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Eventually positive elements in ordered Banach algebras

GERD HERZOG, PEER C. KUNSTMANN

Abstract. In ordered Banach algebras, we introduce eventually and asymptotically positive elements. We give conditions for the following spectral properties: the spectral radius belongs to the spectrum (Perron–Frobenius property); the spectral radius is the only element in the peripheral spectrum; there are positive (approximate) eigenvectors for the spectral radius. Recently such types of results have been shown for operators on Banach lattices. Our results can be viewed as a complement, since our structural assumptions on the ordered Banach algebra are much weaker.

Keywords: ordered Banach algebra; eventually positive element; spectral property; Perron–Frobenius property

Classification: 46B40, 46H05

1. Introduction

Let $\mathcal{A}$ be a complex Banach algebra with unit $\mathbb{1}$. A set $\emptyset \neq K \subseteq \mathcal{A}$ is called a cone if K is closed, $K + K \subseteq K$, $\lambda K \subseteq K$, $\lambda \geq 0$, and $K \cap (-K) = \{0\}$. By setting $a \leq b := b - a \in K$ we obtain partial order on $\mathcal{A}$. Following the notation in [6] we call $\mathcal{A}$ an ordered Banach algebra if K is an *algebra cone*, that is if K satisfies in addition $\mathbb{1} \in K$ and

$$a, b \in K \Rightarrow ab \in K.$$

In the following we always assume that $\mathcal{A}$ is an ordered Banach algebra, and that K is a normal cone, that is

$$\exists \gamma \geq 1 \forall a, b, c \in \mathcal{A}: a \leq b \leq c \Rightarrow \|b\| \leq \gamma(\|a\| + \|c\|).$$

It is well known [6, Theorem 4.1] that then the spectral radius r is increasing on K , that is

$$\forall a, b \in \mathcal{A}: 0 \leq a \leq b \Rightarrow r(a) \leq r(b).$$

We call the elements of K *positive*. Moreover $a \in \mathcal{A}$ is called *eventually positive* if

$$\exists m \in \mathbb{N}_0 \forall n \geq m+1: a^n \geq 0,$$

and it is called *asymptotically positive* if $r(a) > 0$ and

$$\lim_{n \rightarrow \infty} d_n(a) = 0 \quad \text{where } d_n(a) := \text{dist}\left(\frac{a^n}{r(a)^n}, K\right), \quad n \in \mathbb{N}_0.$$

Let $\sigma(a)$ denote the spectrum of $a \in \mathcal{A}$. The Perron–Frobenius property “ $r(a) \in \sigma(a)$ for $a \in K$ ” was proved by R. Raubenheimer and S. Rode in [6, Theorem 5.2]. For eventually positive and asymptotically positive elements various results for matrices and operators in ordered Banach spaces are known. Recently J. Glück in [3] investigated the Perron–Frobenius property under several asymptotic positivity conditions for operators in Banach lattices. For the matrix case we refer to [7] and the references given there.

In this paper we will prove that the Perron–Frobenius property holds for $a \in \mathcal{A}$ with $r(a) > 0$ if $\sqrt{n} d_n(a) \rightarrow 0$, $n \rightarrow \infty$. In particular, one has $r(a) \in \sigma(a)$ if a is eventually positive. Theorem 4.1 in [3] applied to the operator $x \mapsto Tx = ax$ shows that the weaker condition $d_n(a) \rightarrow 0$, $n \rightarrow \infty$, is sufficient for $r(a) \in \sigma(a)$ if $\mathcal{A}$ is in addition a Banach lattice with respect to the order given by K . It might be that this weaker condition is also sufficient in the non-lattice case. We leave this here as an open question.

Moreover we will give some results on the peripheral spectrum and the existence of eigenvectors and approximative eigenvectors of eventually positive elements.

The following lemma from elementary number theory will turn out to be useful for the construction of examples.

Lemma 1. *Let $n_1, n_2 \in \mathbb{N}$ with $\gcd(n_1, n_2) = 1$. Then to each $n > n_1 n_2$ there exist $j_1, j_2 \in \mathbb{N}$ such that $n = j_1 n_1 + j_2 n_2$.*

Note that $a^{n_1}, a^{n_2} \geq 0$ and $n = j_1 n_1 + j_2 n_2$ yields

$$a^n = (a^{n_1})^{j_1} (a^{n_2})^{j_2} \geq 0.$$

Thus, we get from Lemma 1 that $a \in \mathcal{A}$ is eventually positive if and only if there exist $n_1, n_2 \in \mathbb{N}$ with $\gcd(n_1, n_2) = 1$ and $a^{n_1}, a^{n_2} \geq 0$. As an example consider $\mathcal{A} = \mathbb{C}^{3 \times 3}$ ordered by the cone of nonnegative matrices and

$$A = \begin{pmatrix} -2 & 3 & -1 \\ 1 & 2 & 1 \\ 2 & 3 & -1 \end{pmatrix}.$$

Here

$$A^7 = \begin{pmatrix} 851 & 891 & 478 \\ 762 & 3434 & 452 \\ 439 & 2751 & 838 \end{pmatrix}, \quad A^9 = \begin{pmatrix} 4603 & 10659 & 6062 \\ 9322 & 36122 & 5476 \\ 5183 & 33735 & 4310 \end{pmatrix}$$

and $\gcd(7, 9) = 1$. Thus A is eventually positive.

For a second example consider the Wiener algebra

$$\mathcal{A} = A(\mathbb{T}) = \left\{ f: f(t) = \sum_{k \in \mathbb{Z}} c_k e^{ikt}, \sum_{k \in \mathbb{Z}} |c_k| < \infty \right\}$$

with norm $\|f\| = \sum_{k \in \mathbb{Z}} |c_k|$ and ordered by the normal algebra cone

$$K = \{f: c_k \geq 0, k \in \mathbb{Z}\}.$$

The trigonometric polynomial

$$t \mapsto f(t) = \exp(-i2t) + \exp(-it) - \frac{3}{4} + \exp(it) + \exp(i2t).$$

is an eventually positive element of $\mathcal{A}$ since $f^{10}, f^{11} \in K$ (whereas $f^k \notin K$, $k = 1, \dots, 9$).

2. The Perron–Frobenius property

Theorem 1. *Let $a \in \mathcal{A}$ with $r(a) > 0$ be such that*

$$\lim_{n \rightarrow \infty} \sqrt{n} d_n(a) = 0.$$

Then $r(a) \in \sigma(a)$.

PROOF: We assume without loss of generality that $r(a) = 1$. Then $d_n(a) = \text{dist}(a^n, K)$ and by assumption there is a sequence $(b_n)_{n \in \mathbb{N}_0}$ with $b_0 = 0$ such that

$$a^n + b_n \geq 0, \quad n \in \mathbb{N}_0, \quad \text{and} \quad \beta_n := \sqrt{n} \|b_n\| \rightarrow 0, \quad n \rightarrow \infty.$$

Assume by contradiction that $1 \notin \sigma(a)$. Then we can choose $\alpha \in (0, 1)$ such that

$$\max\{\Re \lambda: \lambda \in \sigma(a)\} < \alpha.$$

Since

$$\max\{\Re \lambda: \lambda \in \sigma(a)\} = \lim_{t \rightarrow \infty} \frac{1}{t} \log(\|e^{ta}\|),$$

see [1, Section 3], there is some constant $\delta \geq 1$ such that

$$\|e^{ta}\| \leq \delta e^{t\alpha}, \quad t \geq 0.$$

For each $n \in \mathbb{N}_0$ and $t > 0$ we have

$$0 \leq \frac{t^n}{n!}(a^n + b_n) \leq \sum_{k=0}^{\infty} \frac{t^k}{k!}(a^k + b_k) = e^{ta} + \sum_{k=0}^{\infty} \frac{t^k}{k!}b_k,$$

hence

$$-\frac{t^n}{n!}b_n \leq \frac{t^n}{n!}a^n \leq e^{ta} + \sum_{n \neq k=0}^{\infty} \frac{t^k}{k!}b_k,$$

and since K is normal, $r(a^n) = 1 \leq \|a^n\|$, and $b_0 = 0$, we get the inequality

$$\begin{aligned} \frac{t^n}{n!} &\leq \frac{t^n}{n!}\|a^n\| \leq \gamma \left(\left\| e^{ta} + \sum_{n \neq k=0}^{\infty} \frac{t^k}{k!}b_k \right\| + \left\| \frac{t^n}{n!}b_n \right\| \right) \leq \gamma \left(\delta e^{t\alpha} + \sum_{k=0}^{\infty} \frac{t^k}{k!}\|b_k\| \right) \\ &= \gamma \left(\delta e^{t\alpha} + \sum_{k=1}^{\infty} \frac{t^k}{k!} \frac{\beta_k}{\sqrt{k}} \right) \leq q \left(e^{t\alpha} + \sum_{k=1}^{\infty} \frac{t^k}{k!} \frac{\beta_k}{\sqrt{k}} \right), \end{aligned}$$

with $q := \gamma\delta$. Multiplication with $\sqrt{t}e^{-t}$ and inserting $t = n$ yields

$$\frac{1}{q} \frac{n^{n+1/2}e^{-n}}{n!} \leq \sqrt{n}e^{(\alpha-1)n} + \sqrt{n}e^{-n} \sum_{k=1}^{\infty} \frac{n^k}{k!} \frac{\beta_k}{\sqrt{k}}.$$

By Stirling's formula the left hand side tends to $1/(q\sqrt{2\pi})$, $n \rightarrow \infty$, and by $\alpha \in (0, 1)$ we have $\sqrt{n}e^{(\alpha-1)n} \rightarrow 0$, $n \rightarrow \infty$. Thus we obtain a contradiction by showing that

$$h_n := \sqrt{n}e^{-n} \sum_{k=1}^{\infty} \frac{n^k}{k!} \frac{\beta_k}{\sqrt{k}} \rightarrow 0, \quad n \rightarrow \infty.$$

To see this, consider

$$\begin{aligned} h_n &= e^{-n} \sum_{k=1}^{\infty} \frac{n^{k+1/2}}{k!} \frac{\beta_k}{\sqrt{k}} = e^{-n} \sum_{k=1}^{\infty} \sqrt{\frac{n^k}{k!}} \sqrt{\frac{n^{k+1}}{k!k}} \beta_k^2 \\ &\leq e^{-n} \left(\sum_{k=1}^{\infty} \frac{n^k}{k!} \right)^{1/2} \left(\sum_{k=1}^{\infty} \frac{n^{k+1}}{(k+1)!} \frac{k+1}{k} \beta_k^2 \right)^{1/2} \\ &\leq e^{-n} e^{n/2} \left(\sum_{k=1}^{\infty} \frac{n^{k+1}}{(k+1)!} \frac{k+1}{k} \beta_k^2 \right)^{1/2} = \left(e^{-n} \sum_{k=1}^{\infty} \frac{n^{k+1}}{(k+1)!} \frac{k+1}{k} \beta_k^2 \right)^{1/2}. \end{aligned}$$

Since

$$\frac{k+1}{k} \beta_k^2 \rightarrow 0, \quad k \rightarrow \infty,$$

we have

$$e^{-n} \sum_{k=1}^{\infty} \frac{n^{k+1}}{(k+1)!} \frac{k+1}{k} \beta_k^2 \rightarrow 0, \quad n \rightarrow \infty,$$

which finishes the proof. $\square$

Corollary 1. *Let $\mathcal{B}$ be a Banach algebra, $a \in \mathcal{B}$ with $r(a) = 1$ and assume that there exists a multiplicative linear functional $\varphi: \mathcal{B} \rightarrow \mathbb{C}$ and a sequence (b_n) in $\mathcal{B}$ with*

$$\|b_n\| = \varphi(b_n) = 1, \quad n \in \mathbb{N}, \quad \lim_{n \rightarrow \infty} \sqrt{n} \|a^n - b_n\| = 0.$$

Then $1 \in \sigma(a)$.

PROOF: Let $\mathcal{A}$ be the Banach algebra $\mathcal{B}$ ordered by the cone

$$K := \{b \in \mathcal{B}: \|b\| \leq \varphi(b)\}.$$

Note that a multiplicative linear functional is nontrivial by definition and that $\varphi(\mathbb{1}) = 1$ and $\|\varphi\| = 1$. As a consequence K is a normal algebra cone with $\gamma = 1$. Now $b_n \geq 0$, $n \in \mathbb{N}$, thus

$$\sqrt{n} d_n(a) \leq \sqrt{n} \|a^n - b_n\| \rightarrow 0, \quad n \rightarrow \infty.$$

Application of Theorem 1 proves $1 \in \sigma(a)$. $\square$

To cherish the following result we consider two examples which show that “eventually positive” is badly compatible with addition:

1. Let

$$A = \begin{pmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 2 & -2 & 1 & 1 \\ -2 & 2 & 1 & 1 \end{pmatrix}.$$

Then A is eventually positive, but $A + \lambda I$ fails to be eventually positive for each $\lambda > 0$, see [2, Exercise 3.13].

2. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

The A is positive, B is eventually positive (even nilpotent) but $r(A + B) \notin \sigma(A + B)$, see [7, Lemma 3.3].

As an application of Theorem 1 we can prove the following result.

Theorem 2. *Let $a, b \in \mathcal{A}$ be eventually positive, $ab = ba$ and*

$$\max\{r(a), r(b)\} < r(a + b).$$

Then $r(a + b) \in \sigma(a + b)$.

PROOF: Choose $m \in \mathbb{N}$ such that $a^n, b^n \geq 0$, $n \geq m+1$, and let $c := (a+b)/r(a+b)$. For $n \geq 2(m+1)$ we get

$$\begin{aligned} r(a+b)^n c^n &= (a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} \\ &= \sum_{k=0}^m \binom{n}{k} a^k b^{n-k} + \sum_{k=m+1}^{n-m-1} \binom{n}{k} a^k b^{n-k} + \sum_{k=n-m}^n \binom{n}{k} a^k b^{n-k}, \end{aligned}$$

with

$$\sum_{k=m+1}^{n-m-1} \binom{n}{k} a^k b^{n-k} \geq 0.$$

Hence

$$\begin{aligned} d_n((a+b)^n) &= \text{dist}(c^n, K) \\ &\leq \frac{1}{r(a+b)^n} \left(\sum_{k=0}^m \binom{n}{k} \|a^k\| \|b^{n-k}\| + \sum_{k=n-m}^n \binom{n}{k} \|a^k\| \|b^{n-k}\| \right) \\ &= \frac{1}{r(a+b)^n} \left(\sum_{k=0}^m \binom{n}{k} \|a^k\| \|b^{n-k}\| + \sum_{k=0}^m \binom{n}{k} \|a^{n-k}\| \|b^k\| \right). \end{aligned}$$

Let $\varepsilon > 0$ be such that

$$\frac{r(a) + \varepsilon}{r(a+b)} < 1 \quad \text{and} \quad \frac{r(b) + \varepsilon}{r(a+b)} < 1$$

and choose $n_0 \geq 2(m+1)$ such that for each $j \geq n_0 - m$

$$\|a^j\|^{1/j} \leq r(a) + \varepsilon \quad \text{and} \quad \|b^j\|^{1/j} \leq r(b) + \varepsilon.$$

Then, for $n \geq n_0$, we have

$$\sqrt{n} d_n((a+b)^n) \leq \sum_{k=0}^m \sqrt{n} \binom{n}{k} \|a^k\| \frac{(r(b) + \varepsilon)^{n-k}}{r(a+b)^n} + \sum_{k=0}^m \sqrt{n} \binom{n}{k} \|b^k\| \frac{(r(a) + \varepsilon)^{n-k}}{r(a+b)^n}.$$

Now, if $\mu \in (0, 1)$ then

$$\sqrt{n} \binom{n}{k} \mu^n \rightarrow 0, \quad n \rightarrow \infty,$$

for each $k = 0, \dots, m$. Thus $\sqrt{n} d_n((a+b)^n) \rightarrow 0$, $n \rightarrow \infty$, and Theorem 1 applies. $\square$

3. The peripheral spectrum

Let $\sigma_{\text{per}}(a)$ denote the peripheral spectrum of a , i.e.

$$\sigma_{\text{per}}(a) := \{\lambda \in \sigma(a) : |\lambda| = r(a)\}.$$

Theorem 3. *Let $a \in \mathcal{A}$. If there exist $n_1, n_2 \in \mathbb{N}$ with $\gcd(n_1, n_2) = 1$ and $\alpha > 0$ such that*

$$a^{n_1} \geq \alpha \mathbb{1}, \quad a^{n_2} \geq \alpha \mathbb{1},$$

then $\sigma_{\text{per}}(a) = \{r(a)\}$.

PROOF: Since $r(a^{n_1}) \geq r(\alpha \mathbb{1}) = \alpha > 0$ we have $r(a) > 0$. According to [4, Theorem 4.25]

$$\sigma_{\text{per}}(a^{n_1}) = \{r(a)^{n_1}\}, \quad \sigma_{\text{per}}(a^{n_2}) = \{r(a)^{n_2}\}.$$

Let $\lambda \in \sigma_{\text{per}}(a)$. Then $\lambda^{n_k} \in \sigma_{\text{per}}(a^{n_k})$, hence $\lambda^{n_k} = r(a)^{n_k}$, $k = 1, 2$. Let $j_1, j_2 \in \mathbb{Z}$ be such that $1 = j_1 n_1 + j_2 n_2$. Then

$$\lambda = \lambda^{j_1 n_1 + j_2 n_2} = (\lambda^{n_1})^{j_1} (\lambda^{n_2})^{j_2} = (r(a)^{n_1})^{j_1} (r(a)^{n_2})^{j_2} = r(a).$$

Thus $\sigma_{\text{per}}(a) = \{r(a)\}$. □

Example 1. Consider the ordered Wiener algebra from the introduction. Note that $\sigma(f) = f([0, 2\pi])$, $f \in A(\mathbb{T})$. Let $f \in A(\mathbb{T})$ be eventually positive and let $m \in \mathbb{N}_0$ be such that $f^n \in K$, $n \geq m + 1$. Then for $n \geq m + 1$ we have $f^n(0) = r(f^n)$, hence $|f(0)| = r(f)$. So, if $f(0) = 0$ then $\sigma(f) = \{0\}$ and $f = 0$. If $f(0) \neq 0$ choose $n_1, n_2 \geq m + 1$ with $\gcd(n_1, n_2) = 1$ and let $j_1, j_2 \in \mathbb{Z}$ satisfy $j_1 n_1 + j_2 n_2 = 1$. Then

$$f(0) = (f(0)^{n_1})^{j_1} (f(0)^{n_2})^{j_2} > 0.$$

In particular $r(f) = f(0)$. For

$$f(t) = \exp(-i2t) + \exp(-it) - \frac{3}{4} + \exp(it) + \exp(i2t)$$

we have

$$f^{10} \geq 18535 \cdot \mathbb{1}, \quad f^{11} \geq 2038 \cdot \mathbb{1},$$

hence, by Theorem 3,

$$\sigma_{\text{per}}(f) = \{r(f)\} = \{f(0)\} = \left\{\frac{13}{4}\right\}.$$

4. Krein–Rutman type theorems

If $r(a)$ is a pole of the resolvent we obtain the following variant of the Krein–Rutman theorem for eventually positive elements. For various versions of this theorem for positive elements in ordered Banach algebras, see [5, Section 3].

Theorem 4. *Let $a \in \mathcal{A}$ be eventually positive. Moreover let $r(a) > 0$ be a pole of the resolvent:*

$$(*) \quad (\lambda \mathbb{1} - a)^{-1} = \sum_{n=-q}^{\infty} b_n (\lambda - r(a))^n \quad \text{where } q \in \mathbb{N}, b_{-q} \neq 0.$$

Then $b_{-q} > 0$ and $ab_{-q} = b_{-q}a = r(a)b_{-q}$.

PROOF: Choose $m \in \mathbb{N}_0$ such that $a^n \geq 0$, $n \geq m+1$. Note that

$$(\lambda \mathbb{1} - a)^{-1} = \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}}, \quad |\lambda| > r(a).$$

We have

$$(**) \quad (\lambda - r(a))^q (\lambda \mathbb{1} - a)^{-1} \rightarrow b_{-q}, \quad \lambda \rightarrow r(a).$$

For $\lambda > r(a)$ we have

$$\begin{aligned} (\lambda - r(a))^q (\lambda \mathbb{1} - a)^{-1} &= (\lambda - r(a))^q \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}} \\ &\geq (\lambda - r(a))^q \sum_{n=0}^m \frac{a^n}{\lambda^{n+1}} \rightarrow 0, \quad \lambda \rightarrow r(a)^+. \end{aligned}$$

Thus $b_{-q} > 0$. By

$$a(\lambda \mathbb{1} - a)^{-1} = (a - \lambda \mathbb{1} + \lambda \mathbb{1})(\lambda \mathbb{1} - a)^{-1} = \lambda(\lambda \mathbb{1} - a)^{-1} - \mathbb{1}$$

and by (**) we have

$$\begin{aligned} (\lambda - r(a))^q (\lambda(\lambda \mathbb{1} - a)^{-1} - \mathbb{1}) &\rightarrow r(a)b_{-q}, & \lambda \rightarrow r(a), \\ (\lambda - r(a))^q a(\lambda \mathbb{1} - a)^{-1} &\rightarrow ab_{-q} = b_{-q}a, & \lambda \rightarrow r(a). \end{aligned}$$

Therefore $ab_{-q} = b_{-q}a = r(a)b_{-q}$. □

Concerning the case $r(a) = 0$ note that a is nilpotent if 0 is a pole of the resolvent as in (*). In this case $a^q = 0$ and $b_{-q} = a^{q-1}$, so $b_{-q} > 0$ does not follow from a being eventually positive, as each nilpotent element shares this property.

If $r(a)$ is not a pole of the resolvent we at least get an approximative version of Theorem 4. Let $\sigma_{ap}(a)$ denote the approximative point spectrum of a . As $\partial\sigma(a) \subseteq \sigma_{ap}(a)$, $a \in \mathcal{A}$, we have $r(a) \in \sigma_{ap}(a)$ if a is eventually positive. The next result shows that in this case the almost eigenvectors of $x \mapsto ax$ can be chosen in K .

Theorem 5. *Let $a \in \mathcal{A}$ be eventually positive and $r(a) > 0$. Then there exists a sequence (b_n) in K with $\|b_n\| = 1$, $b_n a = ab_n$, $n \in \mathbb{N}$, and*

$$(r(a)\mathbb{1} - a)b_n \rightarrow 0, \quad n \rightarrow \infty.$$

PROOF: Let $m \in \mathbb{N}_0$ be such that $a^n \geq 0$, $n \geq m+1$, and let $\lambda_n := r(a) + 1/n$, $n \in \mathbb{N}$. Since $r(a) \in \sigma(a)$ we have

$$\|(\lambda_n \mathbb{1} - a)^{-1}\| \rightarrow \infty, \quad n \rightarrow \infty.$$

Note that $(\lambda_n \mathbb{1} - a)^{-1} = \sum_{k=0}^{\infty} a^k / \lambda_n^{k+1}$. Set

$$x_n := (\lambda_n \mathbb{1} - a)^{-1} - \sum_{k=0}^m \frac{a^k}{\lambda_n^{k+1}} = \sum_{k=m+1}^{\infty} \frac{a^k}{\lambda_n^{k+1}}, \quad n \in \mathbb{N}.$$

Then $x_n \geq 0$ and (as $r(a) > 0$)

$$\|x_n\| \geq \|(\lambda_n \mathbb{1} - a)^{-1}\| - \left\| \sum_{k=0}^m \frac{a^k}{\lambda_n^{k+1}} \right\| \rightarrow \infty, \quad n \rightarrow \infty.$$

Set $b_n = x_n / \|x_n\|$, $n \in \mathbb{N}$. Clearly $b_n a = ab_n$, $\|b_n\| = 1$ and $b_n \geq 0$, $n \in \mathbb{N}$. Moreover

$$\begin{aligned} (r(a)\mathbb{1} - a)b_n &= \frac{1}{\|x_n\|} \left((r(a)\mathbb{1} - a)(\lambda_n \mathbb{1} - a)^{-1} - (r(a)\mathbb{1} - a) \sum_{k=0}^m \frac{a^k}{\lambda_n^{k+1}} \right) \\ &= \frac{1}{\|x_n\|} \left((r(a) - \lambda_n)(\lambda_n \mathbb{1} - a)^{-1} + \mathbb{1} - (r(a)\mathbb{1} - a) \sum_{k=0}^m \frac{a^k}{\lambda_n^{k+1}} \right) \\ &= -\frac{(\lambda_n \mathbb{1} - a)^{-1}}{n\|x_n\|} + \frac{1}{\|x_n\|} \left(\mathbb{1} - (r(a)\mathbb{1} - a) \sum_{k=0}^m \frac{a^k}{\lambda_n^{k+1}} \right) \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

□

Concerning the case $r(a) = 0$ assume that there is some $\varepsilon > 0$ such that $a + t\mathbb{1}$ is eventually positive for each $0 \leq t < \varepsilon$. Then the assertion of Theorem 5 still holds: For $n \in \mathbb{N}$, $n > 1/\varepsilon$, set $a_n := a + (1/n)\mathbb{1}$. Then $r(a_n) = 1/n > 0$, so by Theorem 5 there exists $b_n \geq 0$ with $\|b_n\| = 1$, $b_n a = ab_n$ and $\|(r(a_n)\mathbb{1} - a_n)b_n\| \leq 1/n$. Thus

$$\|(r(a)\mathbb{1} - a)b_n\| = \left\| \left(\frac{1}{n}\mathbb{1} - a - \frac{1}{n}\mathbb{1} \right) b_n \right\| = \|(r(a_n)\mathbb{1} - a_n)b_n\| \rightarrow 0, \quad n \rightarrow \infty.$$

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