3). Recall that the φ -transform S_{φ} is defined by setting $(S_{\varphi}f)_{k,m} := \langle f, \varphi_{k,m} \rangle$ where

$$\varphi_{k,m}(x) = 2^{k(n/2)} \varphi(2^k x - m), \quad m \in \mathbb{Z}^n, \text{ and } k \in \mathbb{Z}.$$

The inverse ψ -transform T_{ψ} is defined by

$$T_{\psi}\lambda := \sum_{k=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{k,m} \psi_{k,m},$$

where $\lambda = \{\lambda_{k,m}\}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n} \subset \mathbb{C}$, see [19].

Now we introduce the following sequence spaces.

Definition 3.4. Let $0 < p \leq \infty$ and $0 < q \leq \infty$. Let $\{t_k\}$ be a p -admissible weight sequence. Then for all complex valued sequences $\lambda = \{\lambda_{k,m}\}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n} \subset \mathbb{C}$ we define

$$\dot{b}_{p,q}(\mathbb{R}^n, \{t_k\}) := \{\lambda: \|\lambda| \dot{b}_{p,q}(\mathbb{R}^n, \{t_k\})\| < \infty\},$$

where

$$\|\lambda| \dot{b}_{p,q}(\mathbb{R}^n, \{t_k\})\| := \left(\sum_{k=-\infty}^{\infty} 2^{knq/2} \left\| \sum_{m \in \mathbb{Z}^n} t_k \lambda_{k,m} \chi_{k,m} \right\|_{L_p(\mathbb{R}^n)}^q \right)^{1/q}$$

and

$$\dot{f}_{p,q}(\mathbb{R}^n, \{t_k\}) := \{\lambda: \|\lambda| \dot{f}_{p,q}(\mathbb{R}^n, \{t_k\})\| < \infty\}$$

with $0 < p < \infty$, where

$$\|\lambda| \dot{f}_{p,q}(\mathbb{R}^n, \{t_k\})\| := \left\| \left(\sum_{k=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{knq/2} t_k^q |\lambda_{k,m}|^q \chi_{k,m} \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)}.$$

Allowing the smoothness t_k , $k \in \mathbb{Z}$, to vary from point to point will raise extra difficulties to study these function spaces. But by the following lemma the problem can be reduced to the case of fixed smoothness.

Proposition 3.5. Let $0 < p \leq \infty$ and $0 < q \leq \infty$. Let $\{t_k\}$ be a p -admissible weight sequence.

(i) Then

$$\|\lambda| \dot{b}_{p,q}(\mathbb{R}^n, \{t_k\})\|^* := \left(\sum_{k=-\infty}^{\infty} 2^{knq/2} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{k,m}|^p t_{k,m,p}^p \right)^{q/p} \right)^{1/q}$$

is an equivalent quasi-norm in $\dot{b}_{p,q}(\mathbb{R}^n, \{t_k\})$.

(ii) Let $0 < \theta \leq p < \infty$, $0 < q < \infty$ and $0 < \kappa \leq 1$. Assume that $\{t_k\}$ satisfies (2.3) with $\sigma_1 = \theta(p/\theta)'$ and $j = k$. Then

$$\begin{aligned} & \|\lambda| \dot{f}_{p,q,\kappa}(\mathbb{R}^n, \{t_k\})\|^* \\ &:= \left\| \left(\sum_{k=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{knq(1/2+1/(\kappa p))} t_{k,m,\kappa p}^q |\lambda_{k,m}|^q \chi_{k,m} \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)} \end{aligned}$$

is an equivalent quasi-norm in $\dot{f}_{p,q}(\mathbb{R}^n, \{t_k\})$, where

$$t_{k,m,\kappa p} := \|t_k| L_{\kappa p}(Q_{k,m})\|, \quad k \in \mathbb{Z}, \quad m \in \mathbb{Z}^n.$$

PROOF: We prove only (ii) since (i) is obvious. We will proceed in two steps.

Step 1. Let us prove that

$$(3.4) \quad \|\lambda| \dot{f}_{p,q}(\mathbb{R}^n, \{t_k\})\| \lesssim \|\lambda| \dot{f}_{p,q,\kappa}(\mathbb{R}^n, \{t_k\})\|^*.$$

Let $0 < \eta < \min(\theta, q)$. By duality, $\|\lambda|f_{p,q}(\mathbb{R}^n, \{t_k\})\|^\eta$ is just

$$\sup \sum_{k=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{(n\eta/2)k} |\lambda_{k,m}|^\eta \int_{Q_{k,m}} t_k^\eta(x) |g_k(x)| dx = \sup \sum_{k=-\infty}^{\infty} S_k,$$

where the supremum is taking over all sequence of functions $\{g_k\} \in L_{(p/\eta)'}(l_{(q/\eta)'})$ with

$$(3.5) \quad \|\{g_k\}\|_{L_{(p/\eta)'}(l_{(q/\eta)'})} \leq 1.$$

By Hölder's inequality,

$$1 = \left(\frac{1}{|Q_{k,m}|} \int_{Q_{k,m}} t_k^{-\eta h}(x) t_k^{\eta h}(x) dx \right)^{1/h} \leq M_{Q_{k,m}, \varrho}(t_k^{-\eta})(M_{Q_{k,m}, \kappa p}(t_k))^\eta$$

for any $h > 0$, with $1/h = 1/\varrho + \eta/(\kappa p)$. Therefore,

$$\int_{Q_{k,m}} t_k^\eta(x) |g_k(x)| dx \lesssim |Q_{k,m}| (M_{Q_{k,m}, \kappa p}(t_k))^\eta M_{Q_{k,m}, \varrho}(t_k^{-\eta} \mathcal{M}(t_k^\eta g_k)).$$

We set

$$f_k(x) := \sum_{m \in \mathbb{Z}^n} 2^{kn/2} |\lambda_{k,m}| |Q_{k,m}|^{-1/(\kappa p)} t_{k,m, \kappa p} \chi_{k,m}(x), \quad x \in \mathbb{R}^n, \quad k \in \mathbb{Z}.$$

Take $0 < \varrho < \min((p/\eta)', (q/\eta)'),$ we find by Hölder's inequality that the sum $\sum_{k=-\infty}^{\infty} S_k$ can be estimated from above by

$$\begin{aligned} c \int_{\mathbb{R}^n} \sum_{k=-\infty}^{\infty} f_k^\eta(x) \mathcal{M}_\varrho(t_k^{-\eta} \mathcal{M}(t_k^\eta g_k))(x) dx &\lesssim \left\| \left(\sum_{k=-\infty}^{\infty} f_k^q \right)^{\eta/q} \right\|_{L_{p/\eta}(\mathbb{R}^n)} \\ &\times \left\| \left(\sum_{k=-\infty}^{\infty} (\mathcal{M}_\varrho(t_k^{-\eta} \mathcal{M}(t_k^\eta g_k)))^{(q/\eta)'} \right)^{1/(q/\eta)'} \right\|_{L_{(p/\eta)' }(\mathbb{R}^n)}. \end{aligned}$$

Observe that

$$t_k^p \in A_{p/\theta} \subset A_{p/\eta}, \quad k \in \mathbb{Z},$$

which yields that

$$t_k^{-\eta(p/\eta)'} \in A_{(p/\eta)'}, \quad k \in \mathbb{Z},$$

by Lemma 2.3 (i)–(ii). By the vector-valued maximal inequality of Fefferman and Stein (2.5), Lemma 2.9 and (3.5), we obtain that

$$\sum_{k=-\infty}^{\infty} S_k \lesssim \left\| \left(\sum_{k=-\infty}^{\infty} f_k^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)}^\eta,$$

which yields (3.4).

Step 2. We prove the opposite inequality of (3.4). We can write

$$|\lambda_{k,m}|^{\nu\theta} = \frac{1}{|Q_{k,m}|} \int_{Q_{k,m}} |\lambda_{k,m}|^{\nu\theta} \chi_{k,m}(y) dy, \quad k \in \mathbb{Z}, m \in \mathbb{Z}^n.$$

Again by Hölder's inequality, since $\theta/\sigma_1 + \theta/p = 1$,

$$\begin{aligned} |\lambda_{k,m}| &\leq M_{Q_{k,m},\nu p}(\lambda_{k,m} t_k) M_{Q_{k,m},\nu\sigma_1}(t_k^{-1}) \\ &\lesssim \mathcal{M}_{\nu p} \left(\sum_{m \in \mathbb{Z}^n} \lambda_{k,m} t_k \chi_{k,m} \right) (x) M_{Q_{k,m},\sigma_1}(t_k^{-1}) \end{aligned}$$

for any $x \in Q_{k,m}$ with $0 < \nu < \min(1, q/p)$. Observe that

$$M_{Q_{k,m},\kappa p}(t_k) \leq M_{Q_{k,m},p}(t_k), \quad k \in \mathbb{Z}, m \in \mathbb{Z}^n.$$

Hence by (2.3), we find that

$$|\lambda_{k,m}| \lesssim 2^{-kn/(\kappa p)} t_{k,m,\kappa p}^{-1} \mathcal{M}_{\nu p} \left(\sum_{m \in \mathbb{Z}^n} t_k \lambda_{k,m} \chi_{k,m} \right) (x)$$

for any $x \in Q_{k,m}$. Therefore,

$$\begin{aligned} \|\lambda| \dot{f}_{p,q,\kappa}(\mathbb{R}^n, \{t_k\})\|^* &\lesssim \left\| \left(\sum_{k=-\infty}^{\infty} \left(\mathcal{M}_{\nu p} \left(\sum_{m \in \mathbb{Z}^n} 2^{kn/2} t_k \lambda_{k,m} \chi_{k,m} \right) \right)^q \right)^{1/q} \Big|_{L_p(\mathbb{R}^n)} \right\| \\ &\lesssim \|\lambda| \dot{f}_{p,q}(\mathbb{R}^n, \{t_k\})\|, \end{aligned}$$

where we used the vector-valued maximal inequality of Fefferman and Stein (2.5). The proof is complete. $\square$

For simplicity in what follows, we use $\dot{a}_{p,q}(\mathbb{R}^n, \{t_k\})$ instead of $\dot{b}_{p,q}(\mathbb{R}^n, \{t_k\})$ or $\dot{f}_{p,q}(\mathbb{R}^n, \{t_k\})$. Now we have the following result which is called the φ -transform characterization in the sense of Frazier and Jawerth. It will play an important role in the rest of the paper. The proof is given in [11] and [12].

Theorem 3.6. *Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, $0 < \theta \leq p < \infty$ and $0 < q < \infty$. Let $\{t_k\} \in \dot{X}_{\alpha,\sigma,p}$ be a p -admissible weight sequence with $\sigma = (\sigma_1 = \theta(p/\theta)', \sigma_2 \geq p)$. Let $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (3.1) through (3.3). The operators*

$$S_\varphi: \dot{A}_{p,q}(\mathbb{R}^n, \{t_k\}) \rightarrow \dot{a}_{p,q}(\mathbb{R}^n, \{t_k\})$$

and

$$T_\psi: \dot{a}_{p,q}(\mathbb{R}^n, \{t_k\}) \rightarrow \dot{A}_{p,q}(\mathbb{R}^n, \{t_k\})$$

are bounded. Furthermore, $T_\psi \circ S_\varphi$ is the identity on $\dot{A}_{p,q}(\mathbb{R}^n, \{t_k\})$.

Remark 3.7. This theorem can then be exploited to obtain a variety of results for the $\dot{A}_{p,q}(\mathbb{R}^n, \{t_k\})$ spaces, where arguments can be equivalently transferred

to the sequence space, which is often more convenient to handle. More precisely, under the same hypothesis of the last theorem,

$$\|\{\langle f, \varphi_{k,m} \rangle\}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n} | \dot{A}_{p,q}(\mathbb{R}^n, \{t_k\})\| \approx \|\{f | \dot{A}_{p,q}(\mathbb{R}^n, \{t_k\})\|.$$

Corollary 3.8. *Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, $0 < \theta \leq p < \infty$ and $0 < q < \infty$. Let $\{t_k\} \in \dot{X}_{\alpha, \sigma, p}$ be a p -admissible weight sequence with $\sigma = (\sigma_1 = \theta(p/\theta)', \sigma_2 \geq p)$. The definition of the spaces $\dot{A}_{p,q}(\mathbb{R}^n, \{t_k\})$ is independent of the choices of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (3.1) and (3.2).*

Theorem 3.9. *Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, $0 < \theta \leq p < \infty$ and $0 < q < \infty$. Let $\{t_k\} \in \dot{X}_{\alpha, \sigma, p}$ be a p -admissible weight sequence with $\sigma = (\sigma_1 = \theta(p/\theta)', \sigma_2 \geq p)$. Then $\dot{A}_{p,q}(\mathbb{R}^n, \{t_k\})$ are quasi-Banach spaces. They are Banach spaces if $1 \leq p < \infty$ and $1 \leq q < \infty$.*

Theorem 3.10. *Let $0 < \theta \leq p < \infty$ and $0 < q < \infty$. Let $\{t_k\} \in \dot{X}_{\alpha, \sigma, p}$ be a p -admissible weight sequence with $\sigma = (\sigma_1 = \theta(p/\theta)', \sigma_2 \geq p)$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. We have the embedding*

$$\mathcal{S}_\infty(\mathbb{R}^n) \hookrightarrow \dot{A}_{p,q}(\mathbb{R}^n, \{t_k\}) \hookrightarrow \mathcal{S}'_\infty(\mathbb{R}^n).$$

In addition $\mathcal{S}_\infty(\mathbb{R}^n)$ is dense in $\dot{A}_{p,q}(\mathbb{R}^n, \{t_k\})$.

The proof is given in [11] and [12]. Now we recall the following spaces, see [13] and [35], when $\{t_k\} = \{2^{sk}w\}$, $s \in \mathbb{R}$, with a weight w .

Definition 3.11. Let $0 < q < \infty$. Let $\{t_k\}$ be a q -admissible weight sequence and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (3.1) and (3.2) and we put $\varphi_k = 2^{kn}\varphi(2^k \cdot)$, $k \in \mathbb{Z}$. The Triebel–Lizorkin space $\dot{F}_{\infty,q}(\mathbb{R}^n, \{t_k\})$ is the collection of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f | \dot{F}_{\infty,q}(\mathbb{R}^n, \{t_k\})\| := \sup_{P \in \mathcal{Q}} \left(\frac{1}{|P|} \int_P \sum_{k=-\log_2 l(P)}^{\infty} t_k^q(x) |\varphi_k * f(x)|^q dx \right)^{1/q} < \infty.$$

We define $\dot{f}_{\infty,q}(\mathbb{R}^n, \{t_k\})$, the sequence space corresponding to $\dot{F}_{\infty,q}(\mathbb{R}^n, \{t_k\})$ as follows.

Definition 3.12. Let $0 < q < \infty$ and $\{t_k\}$ be a q -admissible sequence. Then for all complex valued sequences $\lambda = \{\lambda_{k,m}\}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n} \subset \mathbb{C}$ we define

$$\dot{f}_{\infty,q}(\mathbb{R}^n, \{t_k\}) := \{\lambda: \|\lambda | \dot{f}_{\infty,q}(\mathbb{R}^n, \{t_k\})\| < \infty\},$$

where

$$(3.6) \quad \|\lambda|\dot{f}_{\infty,q}(\mathbb{R}^n, \{t_k\})\| := \sup_{P \in \mathcal{Q}} \left(\frac{1}{|P|} \int_P \sum_{k=-\log_2 l(P)}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{knq/2} t_k^q(x) |\lambda_{k,m}|^q \chi_{k,m}(x) dx \right)^{1/q}.$$

The quasi-norm (3.6) can be rewritten as follows:

Proposition 3.13. *Let $0 < \theta \leq q < \infty$. Let $\{t_k\}$ be a q -admissible sequence. Then*

$$\begin{aligned} & \|\lambda|\dot{f}_{\infty,q}(\mathbb{R}^n, \{t_k\})\| \\ &= \sup_{P \in \mathcal{Q}} \left(\frac{1}{|P|} \int_P \sum_{k=-\log_2 l(P)}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{knq(1/2+1/q)} t_{k,m,q}^q |\lambda_{k,m}|^q \chi_{k,m}(x) dx \right)^{1/q}, \end{aligned}$$

where

$$t_{k,m,q} := \|t_k|L_q(Q_{k,m})\|, \quad k \in \mathbb{Z}, m \in \mathbb{Z}^n.$$

PROOF: Let $Q_{k,m} \subset P \in \mathcal{Q}$, $k \in \mathbb{Z}$, $m \in \mathbb{Z}^n$. The claim is a simple consequence of the fact that

$$\begin{aligned} & \int_P \sum_{k=-\log_2 l(P)}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{knq/2} t_k^q(x) |\lambda_{k,m}|^q \chi_{k,m}(x) dx \\ &= \sum_{k=-\log_2 l(P)}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{knq/2} |\lambda_{k,m}|^q \int_P t_k^q(x) \chi_{k,m}(x) dx \\ &= \sum_{k=-\log_2 l(P)}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{knq/2} |\lambda_{k,m}|^q t_{k,m,q}^q \\ &= \sum_{k=-\log_2 l(P)}^{\infty} \sum_{m \in \mathbb{Z}^n} |Q_{k,m}|^{-1} \int_{Q_{k,m}} 2^{knq/2} |\lambda_{k,m}|^q t_{k,m,q}^q dx \\ &= \int_P \sum_{k=-\log_2 l(P)}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{knq(1/2+1/q)} t_{k,m,q}^q |\lambda_{k,m}|^q \chi_{k,m}(x) dx. \end{aligned}$$

The proof is complete. $\square$

We have, see [19],

$$\dot{F}_{\infty,2}(\mathbb{R}^n, \{1\}) = BMO(\mathbb{R}^n).$$

Notice that Theorem 3.6 is true for the spaces $\dot{F}_{\infty,q}(\mathbb{R}^n, \{t_k\})$ and $\dot{f}_{\infty,q}(\mathbb{R}^n, \{t_k\})$, see [13]. More precisely, we have

Theorem 3.14. Let $0 < \theta \leq q < \infty$. Let $\{t_k\} \in \dot{X}_{\alpha, \sigma, q}$ be a q -admissible weight sequence with $\sigma = (\sigma_1 = \theta(q/\theta)', \sigma_2 \geq q)$. Let $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (3.1) through (3.3). The operators

$$S_\varphi: \dot{F}_{\infty, q}(\mathbb{R}^n, \{t_k\}) \rightarrow \dot{f}_{\infty, q}(\mathbb{R}^n, \{t_k\})$$

and

$$T_\psi: \dot{f}_{\infty, q}(\mathbb{R}^n, \{t_k\}) \rightarrow \dot{F}_{\infty, q}(\mathbb{R}^n, \{t_k\})$$

are bounded. Furthermore, $T_\psi \circ S_\varphi$ is the identity on $\dot{F}_{\infty, q}(\mathbb{R}^n, \{t_k\})$.

Corollary 3.15. Let $0 < \theta \leq q < \infty$. Let $\{t_k\} \in \dot{X}_{\alpha, \sigma, q}$ be a q -admissible weight sequence with $\sigma = (\sigma_1 = \theta(q/\theta)', \sigma_2 \geq q)$. The definition of the spaces $\dot{F}_{\infty, q}(\mathbb{R}^n, \{t_k\})$ is independent of the choices of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (3.1) and inequalities (3.2).

As in [19], see also [13], we obtain the following statement.

Proposition 3.16. Let $0 < \theta \leq q < \infty$. Let $\{t_k\}$ be a q -admissible weight sequence satisfying (2.3) with $\sigma_1 = \theta(q/\theta)'$, $p = q$ and $j = k$. Then $\lambda = \{\lambda_{k, m}\}_{k \in \mathbb{Z}, m \in \mathbb{Z}^n} \in \dot{f}_{\infty, q}(\mathbb{R}^n, \{t_k\})$ if and only if for each dyadic cube $Q_{k, m}$ there is a subset $E_{Q_{k, m}} \subset Q_{k, m}$ with $|E_{Q_{k, m}}| > |Q_{k, m}|/2$ (or any other, fixed, number $0 < \varepsilon < 1$) such that

$$\left\| \left(\sum_{k=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{knq(1/2+1/q)} t_{k, m, q}^q |\lambda_{k, m}|^q \chi_{E_{Q_{k, m}}} \right)^{1/q} \right\|_{L_\infty(\mathbb{R}^n)} < \infty.$$

Moreover, the infimum of this expression over all such collections $\{E_{Q_{k, m}}\}_{k, m}$ is equivalent to $\|\lambda\|_{\dot{f}_{\infty, q}(\mathbb{R}^n, \{t_k\})}$.

4. Complex interpolation

In this section we study complex interpolation of the above function spaces using Calderón product method. We follow the approach of M. Frazier and B. Jawerth in [19], see also [34]. We start by defining the Calderón product of two quasi-Banach lattices. Let $(\mathcal{A}, S, \mu)$ be a σ -finite measure space and let $\mathfrak{M}$ be the class of all complex-valued, μ -measurable functions on $\mathcal{A}$. Then a quasi-Banach space $X \subset \mathfrak{M}$ is called a quasi-Banach lattice of functions if for every $f \in X$ and $g \in \mathfrak{M}$ with $|g(x)| \leq |f(x)|$ for μ -a.e. $x \in X$ one has $g \in X$ and $\|g\|_X \leq \|f\|_X$.

Definition 4.1. Let $(\mathcal{A}, S, \mu)$ be a σ -finite measure space and let $\mathfrak{M}$ be the class of all complex-valued, μ -measurable functions on $\mathcal{A}$. Suppose that X_0 and X_1 are quasi-Banach lattices on $\mathfrak{M}$. Given $0 < \theta < 1$, define the Calderón product

$X_0^{1-\theta} \cdot X_1^\theta$ as the collection of all functions $f \in \mathfrak{M}$ such that

$$\|f\|_{X_0^{1-\theta} \cdot X_1^\theta} := \inf \{ \|g\|_{X_0}^{1-\theta} \|h\|_{X_1}^\theta : |f| \leq |g|^{1-\theta} |h|^\theta, \mu\text{-a.e.}, g \in X_0, h \in X_1 \} < \infty.$$

Remark 4.2. Calderón products have been introduced by A. P. Calderón in [8] (in a little bit different form which coincides with the above one). For further properties we refer to [19] and [44].

We need a few useful properties, see [44].

Lemma 4.3. *Let $(\mathcal{A}, S, \mu)$ be a σ -finite measure space and let $\mathfrak{M}$ be the class of all complex-valued, μ -measurable functions on $\mathcal{A}$. Suppose that X_0 and X_1 are quasi-Banach lattices on $\mathfrak{M}$. Let $0 < \theta < 1$.*

- (i) *Then the Calderón product $X_0^{1-\theta} \cdot X_1^\theta$ is a quasi-Banach space.*
- (ii) *Define the Calderón product $(X_0^{1-\theta} \cdot X_1^\theta)$ as the collection of all functions $f \in \mathfrak{M}$ such that there exist a positive real number M and $g \in X_0$ and $h \in X_1$ satisfying*

$$|f| \leq M |g|^{1-\theta} |h|^\theta, \quad \|g\|_{X_0} \leq 1, \quad \|h\|_{X_1} \leq 1.$$

We put

$$\|f\|_{(X_0^{1-\theta} \cdot X_1^\theta)} := \inf \{ M > 0 : |f| \leq M |g|^{1-\theta} |h|^\theta, \|g\|_{X_0} \leq 1, \|h\|_{X_1} \leq 1 \}.$$

Then $(X_0^{1-\theta} \cdot X_1^\theta) = X_0^{1-\theta} \cdot X_1^\theta$ follows with equality of quasi-norms.

In the sequel we will need the following lemma:

Lemma 4.4. *Let $0 < \theta < 1$, $0 < 1 - \theta < q_0 < \infty$, $0 < \theta < q_1 < \infty$ and Q be a cube. Let $\{t_k\} \subset L_{q_0}^{\text{loc}}$ and $\{w_k\} \subset L_{q_1}^{\text{loc}}$ be two weight sequences satisfying (2.3) with $(\sigma_1, p, j) = ((1-\theta)(q_0/(1-\theta)))', p = q_0, k)$ and $(\sigma_1, p, j) = (\theta(q_1/\theta))', p = q_1, k)$, respectively. We put*

$$\frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \quad \text{and} \quad \omega_k := t_k^{1-\theta} w_k^\theta, \quad k \in \mathbb{Z}.$$

Then

$$\left(\int_E \omega_k^q(x) \, dx \right)^{1/q} \approx \left(\int_E t_k^{q_0}(x) \, dx \right)^{(1-\theta)/q_0} \left(\int_E w_k^{q_1}(x) \, dx \right)^{\theta/q_1}$$

for any $E \subset Q$ such that $|E| \geq \varepsilon |Q|$, with $0 < \varepsilon < 1$.

PROOF: First this lemma with $E = Q$ is given in [34, Proposition 3.14, Step 5]. Let $0 < \delta, \mu < 1$, and assume that $q_0 < q$. Since, $1/q_0 + 1/\sigma_1 = 1/(1-\theta)$,

Hölder's inequality implies the following estimate

$$\begin{aligned} 1 &= \left(\frac{1}{|E|} \int_E t_k^{\delta(1-\theta)}(x) t_k^{-\delta(1-\theta)}(x) \, dx \right)^{1/\delta} \\ &\leq \frac{1}{\varepsilon^{1/\delta}} \left(\frac{1}{|Q|} \int_Q t_k^{\delta q_0}(x) \chi_E(x) \, dx \right)^{(1-\theta)/\delta q_0} \left(\frac{1}{|Q|} \int_Q t_k^{-\delta \sigma_1}(x) \chi_E(x) \, dx \right)^{(1-\theta)/\delta \sigma_1}. \end{aligned}$$

Since $\{t_k\}$ is a q_0 -admissible sequence satisfying (2.3) with $\sigma_1 = (q_0/(1-\theta))' \times (1-\theta)$ and $p = q_0$, we estimate the above expression by

$$\frac{c}{\varepsilon^{1/\delta}} \left(\frac{1}{|Q|} \int_Q t_k^{\delta q_0}(x) \chi_E(x) \, dx \right)^{(1-\theta)/(\delta q_0)} \left(\frac{1}{|Q|} \int_Q t_k^{q_0}(x) \chi_E(x) \, dx \right)^{-(1-\theta)/q_0},$$

where the positive constant c does not depend on k and Q . Similarly for w_k , $k \in \mathbb{Z}$. Therefore

$$\begin{aligned} (4.1) \quad &|Q|^{-1/q} \left(\int_Q t_k^{q_0}(x) \chi_E(x) \, dx \right)^{(1-\theta)/q_0} \left(\int_Q w_k^{q_1}(x) \chi_E(x) \, dx \right)^{\theta/q_1} \\ &\lesssim \left(\frac{1}{|Q|} \int_Q t_k^{\delta q_0}(x) \chi_E(x) \, dx \right)^{(1-\theta)/(\delta q_0)} \left(\frac{1}{|Q|} \int_Q w_k^{\mu q_1}(x) \chi_E(x) \, dx \right)^{\theta/(\mu q_1)}. \end{aligned}$$

Take $0 < \mu < \theta(1-\theta)/q_1$. Using the fact that

$$\begin{aligned} \left(\frac{1}{|Q|} \int_Q w_k^{\mu q_1}(x) \chi_E(x) \, dx \right)^{\theta/(\mu q_1)} &\leq \left(\frac{1}{|Q|} \int_Q w_k^{\theta(1-\theta)}(x) \chi_E(x) \, dx \right)^{1/(1-\theta)} \\ &\lesssim \mathcal{M}_{1-\theta}(w_k^\theta \chi_E)(y) \end{aligned}$$

for any $y \in Q$ and taking $\delta = (1-\theta)q/q_0$, we find that (4.1) can be estimated from above by

$$\begin{aligned} &c \left(\frac{1}{|Q|} \right)^{(1-\theta)/\delta q_0} \|t_k^{1-\theta} \mathcal{M}_{1-\theta}(w_k^\theta \chi_E) \|_{L_{\delta q_0/(1-\theta)}(\mathbb{R}^n)} \\ &= c |Q|^{-1/q} \|t_k^{1-\theta} \mathcal{M}_{1-\theta}(w_k^\theta \chi_E) \|_{L_q(\mathbb{R}^n)}. \end{aligned}$$

Observe that

$$t_k^{q_0} \in A_{q_0/(1-\theta)} \subset A_{q/(1-\theta)}, \quad k \in \mathbb{Z}.$$

Therefore, since $0 < (1-\theta)q/q_0 < 1$, we obtain

$$t_k^{(1-\theta)q} \in A_{q/(1-\theta)}, \quad k \in \mathbb{Z}.$$

From Lemma 2.9 combined with Remark 2.10 (iii), it follows

$$\|t_k^{1-\theta} \mathcal{M}_{1-\theta}(w_k^\theta \chi_E) |L_q(\mathbb{R}^n)\| \lesssim \|t_k^{1-\theta} w_k^\theta \chi_E |L_q(\mathbb{R}^n)\| = c \left(\int_E \omega_k^q(x) dx \right)^{1/q}.$$

Then

$$(4.2) \quad \left(\int_Q t_k^{q_0}(x) \chi_E(x) dx \right)^{(1-\theta)/q_0} \left(\int_Q w_k^{q_1}(x) \chi_E(x) dx \right)^{\theta/q_1} \lesssim \left(\int_E \omega_k^q(x) dx \right)^{1/q}.$$

The opposite inequality in (4.2) follows from Hölder's inequality. $\square$

Now we turn to the investigation of the Calderón products of the sequence spaces $\dot{a}_{p,q}(\mathbb{R}^n, \{t_k\})$.

Theorem 4.5. *Let $0 < \theta < 1$, $1 \leq p_0, p_1 < \infty$ and $1 \leq q_0, q_1 < \infty$. Let $\{t_k\} \subset L_{p_0}^{\text{loc}}$ and $\{w_k\} \subset L_{p_1}^{\text{loc}}$ be two weight sequences satisfying (2.3) with $(\sigma_1, p, j) = ((1-\theta)(p_0/(1-\theta))', p = p_0, k)$ and $(\sigma_1, p, j) = (\theta(p_1/\theta)', p = p_1, k)$, respectively. We put*

$$(4.3) \quad \frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \omega_k := t_k^{1-\theta} w_k^\theta, \quad k \in \mathbb{Z}.$$

Then

$$(\dot{a}_{p_0, q_0}(\mathbb{R}^n, \{t_k\}))^{1-\theta} (\dot{a}_{p_1, q_1}(\mathbb{R}^n, \{w_k\}))^\theta = \dot{a}_{p, q}(\mathbb{R}^n, \{\omega_k\})$$

holds in the sense of equivalent norms.

PROOF: Obviously we can assume that $1 < p_0, p_1, q_0, q_1 < \infty$. Put

$$t_{k,m,p_0} := \|t_k |L_{p_0}(Q_{k,m})\|, \quad w_{k,m,p_1} := \|w_k |L_{p_1}(Q_{k,m})\|$$

and

$$\omega_{k,m,p} := \|\omega_k |L_p(Q_{k,m})\|, \quad k \in \mathbb{Z}, \quad m \in \mathbb{Z}^n.$$

Step 1. We deal with the case of $\dot{f}$ -spaces. To prove this we do it in two Sub-steps 1.1 and 1.2.

Substep 1.1. We shall prove

$$(\dot{f}_{p_0, q_0}(\mathbb{R}^n, \{t_k\}))^{1-\theta} (\dot{f}_{p_1, q_1}(\mathbb{R}^n, \{w_k\}))^\theta \hookrightarrow \dot{f}_{p, q}(\mathbb{R}^n, \{\omega_k\}).$$

We suppose that sequences $\lambda := (\lambda_{k,m})_{k,m}$, $\lambda^i := (\lambda_{k,m}^i)_{k,m}$, $i = 0, 1$, are given and that

$$|\lambda_{k,m}| \leq |\lambda_{k,m}^0|^{1-\theta} |\lambda_{k,m}^1|^\theta$$

holds for all $k \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$. Let

$$g := \left(\sum_{k=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{k(n/2)q} \omega_k^q |\lambda_{k,m}|^q \chi_{k,m} \right)^{1/q}.$$

Since we have

$$2^{k(n/2)q} \omega_k^q |\lambda_{k,m}|^q \chi_{k,m} \leq (2^{k(n/2)} t_k |\lambda_{k,m}^0| \chi_{k,m})^{q(1-\theta)} (2^{k(n/2)} w_k |\lambda_{k,m}^1| \chi_{k,m})^{q\theta},$$

Hölder's inequality implies that g can be estimated from above by

$$\begin{aligned} & \left(\sum_{k=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} (2^{k(n/2)} t_k |\lambda_{k,m}^0| \chi_{k,m})^{q_0} \right)^{(1-\theta)/q_0} \\ & \quad \times \left(\sum_{k=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} (2^{k(n/2)} w_k |\lambda_{k,m}^1| \chi_{k,m})^{q_1} \right)^{\theta/q_1}. \end{aligned}$$

Applying Hölder's inequality again with conjugate indices $p_0/(1-\theta)$ and p_1/θ , we obtain

$$\|\lambda| \dot{f}_{p,q}(\mathbb{R}^n, \{\omega_k\})\| \leq \|\lambda^0| \dot{f}_{p_0,q_0}(\mathbb{R}^n, \{t_k\})\|^{1-\theta} \|\lambda^1| \dot{f}_{p_1,q_1}(\mathbb{R}^n, \{w_k\})\|^\theta.$$

Substep 1.2. Now we turn to the proof of

$$\dot{f}_{p,q}(\mathbb{R}^n, \{\omega_k\}) \hookrightarrow (\dot{f}_{p_0,q_0}(\mathbb{R}^n, \{t_k\}))^{1-\theta} (\dot{f}_{p_1,q_1}(\mathbb{R}^n, \{w_k\}))^\theta.$$

Substep 1.2.1. We consider the case $\gamma = p/p_0 - q/q_0 > 0$. Let the sequence $\lambda \in \dot{f}_{p,q}(\mathbb{R}^n, \{\omega_k\})$ be given. We have to find sequences λ^0 and λ^1 such that

$$|\lambda_{k,m}| \leq M |\lambda_{k,m}^0|^{1-\theta} |\lambda_{k,m}^1|^\theta$$

for every $k \in \mathbb{Z}$, $m \in \mathbb{Z}^n$ and

$$(4.4) \quad \|\lambda^0| \dot{f}_{p_0,q_0}(\mathbb{R}^n, \{t_k\})\|^{1-\theta} \|\lambda^1| \dot{f}_{p_1,q_1}(\mathbb{R}^n, \{w_k\})\|^\theta \leq c \|\lambda| \dot{f}_{p,q}(\mathbb{R}^n, \{\omega_k\})\|,$$

with some constant c independent of λ .

Preparation. We set

$$g := \left(\sum_{k=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{kn(1/p+1/2)q} \omega_{k,m,p}^q |\lambda_{k,m}|^q \chi_{Q_{k,m}} \right)^{1/q}.$$

We follow ideas of the proof of Theorem 8.2 in [19], see also [34]. Set

$$A_l := \{x \in \mathbb{R}^n : g(x) > 2^l\},$$

with $l \in \mathbb{Z}$. Obviously $A_{l+1} \subset A_l$, with $l \in \mathbb{Z}$. Now we introduce a (partial) decomposition of $\mathbb{Z} \times \mathbb{Z}^n$ by taking

$$C_l := \left\{ (k, m) : |Q_{k,m} \cap A_l| > \frac{|Q_{k,m}|}{2} \text{ and } |Q_{k,m} \cap A_{l+1}| \leq \frac{|Q_{k,m}|}{2} \right\}, \quad l \in \mathbb{Z}.$$

The sets C_l are pairwise disjoint, i.e., $C_l \cap C_v = \emptyset$ if $l \neq v$. Let us prove that $\lambda_{k,m} = 0$ holds for all tuples $(k, m) \notin \bigcup_{l \in \mathbb{Z}} C_l$. Let us consider one such tuple (k_0, m_0) and let us choose $l_0 \in \mathbb{Z}$ arbitrarily. First suppose that $(k_0, m_0) \notin C_{l_0}$, then either

$$(4.5) \quad |Q_{k_0, m_0} \cap A_{l_0}| \leq \frac{|Q_{k_0, m_0}|}{2} \quad \text{or} \quad |Q_{k_0, m_0} \cap A_{l_0+1}| > \frac{|Q_{k_0, m_0}|}{2}.$$

Let us assume for the moment that the second condition is satisfied. By induction on l it follows

$$(4.6) \quad |Q_{k_0, m_0} \cap A_{l+1}| > \frac{|Q_{k_0, m_0}|}{2} \quad \text{for all } l \geq l_0.$$

Let $D := \bigcap_{l \geq l_0} Q_{k_0, m_0} \cap A_{l+1}$. The family $\{Q_{k_0, m_0} \cap A_l\}_{l \in \mathbb{Z}}$ is a decreasing family of sets, i.e., $Q_{k_0, m_0} \cap A_{l+1} \subset Q_{k_0, m_0} \cap A_l$. Therefore, in view of (4.6), the measure of the set D is larger than or equal to $|Q_{k_0, m_0}|/2$. Hence

$$(4.7) \quad \begin{aligned} \|\lambda|f_{p,q}(\mathbb{R}^n, \{\omega_k\})\| &\geq \left(\int_{Q_{k_0, m_0} \cap A_l} g^p(x) \, dx \right)^{1/p} \\ &\geq 2^l |Q_{k_0, m_0} \cap A_l|^{1/p} \geq 2^l |D|^{1/p}, \quad l \geq \max(l_0, 0). \end{aligned}$$

Since $|D| \geq |Q_{k_0, m_0}|/2 > 0$, the term on the right-hand side of (4.7) is strictly greater than 0. Now the norm $\|\lambda|\dot{f}_{p,q}(\mathbb{R}^n, \{\omega_k\})\|$ is finite since $\lambda \in \dot{f}_{p,q}(\mathbb{R}^n, \{\omega_k\})$. In (4.7) letting l tends to infinity we get a contradiction. Hence, we have to turn in (4.5) to the situation where the first condition is satisfied. We claim that

$$|Q_{k_0, m_0} \cap A_l| \leq \frac{|Q_{k_0, m_0}|}{2} \quad \text{for all } l \in \mathbb{Z}.$$

Obviously this yields

$$(4.8) \quad |Q_{k_0, m_0} \cap A_l^c| > \frac{|Q_{k_0, m_0}|}{2} \quad \text{for all } l \in \mathbb{Z},$$

again this claim follows by induction on l using $(k_0, m_0) \notin \bigcup_{l \in \mathbb{Z}} C_l$. Set

$$E = \bigcap_{l \geq \max(0, -l_0)} Q_{k_0, m_0} \cap A_{-l}^c = \bigcap_{l \geq \max(0, -l_0)} h_l.$$

The family $\{h_l\}_l$ is a decreasing family of sets, i.e., $h_{l+1} \subset h_l$. Therefore, in view of (4.8), the measure of the set E is larger than or equal to $|Q_{k_0, m_0}|/2$. By

selecting a point $x \in E$ we obtain

$$(4.9) \quad \omega_{k_0, m_0} |\lambda_{k_0, m_0}| \leq g(x) \leq 2^{-l}.$$

Now, in (4.9) if l tends to ∞ then the claim, namely $\lambda_{k_0, m_0} = 0$, follows.

The choices of $\lambda_{k, m}^0$ and $\lambda_{k, m}^1$. If $(k, m) \notin \cup_{l \in \mathbb{Z}} C_l$, then we define $\lambda_{k, m}^0 = \lambda_{k, m}^1 = 0$. If $(k, m) \in C_l$, we put

$$(4.10) \quad u := n \left(\frac{q}{q_0 p} - \frac{1}{p_0} \right) + \frac{n}{2} \left(\frac{q}{q_0} - B \right), \quad \vartheta_{k, m} := w_{k, m, p_1}^{\theta q / q_0} t_{k, m, p_0}^{-\theta q / q_1},$$

and

$$(4.11) \quad v := n \left(\frac{q}{q_1 p} - \frac{1}{p_1} \right) + \frac{n}{2} \left(\frac{q}{q_1} - 1 \right), \quad \varepsilon_{k, m} := t_{k, m, p_0}^{(1-\theta)q/q_1} w_{k, m, p_1}^{-(1-\theta)q/q_0}.$$

Let $\delta = p/p_1 - q/q_1$. We put

$$(4.12) \quad \lambda_{k, m}^0 := \vartheta_{k, m} 2^{ku} 2^{l\gamma} |\lambda_{k, m}|^{q/q_0}.$$

Also, set

$$(4.13) \quad \lambda_{k, m}^1 := \varepsilon_{k, m} 2^{kv} 2^{l\delta} |\lambda_{k, m}|^{q/q_1}.$$

Some elementary calculations, based on (4.10), (4.11), (4.12) and (4.13), yield

$$(1 - \theta)u + \theta v = 0, \quad (1 - \theta)\gamma + \theta\delta = 0$$

and

$$|\lambda_{k, m}| = (\lambda_{k, m}^0)^{1-\theta} (\lambda_{k, m}^1)^\theta,$$

which holds now for all pairs (k, m) .

Proof of (4.4). It will be sufficient to establish the following two inequalities

$$(4.14) \quad \|\lambda^0 | \dot{f}_{p_0, q_0}(\mathbb{R}^n, \{t_k\}) \| \leq c \|\lambda | \dot{f}_{p, q}(\mathbb{R}^n, \{\omega_k\}) \|^{p/p_0},$$

$$(4.15) \quad \|\lambda^1 | \dot{f}_{p_1, q_1}(\mathbb{R}^n, \{w_k\}) \| \leq c \|\lambda | \dot{f}_{p, q}(\mathbb{R}^n, \{\omega_k\}) \|^{p/p_1}.$$

Let us prove (4.14). Write

$$\begin{aligned} & \sum_{k=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} t_{k, m, p_0}^{q_0} 2^{kn(1/p_0 + 1/2)q_0} (\lambda_{k, m}^0)^{q_0} \chi_{k, m} \\ &= \sum_{l=-\infty}^{\infty} \sum_{(k, m) \in C_l} t_{k, m, p_0}^{q_0} 2^{kn(1/p_0 + 1/2)q_0} (\lambda_{k, m}^0)^{q_0} \chi_{k, m} = I. \end{aligned}$$

Obviously,

$$(4.16) \quad \begin{aligned} \chi_{k,m}(x) &\lesssim \frac{1}{|Q_{k,m} \cap A_l|} \int_{Q_{k,m} \cap A_l} dy \lesssim \frac{1}{|Q_{k,m}|} \int_{Q_{k,m} \cap A_l} dy \\ &\lesssim \mathcal{M}(\chi_{Q_{k,m} \cap A_l})(x), \quad (k, m) \in C_l, \end{aligned}$$

where the implicit positive constant does not depend on x, l, m and k . We can apply Theorem 2.5 and obtain that

$$\|I^{1/q_0}|L_{p_0}(\mathbb{R}^n)\|$$

is bounded by

$$\begin{aligned} c &\left\| \left(\sum_{l=-\infty}^{\infty} \sum_{(k,m) \in C_l} (\mathcal{M}(t_{k,m,p_0} 2^{kn(1/p_0+1/2)} \lambda_{k,m}^0 \chi_{Q_{k,m} \cap A_l}))^{q_0} \right)^{1/q_0} |L_{p_0}(\mathbb{R}^n) \right\| \\ &\lesssim \left\| \left(\sum_{l=-\infty}^{\infty} \sum_{(k,m) \in C_l} t_{k,m,p_0}^{q_0} 2^{kn(1/p_0+1/2)q_0} (\lambda_{k,m}^0)^{q_0} \chi_{Q_{k,m} \cap A_l} \right)^{1/q_0} |L_{p_0}(\mathbb{R}^n) \right\|. \end{aligned}$$

We have

$$q_0 - \frac{\theta q}{q_1} q_0 = (1 - \theta)q,$$

hence by application of Lemma 4.4, we obtain

$$\vartheta_{k,m}^{q_0} t_{k,m,p_0}^{q_0} = w_{k,m,p_1}^{\theta q} t_{k,m,p_0}^{(1-\theta)q} \approx \omega_{k,m,p}^q.$$

We see that

$$2^{l\gamma} \leq \left(\sum_{l=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} \omega_{k,m,p}^q 2^{kn(1/p+1/2)q} |\lambda_{k,m}|^q \chi_{Q_{k,m}}(x) \right)^{\gamma/q},$$

if $x \in A_l$. Therefore,

$$\begin{aligned} &\|I^{1/q_0}|L_{p_0}(\mathbb{R}^n)\| \\ &\lesssim \left\| \left(\sum_{k=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{kn(1/p+1/2)q} \omega_{k,m,p}^q |\lambda_{k,m}|^q \chi_{Q_{k,m}} \right)^{1/q} |L_p(\mathbb{R}^n) \right\|^{p/p_0}. \end{aligned}$$

Let us prove (4.15). We only make some comments concerning necessary modifications. Again, we write

$$\begin{aligned} &\sum_{k=0}^{\infty} \sum_{m \in \mathbb{Z}^n} w_{k,m,p_1}^{q_1} 2^{kn(1/p_1+1/2)q_1} (\lambda_{k,m}^1)^{q_1} \chi_{k,m} \\ &= \sum_{l=-\infty}^{\infty} \sum_{(k,m) \in C_l} w_{k,m,p_1}^{q_1} 2^{kn(1/p_1+1/2)q_1} (\lambda_{k,m}^1)^{q_1} \chi_{k,m}. \end{aligned}$$

Replacing t_{k,m,p_0}, q_0, p_0, u and γ by w_{k,m,p_1}, q_1, p_1, v and δ , respectively, and $Q_{k,m} \cap A_l$ by $Q_{k,m} \cap A_{l+1}^c$, this leads to the desired inequality in view that $\delta + q/q_1 = p/p_1$.

Substep 1.2.2. We consider the case $\gamma < 0$. We use an argument similar to the above with $Q_{k,m} \cap A_l$, respectively, $Q_{k,m} \cap A_{l+1}^c$, replaced by $Q_{k,m} \cap A_{l+1}^c$, respectively, $Q_{k,m} \cap A_l$.

Substep 1.2.3. We consider the case $\gamma = 0$. Then $\delta = 0$. This case can be easily solved.

Step 2. We deal with the case of $\dot{b}$ -spaces. We prove only the embedding

$$\dot{b}_{p,q}(\mathbb{R}^n, \{\omega_k\}) \hookrightarrow (\dot{b}_{p_0,q_0}(\mathbb{R}^n, \{t_k\}))^{1-\theta} (\dot{b}_{p_1,q_1}(\mathbb{R}^n, \{w_k\}))^\theta,$$

since the opposite embedding follows from Hölder's inequality. Assume that $1 \leq q_0 < q_1 < \infty$. Let $\mu = q/q_0 - p/p_0$, $\tau = q/q_1 - p/p_1$,

$$u := \frac{n}{2} \left(\frac{q}{q_0} - 1 \right), \quad v := \frac{n}{2} \left(\frac{q}{q_1} - 1 \right), \quad \vartheta_{k,m} := t_{k,m,p_0}^{-\theta p/p_1} w_{k,m,p_1}^{\theta p/p_0},$$

and

$$\varepsilon_{k,m} := t_{k,m,p_0}^{(1-\theta)p/p_1} w_{k,m,p_1}^{-(1-\theta)p/p_0}.$$

We put

$$\lambda_{k,m}^0 := \vartheta_{k,m} 2^{ku} |\lambda_{k,m}|^{p/p_0} \left(\sum_{h \in \mathbb{Z}^n} |\lambda_{k,h}|^p \omega_{k,h,p}^p \right)^{\mu/p},$$

if $\lambda_{k,m} \neq 0$ and

$$\lambda_{k,m}^0 := 0 \quad \text{if } \lambda_{k,m} = 0.$$

Also, set

$$\lambda_{k,m}^1 := \varepsilon_{k,m} 2^{kv} |\lambda_{k,m}|^{p/p_1} \left(\sum_{h \in \mathbb{Z}^n} |\lambda_{k,h}|^p \omega_{k,h,p}^p \right)^{\tau/p},$$

if $\lambda_{k,m} \neq 0$ and

$$\lambda_{k,m}^1 := 0 \quad \text{if } \lambda_{k,m} = 0.$$

Observe that

$$|\lambda_{k,m}| = (\lambda_{k,m}^0)^{1-\theta} (\lambda_{k,m}^1)^\theta,$$

which holds now for all pairs (k, m) . Lemma 4.4,

$$p_0 - \frac{\theta p}{p_1} p_0 = (1 - \theta)p \quad \text{and} \quad p_1 - \frac{(1 - \theta)p}{p_0} p_1 = \theta p,$$

imply

$$\vartheta_{k,m}^{p_0} t_{k,m,p_0}^{p_0} = w_{k,m,p_1}^{\theta p} t_{k,m,p_0}^{(1-\theta)p} \approx \omega_{k,m,p}^p$$

and

$$\varepsilon_{k,m}^{p_1} w_{k,m,p_1}^{p_1} = w_{k,m,p_1}^{\theta p} t_{k,m,p_0}^{(1-\theta)p} \approx \omega_{k,m,p}^p.$$

Simple calculation gives

$$\|\lambda^i |\dot{b}_{p_i, q_i}(\mathbb{R}^n, \{t_k\})|\| \lesssim \|\lambda |\dot{b}_{p, q}(\mathbb{R}^n, \{\omega_k\})|\|^{q/q_i}, \quad i = 0, 1.$$

The proof is complete. $\square$

Next we study the Calderón product of $\dot{f}_{p_0, q_0}(\mathbb{R}^n, \{t_k\})$ and $\dot{f}_{\infty, q_1}(\mathbb{R}^n, \{w_k\})$. For $\{t_k\} = \{w_k\} = \{1\}$, we refer to [19].

Theorem 4.6. *Let $0 < \theta < 1$, $1 \leq p_0 < \infty$ and $1 \leq q_0, q_1 < \infty$. Let $\{t_k\} \subset L_{p_0}^{\text{loc}}$ and $\{w_k\} \subset L_{q_1}^{\text{loc}}$ be two weight sequences satisfying (2.3) with $(\sigma_1, p, j) = ((1 - \theta)(p_0/(1 - \theta))', p = p_0, k)$ and $(\sigma_1, p, j) = (\theta(q_1/\theta)', p = q_1, k)$, respectively. We put*

$$\frac{1}{p} := \frac{1 - \theta}{p_0}, \quad \frac{1}{q} := \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}, \quad \omega_k := t_k^{1-\theta} w_k^\theta, \quad k \in \mathbb{Z}.$$

Then

$$(\dot{f}_{p_0, q_0}(\mathbb{R}^n, \{t_k\}))^{1-\theta} (\dot{f}_{\infty, q_1}(\mathbb{R}^n, \{w_k\}))^\theta = \dot{f}_{p, q}(\mathbb{R}^n, \{\omega_k\})$$

holds in the sense of equivalent norms.

PROOF: Obviously we can assume that $1 < p_0, p_1, q_0, q_1 < \infty$.

Step 1. In this step we prove the embedding

$$(\dot{f}_{p_0, q_0}(\mathbb{R}^n, \{t_k\}))^{1-\theta} (\dot{f}_{\infty, q_1}(\mathbb{R}^n, \{w_k\}))^\theta \hookrightarrow \dot{f}_{p, q}(\mathbb{R}^n, \{\omega_k\}).$$

We suppose that sequences $\lambda := (\lambda_{k, m})_{k, m}$, $\lambda^i := (\lambda_{k, m}^i)_{k, m}$, $i = 0, 1$, are given and that

$$|\lambda_{k, m}| \leq |\lambda_{k, m}^0|^{1-\theta} |\lambda_{k, m}^1|^\theta$$

holds for all $k \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$. Let $0 < \kappa < 1$ be such that

$$\frac{1}{\kappa p} = \frac{1 - \theta}{p_0} + \frac{\theta}{q_1}.$$

Let $Q_{k, m}$, $k \in \mathbb{Z}$, $m \in \mathbb{Z}^n$, be a dyadic cube. By Proposition 3.16 there exists $E_{Q_{k, m}} \subset Q_{k, m}$ such that $|E_{Q_{k, m}}| > |Q_{k, m}|/2$. We set

$$(4.17) \quad g := \left(\sum_{k=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{kn(1/2+1/(\kappa p))q} \omega_{k, m, \kappa p}^q |\lambda_{k, m}|^q \chi_{E_{Q_{k, m}}} \right)^{1/q}$$

and

$$\omega_{k, m, \kappa p} = \|\omega_k|L_{\kappa p}(Q_{k, m})\|, \quad k \in \mathbb{Z}, \quad m \in \mathbb{Z}^n.$$

The inequality $\chi_{Q_{k, m}} \lesssim \mathcal{M}_\varrho(\chi_{E_{Q_{k, m}} \cap Q_{k, m}})$ with $0 < \varrho < \min(1, \theta, q)$, Lemma 2.9 and Proposition 3.5 imply

$$\|\lambda |\dot{f}_{p, q}(\mathbb{R}^n, \{\omega_k\})|\| \lesssim \|g|L_p(\mathbb{R}^n)|\|.$$

Hölder's inequality yields

$$\omega_{k,m,\kappa p} \leq t_{k,m,p_0}^{1-\theta} w_{k,m,q_1}^\theta, \quad k \in \mathbb{Z}, \quad m \in \mathbb{Z}^n,$$

with

$$t_{k,m,p_0} := \|t_k\|_{L_{p_0}(Q_{k,m})} \quad \text{and} \quad w_{k,m,q_1} = \|w_k\|_{L_{q_1}(Q_{k,m})}, \quad k \in \mathbb{Z}, \quad m \in \mathbb{Z}^n.$$

Since

$$\begin{aligned} 2^{kn(1/2+1/(\kappa p))q} \omega_{k,m,\kappa p}^q |\lambda_{k,m}|^q \chi_{E_{Q_{k,m}}}(x) \\ \leq \left(2^{kn(1/2+1/p_0)} t_{k,m,p_0} |\lambda_{k,m}| \chi_{E_{Q_{k,m}}}(x) \right)^{q(1-\theta)} \\ \times \left(2^{kn(1/2+1/q_1)} w_{k,m,q_1} |\lambda_{k,m}| \chi_{E_{Q_{k,m}}}(x) \right)^{q\theta}, \end{aligned}$$

again, Hölder's inequality implies that g can be estimated from above by

$$\begin{aligned} \left(\sum_{k=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{kn(1/2+1/p_0)q_0} t_{k,m,p_0}^{q_0} |\lambda_{k,m}|^{q_0} \chi_{E_{Q_{k,m}}}(x) \right)^{(1-\theta)/q_0} \\ \times \left(\sum_{k=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{kn(1/2+1/q_1)q_1} w_{k,m,q_1}^{q_1} |\lambda_{k,m}|^{q_1} \chi_{E_{Q_{k,m}}}(x) \right)^{\theta q_1}. \end{aligned}$$

We estimate the second factor by its L^∞ -norm. Again Proposition 3.16 guarantees that

$$\|\lambda|\dot{f}_{p,q}(\mathbb{R}^n, \{\omega_k\})\| \leq \|\lambda^0|\dot{f}_{p_0,q_0}(\mathbb{R}^n, \{t_k\})\|^{1-\theta} \|\lambda^1|\dot{f}_{\infty,q_1}(\mathbb{R}^n, \{w_k\})\|^\theta.$$

Step 2. We prove the embedding

$$\dot{f}_{p,q}(\mathbb{R}^n, \{\omega_k\}) \hookrightarrow (\dot{f}_{p_0,q_0}(\mathbb{R}^n, \{t_k\}))^{1-\theta} (\dot{f}_{\infty,q_1}(\mathbb{R}^n, \{w_k\}))^\theta.$$

Let the sequence $\lambda \in \dot{f}_{p,q}(\mathbb{R}^n, \{\omega_k\})$ be given. We have to find sequences λ^0 and λ^1 such that

$$|\lambda_{k,m}| \leq M |\lambda_{k,m}^0|^{1-\theta} |\lambda_{k,m}^1|^\theta$$

for every $k \in \mathbb{Z}$, $m \in \mathbb{Z}^n$ and

$$(4.18) \quad \|\lambda^0|\dot{f}_{p_0,q_0}(\mathbb{R}^n, \{t_k\})\|^{1-\theta} \|\lambda^1|\dot{f}_{\infty,q_1}(\mathbb{R}^n, \{w_k\})\|^\theta \leq c \|\lambda|\dot{f}_{p,q}(\mathbb{R}^n, \{\omega_k\})\|.$$

Let

$$\gamma = \frac{p}{p_0} - \frac{q}{q_0} \quad \text{and} \quad \delta = -\frac{q}{q_1}.$$

We have to subdivide Step 2 into two substeps:

Substep 2.1. We consider the case $\gamma > 0$. Set

$$A_l := \{x \in \mathbb{R}^n : g(x) > 2^l\},$$

with $l \in \mathbb{Z}$ and g as in (4.17). Obviously $A_{l+1} \subset A_l$, with $l \in \mathbb{Z}$. Now we introduce a (partial) decomposition of $\mathbb{Z} \times \mathbb{Z}^n$ by taking

$$C_l := \left\{ (k, m) : |Q_{k,m} \cap A_l| > \frac{|Q_{k,m}|}{2} \text{ and } |Q_{k,m} \cap A_{l+1}| \leq \frac{|Q_{k,m}|}{2} \right\}, \quad l \in \mathbb{Z}.$$

The sets C_l are pairwise disjoint, i.e., $C_l \cap C_v = \emptyset$ if $l \neq v$. As in Theorem 4.6 we can prove that $\lambda_{k,m} = 0$ holds for all tuples $(k, m) \notin \bigcup_{l \in \mathbb{Z}} C_l$. If $(k, m) \in \bigcup_{l \in \mathbb{Z}} C_l$, then we define $\lambda_{k,m}^0 = \lambda_{k,m}^1 = 0$. If $(k, m) \in C_l$, we put

$$u := n \left(\frac{q}{q_0 \kappa p} - \frac{1}{p_0} \right) + \frac{n}{2} \left(\frac{q}{q_0} - 1 \right), \quad \vartheta_{k,m} := t_{k,m,p_0}^{-\theta q/q_1} w_{k,m,q_1}^{\theta q/q_0},$$

and

$$v := n \left(\frac{q}{q_1 \kappa p} - \frac{1}{q_1} \right) + \frac{n}{2} \left(\frac{q}{q_1} - 1 \right), \quad \varepsilon_{k,m} := t_{k,m,p_0}^{(1-\theta)q/q_1} w_{k,m,q_1}^{-(1-\theta)q/q_0}.$$

We put

$$\lambda_{k,m}^0 := \vartheta_{k,m} 2^{ku} 2^{l\gamma} |\lambda_{k,m}|^{q/q_0} \quad \text{and} \quad \lambda_{k,m}^1 := \varepsilon_{k,m} 2^{kv} 2^{l\delta} |\lambda_{k,m}|^{q/q_1}.$$

A simple calculation yields

$$(1 - \theta)u + \theta v = 0, \quad (1 - \theta)\gamma + \theta\delta = 0$$

and

$$|\lambda_{k,m}| = (\lambda_{k,m}^0)^{1-\theta} (\lambda_{k,m}^1)^\theta,$$

which holds now for all pairs (k, m) . To prove (4.18), it will be sufficient to establish the following two inequalities

$$(4.19) \quad \|\lambda^0 | \dot{f}_{p_0, q_0}(\mathbb{R}^n, \{t_k\}) \| \leq c \|\lambda | \dot{f}_{p, q}(\mathbb{R}^n, \{\omega_k\}) \|^{p/p_0},$$

$$(4.20) \quad \|\lambda^1 | \dot{f}_{\infty, q_1}(\mathbb{R}^n, \{w_k\}) \| \leq c.$$

Proof of (4.19). Write

$$\begin{aligned} & \sum_{k=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} t_{k,m,p_0}^{q_0} 2^{kn(1/2+1/p_0)q_0} (\lambda_{k,m}^0)^{q_0} \chi_{k,m} \\ &= \sum_{l=-\infty}^{\infty} \sum_{(k,m) \in C_l} t_{k,m,p_0}^{q_0} 2^{kn(1/2+1/p_0)q_0} (\lambda_{k,m}^0)^{q_0} \chi_{k,m} = I. \end{aligned}$$

Recall that,

$$\chi_{k,m}(x) \lesssim \mathcal{M}(\chi_{Q_{k,m} \cap A_l})(x), \quad (k, m) \in C_l,$$

where the implicit positive constant does not depend on x, l, m and k , see (4.16). We can apply Theorem 2.5 and obtain that

$$\|I^{1/q_0}|_{L_{p_0}(\mathbb{R}^n)}\|$$

is bounded by

$$\begin{aligned} c & \left\| \left(\sum_{l=-\infty}^{\infty} \sum_{(k,m) \in C_l} (\mathcal{M}(t_{k,m,p_0} 2^{kn(1/2+1/p_0)} \lambda_{k,m}^0 \chi_{Q_{k,m} \cap A_l}))^{q_0} \right)^{1/q_0} \Big|_{L_{p_0}(\mathbb{R}^n)} \right\| \\ & \lesssim \left\| \left(\sum_{l=-\infty}^{\infty} \sum_{(k,m) \in C_l} t_{k,m,p_0}^{q_0} 2^{kn(1/2+1/p_0)q_0} (\lambda_{k,m}^0)^{q_0} \chi_{Q_{k,m} \cap A_l} \right)^{1/q_0} \Big|_{L_{p_0}(\mathbb{R}^n)} \right\|. \end{aligned}$$

We have $q_0 - (\theta q/q_1)q_0 = (1 - \theta)q$, which together with Lemma 4.4 yields

$$\vartheta_{k,m}^{q_0} t_{k,m,p_0}^{q_0} = w_{k,m,q_1}^{\theta q} t_{k,m,p_0}^{(1-\theta)q} \approx \omega_{k,m,\kappa p}^q.$$

In addition, we have

$$2^{l\gamma} \leq \left(\sum_{l=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} \omega_{k,m,\kappa p}^q 2^{kn(1/2+1/(\kappa p))q} |\lambda_{k,m}|^q \chi_{Q_{k,m}}(x) \right)^{\gamma/q}$$

if $x \in A_l$. Therefore,

$$\begin{aligned} & \|I^{1/q_0}|_{L_{p_0}(\mathbb{R}^n)}\| \\ & \lesssim \left\| \left(\sum_{k=-\infty}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{kn(1/2+1/(\kappa p))q} \omega_{k,m,\kappa p}^q |\lambda_{k,m}|^q \chi_{Q_{k,m}} \right)^{1/q} \Big|_{L_p(\mathbb{R}^n)} \right\|^{p/p_0} \\ & \lesssim \|\lambda| \dot{f}_{p,q}(\mathbb{R}^n, \{\omega_k\})\|^{p/p_0}. \end{aligned}$$

Proof of (4.20). By Proposition 3.16 with $E_{Q_{k,m}}^l = Q_{k,m} \cap A_{l+1}^c$, we obtain

$$\begin{aligned} & \|\lambda^1| \dot{f}_{\infty,q_1}(\mathbb{R}^n, \{w_k\})\| \\ & \lesssim \left\| \left(\sum_{l=-\infty}^{\infty} \sum_{(k,m) \in C_l} 2^{kn(1/2+1/q_1)q_1} w_{k,m,q_1}^{q_1} |\lambda_{k,m}^1|^{q_1} \chi_{E_{Q_{k,m}}^l} \right)^{1/q_1} \right\|_{\infty}. \end{aligned}$$

The special choice of $\lambda_{k,m}^1$ implies

$$w_{k,m,q_1} \lambda_{k,m}^1 \lesssim \omega_{k,m,\kappa p}^{q/q_1} 2^{l\delta+kv} |\lambda_{k,m}|^{q/q_1} \leq \omega_{k,m,\kappa p}^{q/q_1} 2^{kv} g^{-q/q_1}(x) |\lambda_{k,m}|^{q/q_1}$$

for any $x \in E_{Q_{k,m}}^l$ and

$$v + \frac{n}{q_1} + \frac{n}{2} = \frac{nq}{q_1} \left(\frac{1}{\kappa p} + \frac{1}{2} \right).$$

Consequently, we get

$$\|\lambda^1 | \dot{f}_{\infty, q_1}(\mathbb{R}^n, \{w_k\})\| \lesssim 1.$$

Substep 1.2.2. We consider the case $\gamma < 0$. We use an argument similar to the above with $Q_{k,m} \cap A_l$ replaced by $Q_{k,m} \cap A_{l+1}^c$.

Hence, we complete the proof. $\square$

There are nice connections between complex interpolation spaces and the corresponding Calderón product, see the original paper of A. P. Calderón [8]. Suppose that X_0 and X_1 are Banach lattices on measure space $(\mathcal{M}, \mu)$, and let

$$(4.21) \quad X = X_0^{1-\theta} \cdot X_1^\theta$$

for some $0 < \theta < 1$. Suppose that X has the property

$$f \in X, \quad |f_n(x)| \leq |f(x)|, \quad \mu\text{-a.e.},$$

and

$$\lim_{n \rightarrow \infty} f_n = f, \quad \mu\text{-a.e.} \implies \lim_{n \rightarrow \infty} \|f_n\|_X = \|f\|_X,$$

A. P. Calderón in [8, page 125] then shows that

$$(4.22) \quad X_0^{1-\theta} \cdot X_1^\theta = [X_0, X_1]_\theta.$$

Let $1 \leq p < \infty$, $1 \leq q < \infty$, and $\{t_k\}$ be a p -admissible weight sequence. In (4.21) assume that $X = \dot{a}_{p,q}(\mathbb{R}^n, \{t_k\})$ for some Banach lattices X_0 and X_1 . By dominated convergence theorem X satisfies the Calderón's result (4.22). Using Theorem 4.5 and Calderón's result we get the following theorem.

Theorem 4.7. *Let $0 < \theta < 1$, $1 \leq p_0 < \infty$, $1 \leq p_1 < \infty$ and $1 \leq q_0, q_1 < \infty$. Let $\{t_k\} \subset L_{p_0}^{\text{loc}}$ and $\{w_k\} \subset L_{p_1}^{\text{loc}}$ be two weight sequences satisfying (2.3) with $(\sigma_1, p, j) = ((1-\theta)(p_0/(1-\theta))', p = p_0, k)$ and $(\sigma_1, p, j) = (\theta(p_1/\theta)', p = p_1, k)$, respectively. We put*

$$\frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \quad \text{and} \quad \omega_k := t_k^{1-\theta} w_k^\theta, \quad k \in \mathbb{Z}.$$

Let $1/p := (1-\theta)/p_0 + \theta/p_1$. Then

$$[\dot{a}_{p_0, q_0}(\mathbb{R}^n, \{t_k\}), \dot{a}_{p_1, q_1}(\mathbb{R}^n, \{w_k\})]_\theta = \dot{a}_{p, q}(\mathbb{R}^n, \{\omega_k\})$$

holds in the sense of equivalent norms.

By Theorems 3.6 and 4.7, we easily obtain the following result.

Theorem 4.8. *Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, $\beta = (\beta_1, \beta_2) \in \mathbb{R}^2$, $0 < \theta < 1$, $1 \leq p_0 < \infty$, $1 \leq p_1 < \infty$ and $1 \leq q_0, q_1 < \infty$. Let $\{t_k\} \in \dot{X}_{\alpha, \sigma, p_0}$ be a p_0 -admissible weight*

sequence with $\sigma = ((1 - \theta)(p_0/(1 - \theta))', \sigma_2 \geq p_0)$. Let $\{w_k\} \in \dot{X}_{\beta, \sigma, p_1}$ be a p_1 -admissible weight sequence with $\sigma = (\theta(p_1/\theta)', \sigma_2 \geq p_1)$. We put

$$\frac{1}{p} := \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} := \frac{1 - \theta}{q_0} + \frac{\theta}{q_1} \quad \text{and} \quad \omega_k := t_k^{1-\theta} w_k^\theta, \quad k \in \mathbb{Z}.$$

Then

$$[\dot{A}_{p_0, q_0}(\mathbb{R}^n, \{t_k\}), \dot{A}_{p_1, q_1}(\mathbb{R}^n, \{w_k\})]_\theta = \dot{A}_{p, q}(\mathbb{R}^n, \{\omega_k\})$$

holds in the sense of equivalent norms.

From Theorem 4.6 and Calderón's result we obtain the following statement.

Theorem 4.9. Let $0 < \theta < 1$, $1 \leq p_0 < \infty$ and $1 \leq q_0, q_1 < \infty$. Let $\{t_k\} \subset L_{p_0}^{\text{loc}}$ and $\{w_k\} \subset L_{p_1}^{\text{loc}}$ be two weight sequences satisfying (2.3) with $(\sigma_1, p, j) = ((1 - \theta)(p_0/(1 - \theta))', p = p_0, k)$ and $(\sigma_1, p, j) = (\theta(q_1/\theta)', p = q_1, k)$, respectively. We put

$$\frac{1}{p} := \frac{1 - \theta}{p_0}, \quad \frac{1}{q} := \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}, \quad \omega_k := t_k^{1-\theta} w_k^\theta, \quad k \in \mathbb{Z}.$$

Then

$$[\dot{f}_{p_0, q_0}(\mathbb{R}^n, \{t_k\}), \dot{f}_{\infty, q_1}(\mathbb{R}^n, \{t_k\})]_\theta = \dot{f}_{p, q}(\mathbb{R}^n, \{\omega_k\})$$

holds in the sense of equivalent norms.

Theorems 3.6, 3.14, 4.7 and 4.9 yield the following result.

Theorem 4.10. Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, $\beta = (\beta_1, \beta_2) \in \mathbb{R}^2$, $0 < \theta < 1$, $1 \leq p_1 < \infty$ and $1 \leq q_0, q_1 < \infty$. Let $\{t_k\} \in \dot{X}_{\alpha, \sigma, p_0}$ be a p_0 -admissible weight sequence with $\sigma = ((1 - \theta)(p_0/(1 - \theta))', \sigma_2 \geq p_0)$. Let $\{w_k\} \in \dot{X}_{\beta, \sigma, q_1}$ be a p_1 -admissible weight sequence with $\sigma = (\theta(q_1/\theta)', \sigma_2 \geq q_1)$. We put

$$\frac{1}{p} := \frac{1 - \theta}{p_0}, \quad \frac{1}{q} := \frac{1 - \theta}{q_0} + \frac{\theta}{q_1} \quad \text{and} \quad \omega_k := t_k^{1-\theta} w_k^\theta, \quad k \in \mathbb{Z}.$$

Then

$$[\dot{F}_{p_0, q_0}(\mathbb{R}^n, \{t_k\}), \dot{F}_{\infty, q_1}(\mathbb{R}^n, \{w_k\})]_\theta = \dot{F}_{p, q}(\mathbb{R}^n, \{\omega_k\})$$

holds in the sense of equivalent norms.

Remark 4.11. The methods of [34] are capable of dealing with the spaces $\dot{B}_{p, q}(\mathbb{R}^n, \{t_k\})$ and $\dot{F}_{p, q}(\mathbb{R}^n, \{t_k\})$ with the smoothness t_k being independent of k , $k \in \mathbb{Z}$, but they do not apply in the case of spaces of variable smoothness as in this paper.

Corollary 4.12. Let $s_0, s_1 \in \mathbb{R}$, $0 < \theta < 1$, $1 \leq p_0 < \infty$ and $1 \leq q_0, q_1 < \infty$. Let $\omega_0^{p_0} \in A_{p_0/(1-\theta)}(\mathbb{R}^n)$ and $\omega_1^{q_1} \in A_{q_1/\theta}(\mathbb{R}^n)$,

$$\frac{1}{p} := \frac{1-\theta}{p_0}, \quad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \quad \text{and} \quad s = (1-\theta)s_0 + \theta s_1.$$

Then

$$[\dot{F}_{p_0, q_0}^{s_0}(\mathbb{R}^n, \omega_0), \dot{F}_{\infty, q_1}^{s_1}(\mathbb{R}^n, \omega_1)]_\theta = \dot{F}_{p, q}^s(\mathbb{R}^n, \omega_0^{1-\theta} \omega_1^\theta)$$

holds in the sense of equivalent norms.

PROOF: The proof easily follows Theorem 4.10. $\square$

Remark 4.13. All our results are easily generalized to the inhomogeneous Triebel–Lizorkin spaces $F_{p, q}(\mathbb{R}^n, \{t_k\})$ and the inhomogeneous Besov spaces $B_{p, q}(\mathbb{R}^n, \{t_k\})$.

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