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Aposyndesis in $\mathbb{N}$

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Abstract. We consider the Golomb and the Kirch topologies in the set of natural numbers. Among other results, we show that while with the Kirch topology every arithmetic progression is aposyndetic, in the Golomb topology only for those arithmetic progressions $P(a,b)$ with the property that every prime number that divides a also divides b, it follows that being connected, being Brown, being totally Brown, and being aposyndetic are all equivalent. This characterizes the arithmetic progressions which are aposyndetic in the Golomb space.

Keywords: aposyndesis; arithmetic progression; Golomb topology; Kirch topology; totally Brown space; totally separated space

Classification: 11B25, 54D05, 11A41, 11B05, 54A05, 54D10

1. Introduction

The following paper is a continuation of the work presented in [1] and [2]. We denote by $\mathbb{Z}$ the set of integers, and by $\mathbb{N}$ the set of natural numbers. We define $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and

$$\mathbb{N}_b = \{n \in \mathbb{N} \colon n \geq b\} \qquad \text{for every } b \in \mathbb{N}.$$

For each $a, b \in \mathbb{N}$ we define

$$P(a,b) = \{b + an \colon n \in \mathbb{N}_0\} = b + a\mathbb{N}_0.$$

If $a \in \mathbb{N}$ and $b \in \mathbb{Z}$ we also define

$$P_F(a,b) = \{b + az \colon z \in \mathbb{Z}\} = b + a\mathbb{Z} \qquad \text{and} \qquad M(a) = \{an \colon n \in \mathbb{N}\}.$$

Note that $P(a,b)$ is the arithmetic progression in $\mathbb{N}$ whose initial term is b and the common difference of successive members is a, while $M(a)$ is the arithmetic progression of all multiples of a in $\mathbb{N}$. Hence $P(a,b) \subset \mathbb{N}_b, M(a) = P(a,a)$, $M(1) = \mathbb{N}$, and $P(a,b) = P_F(a,b) \cap \mathbb{N}_b$. Moreover,

$$P(a,b) = P_F(a,b) \cap \mathbb{N}, \qquad \text{if } a \geq b. \tag{1}$$

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The symbol $\mathbb{P}$ denotes the set of prime numbers. We consider that $\mathbb{P} \subset \mathbb{N}$. For each $a \in \mathbb{N}$ we define

$$\Theta(a) = \{p \in \mathbb{P} \colon\ p|a\}. \tag{2}$$

Note that $\Theta(a) = \emptyset$ if and only if $a = 1$. In this paper we consider two topologies τ_G and τ_K on $\mathbb{N}$ and pursue two purposes. As the first one we present, in the respective topological spaces $(\mathbb{N}, \tau_G)$ and $(\mathbb{N}, \tau_K)$, properties of the arithmetic progressions $P(a, b)$ with $\Theta(a) \subset \Theta(b)$.

Let (X, τ) be a topological space and $A \subset X$. The symbols $\mathrm{cl}_X(A)$ and $\mathrm{int}_X(A)$ denote the closure and the interior of A in (X, τ), respectively. If we like to specify the topology τ on X, we write $\mathrm{cl}_{(X,\tau)}(A)$ and $\mathrm{int}_{(X,\tau)}(A)$, respectively.

The following notions appear in [1].

Definition 1.1. Let (X, τ) be a topological space. We say that X is a

(1) *Brown space* if for every nonempty open subsets U and V of X we have $\mathrm{cl}_X(U) \cap \mathrm{cl}_X(V) \neq \emptyset$;
(2) *totally Brown space* if for every $n \in \mathbb{N}_2$ and each nonempty open subsets $U_1, U_2, \ldots, U_n$ of X we have

$$\mathrm{cl}_X(U_1) \cap \mathrm{cl}_X(U_2) \cap \cdots \cap \mathrm{cl}_X(U_n) \neq \emptyset.$$

Definition 1.2. Let X be a topological space and $a, b \in X$ with $a \neq b$. We say that X is *aposyndetic at a with respect to b* if there exists a closed and connected subset M of X such that $a \in \mathrm{int}_X(M)$ and $b \notin M$. If for each $x \in X \setminus \{a\}$, X is aposyndetic at a with respect to x, we say that X is *aposyndetic at a*. Finally, we say that X is *aposyndetic* if X has this property at each of its points.

Note that: totally Brown $\implies$ Brown $\implies$ connected.

In both [1, Section 3] and [2, Subsection 2.1] some properties of Brown and of totally Brown spaces are presented. In [1] it is shown that connected spaces are not necessarily Brown, and that Brown spaces are not necessarily totally Brown. The next result is proved in [1, Theorem 3.6].

Theorem 1.3. *If X is totally Brown (Brown, respectively) and T_2, then X is aposyndetic.*

As our second purpose, in this paper we characterize the arithmetic progressions in $(\mathbb{N}, \tau_G)$ and in $(\mathbb{N}, \tau_K)$ that are aposyndetic.

We divide this paper in four sections. After this one, in Section 2 we present some notions and results that we use in the rest of the paper. In Section 3 we consider the topological space $(\mathbb{N}, \tau_G)$ and present new properties of it, mainly concerning those arithmetic progressions $P(a, b)$ with $\Theta(a) \subset \Theta(b)$. It turns out that those are the only ones that are aposyndetic. In Section 4, we consider the topological space $(\mathbb{N}, \tau_K)$ and show that most of the properties presented in Section 3 for the arithmetic progressions $P(a,b)$ with $\Theta(a) \subset \Theta(b)$, remain valid in $(\mathbb{N}, \tau_K)$. We also show that all arithmetic progressions are aposyndetic in $(\mathbb{N}, \tau_K)$.

2. Notation and first results

Given nonzero integers a and b, the symbol $\langle a, b\rangle$ denotes the greatest common divisor of a and b. For nonzero integers $a_1, a_2, \dots, a_k \in \mathbb{N}$, the symbol $[a_1, a_2, \dots, a_k]$ denotes the least common multiple of $a_1, a_2, \dots, a_k$.

For $a, b \in \mathbb{Z}$, the symbol $a|b$ means that $b = ac$ for some $c \in \mathbb{Z}$. If $a, b \in \mathbb{Z}$ and $m \in \mathbb{N}$, the symbol $a \equiv b (\operatorname{mod} m)$ means that $m|(a-b)$. The next result is proved in [2, Theorem 3.1].

Theorem 2.1. *If* $a, b, c, d \in \mathbb{N}$, *then*

$$P(c,d) \subset P(a,b) \quad \text{if and only if } a|c \text{ and } d \in P(a,b). \tag{3}$$

In particular, $P(a,b) = P(c,d)$ *if and only if* $a = c$ *and* $b = d$.

The following three results are proved in [1, Corollary 4.11, Theorem 4.12 and Theorem 4.14].

Theorem 2.2. *If* $b, a_1, a_2 \dots, a_k \in \mathbb{N}$, *then*

$$P([a_1, a_2, \dots, a_k], b) = \bigcap_{i=1}^{k} P(a_i, b). \tag{4}$$

Theorem 2.3. *For each* $i \in \{1, 2, \dots, k\}$ *let* $a_i, b_i \in \mathbb{N}$ *be such that*

$$\bigcap_{i=1}^{k} P(a_i, b_i) \neq \emptyset.$$

If $z \in \mathbb{N}$, *then*

$$P([a_1, a_2, \dots, a_k], z) = \bigcap_{i=1}^{k} P(a_i, b_i) \tag{5}$$

if and only if $z = \min\left(\bigcap_{i=1}^{k} P(a_i, b_i)\right)$.

Theorem 2.4. *If $a \in \mathbb{N}_2$ and $a = \prod_{i=1}^{k} p_i^{\alpha_i}$ is the standard prime decomposition of a, then*

$$M(a) = \bigcap_{i=1}^{k} M(p_i^{\alpha_i}). \tag{6}$$

In [1, Theorem 4.19] we decompose an arithmetic progression in $\mathbb{N}$ as the union of pairwise disjoint arithmetic progressions in $\mathbb{N}$. Such result, that we present here as Corollary 2.6, can be generalized as follows.

Theorem 2.5. *If $a \in \mathbb{N}_2$ and $b \in \mathbb{N}$ then for each $c \in \mathbb{N}_2$ such that $a|c$ it follows that*

$$P(a,b) = \bigcup_{k=0}^{c/a-1} P(c, ak+b). \tag{7}$$

Moreover, if $c/a \in \mathbb{N}_2$, then the members of the family

$$\mathcal{F} = \left\{P(c, ak+b) \colon k \in \left\{0, 1, \ldots, \frac{c}{a} - 1\right\}\right\}$$

are pairwise disjoint.

Proof: Let $c \in \mathbb{N}_2$ be such that $a|c$, as well as $d \in \mathbb{N}$ so that $c = ad$. If $d = 1$, then clearly (7) is satisfied, so assume that $d \in \mathbb{N}_2$. Define $A = \{0, 1, \ldots, d-1\}$. Take x in the right side of (7). Hence there exist $m \in \mathbb{N}_0$ and $k_0 \in A$ so that $x = cm + (ak_0 + b)$. Note that $dm + k_0 \in \mathbb{N}_0$ and

$$x = cm + (ak_0 + b) = adm + ak_0 + b = a(dm + k_0) + b \in P(a,b).$$

This shows that the right side of (8) is a subset of its left side. Now assume that x is in the left side of (7). Let $l, m \in \mathbb{N}_0$ and $k \in A$ be such that $x = al + b$ and $l = dm + k$. Therefore

$$x = al + b = a(dm + k) + b = (ad)m + (ak + b) = cm + (ak + b) \in P(c, ak+b).$$

Hence x is in the right side of (7) and such equality is satisfied.

To show the second part, let $n_1, n_2 \in \mathbb{N}_0$ and $k_1, k_2 \in A$ be such that $k_1 \neq k_2$ and $cn_1 + (ak_1 + b) = cn_2 + (ak_2 + b)$. Then $d(n_1 - n_2) = k_2 - k_1$. Hence $k_2 \equiv k_1 (\operatorname{mod} d)$. Now, since k_1 and k_2 are distinct members of the set A, which is a complete residue system modulo d, the integers k_1 and k_2 are not congruent modulo d. From this contradiction, we infer that the members of $\mathcal{F}$ are pairwise disjoint. $\square$

Corollary 2.6. *If $a \in \mathbb{N}_2$ and $b \in \mathbb{N}$ then for each $t \in \mathbb{N}_2$*

$$P(a,b) = \bigcup_{k=0}^{a^{t-1}-1} P(a^t, ak+b) \tag{8}$$

and the members of the family

$$\mathcal{F} = \{P(a^t, ak+b) \colon k \in \{0, 1, \ldots, a^{t-1}-1\}\}$$

are pairwise disjoint.

Proof: Take $t \in \mathbb{N}_2$ and define $c = a^t$. Note that $c \in \mathbb{N}_2$, $a|c$, and $c/a = a^{t-1} \in \mathbb{N}_2$. Hence the result follows from this and Theorem 2.5. $\square$

We say that $a \in \mathbb{N}_2$ is *square-free* if it is not divided by the square p^2 of any $p \in \mathbb{P}$. Equivalently a is square-free if its standard prime decomposition is of the form $\prod_{i=1}^{k} p_i$.

The topological space (X, τ) is said to be *totally separated* if for each $x, y \in X$ with $x \neq y$, there exist $U, V \in \tau$ so that $x \in U$, $y \in V$, $X = U \cup V$ and $U \cap V = \emptyset$. We say that X is *hereditarily disconnected* if X does not contain any connected subset of cardinality larger than one. By [4, page 356, Theorem 6.1.22] totally separated spaces are hereditarily disconnected. A space which is hereditarily disconnected but not totally separated is presented in [8, page 91, Example 72].

3. The Golomb space

In both [5, page 663] and [6, page 179], S. W. Golomb showed that the family

$$\mathcal{B}_G = \{P(a,b) \colon (a,b) \in \mathbb{N} \times \mathbb{N} \text{ and } \langle a,b \rangle = 1\}$$

is a base for a topology τ_G in $\mathbb{N}$. In [8, page 82, Examples 60 and 61] this topology is called the relatively prime integer topology. However as it is now more popular in general topology, we call τ_G the *Golomb topology* and refer to the topological space $(\mathbb{N}, \tau_G)$ as the *Golomb space*. Clearly

$$\begin{aligned} \tau_G = \{\emptyset\} \cup \{U \subset \mathbb{N} \colon & \text{ for each } b \in U \text{ there is } a \in \mathbb{N} \\ & \text{so that } \langle a,b \rangle = 1 \text{ and } P(a,b) \subset U\}. \end{aligned}$$

In [5, Theorem 2] it is proved that $(\mathbb{N}, \tau_G)$ is T_2.

In this section we present new properties of $(\mathbb{N}, \tau_G)$, that either generalize or complement the ones presented in [1, Section 5]. In [1, page 205] it is proved, for example that for each $p \in \mathbb{P}$ the arithmetic progression $M(p)$ is closed in $(\mathbb{N}, \tau_G)$.

The following result is proved in [1, Theorem 5.8].

Theorem 3.1. *If $b \in \mathbb{N}$ and $p \in \mathbb{P}$, then*

$$\mathrm{cl}_{(\mathbb{N},\tau_G)}(P(p^n, b)) = M(p) \cup [P_F(p^n, b) \cap \mathbb{N}] \qquad \text{for each } n \in \mathbb{N}. \tag{9}$$

As an application of Theorem 3.1 we have the following.

Theorem 3.2. *If $p \in \mathbb{P}$, then*

$$\mathrm{cl}_{(\mathbb{N},\tau_G)}(M(p^n)) = M(p) \qquad \text{for each } n \in \mathbb{N}. \tag{10}$$

PROOF: By (1) and (9) we have

$$\begin{aligned}\mathrm{cl}_{(\mathbb{N},\tau_G)}(M(p^n)) &= \mathrm{cl}_{(\mathbb{N},\tau_G)}(P(p^n, p^n)) = M(p) \cup [P_F(p^n, p^n) \cap \mathbb{N}] \\ &= M(p) \cup P(p^n, p^n) = M(p) \cup M(p^n) = M(p).\end{aligned}$$

This shows (10). □

Theorem 3.3. *For each $a \in \mathbb{N}_2$, the set $M(a)$ has empty interior in $(\mathbb{N}, \tau_G)$.*

PROOF: Take $a \in \mathbb{N}_2$ and assume that $\mathrm{int}_{(\mathbb{N},\tau_G)}(M(a)) \neq \emptyset$. Hence there exist $c, d \in \mathbb{N}$ such that $\langle c, d\rangle = 1$ and $P(c, d) \subset M(a) = P(a, a)$. This implies, by (3), that $a|c$ and $d \in P(a, a)$. Hence $a|c$ and $a|d$, a contradiction with the fact that $\langle c, d\rangle = 1$. This shows that $\mathrm{int}_{(\mathbb{N},\tau_G)}(M(a)) = \emptyset$. □

The following result is proved in [1, Theorem 5.10]. It originally appears with a different proof in [3, Lemma 2.2].

Theorem 3.4. *Let $a \in \mathbb{N}_2$ and assume that $a = \prod_{i=1}^{k} p_i^{\alpha_i}$ is the standard prime decomposition of a. If $b \in \mathbb{N}$, then*

$$\mathrm{cl}_{(\mathbb{N},\tau_G)}(P(a, b)) = \mathbb{N} \cap \left(\bigcap_{i=1}^{k} [M(p_i) \cup P_F(p_i^{\alpha_i}, b)]\right). \tag{11}$$

If $a \in \mathbb{N}_2$ and $a = \prod_{i=1}^{k} p_i^{\alpha_i}$ is the standard prime decomposition of a then, by (11) we have

$$\mathrm{cl}_{(\mathbb{N},\tau_G)}(M(a)) = \mathbb{N} \cap \left(\bigcap_{i=1}^{k} [M(p_i) \cup P_F(p_i^{\alpha_i}, a)]\right) = \bigcap_{i=1}^{k} [M(p_i) \cup (P_F(p_i^{\alpha_i}, a) \cap \mathbb{N})].$$

Given $a \in \mathbb{N}$ we recall that in (2) we define the set $\Theta(a)$ of prime numbers that divide a. In the following result we calculate the closure in $(\mathbb{N}, \tau_G)$ of an arithmetic progression $P(a, b)$ with $\Theta(a) \subset \Theta(b)$, as well as the closure in $(\mathbb{N}, \tau_G)$ of $M(a)$.

Theorem 3.5. *Let $a \in \mathbb{N}_2$ and $b \in \mathbb{N}$ be such that $\Theta(a) \subset \Theta(b)$. If $a = \prod_{i=1}^{k} p_i^{\alpha_i}$ is the standard prime decomposition of a and $q = p_1 p_2 \cdots p_k$, then*

$$\mathrm{cl}_{(\mathbb{N},\tau_G)}(P(a, b)) = M(q). \tag{12}$$

In particular,

$$\mathrm{cl}_{(\mathbb{N},\tau_G)}(M(a)) = M(q). \tag{13}$$

Proof: Given $i \in \{1,2,\ldots,k\}$ since $p_i|a$ and $\Theta(a) \subset \Theta(b)$, it follows that $p_i|b$. Hence,

$$\mathbb{N} \cap P_F(p_i^{\alpha_i}, b) \subset M(p_i).$$

Using this and Theorem 3.4, we have

$$\begin{aligned}\mathrm{cl}_{(\mathbb{N},\tau_G)}(P(a,b)) &= \mathbb{N} \cap \left(\bigcap_{i=1}^{k} [M(p_i) \cup P_F(p_i^{\alpha_i}, b)] \right) \\ &= \bigcap_{i=1}^{k} [M(p_i) \cup (\mathbb{N} \cap P_F(p_i^{\alpha_i}, b))] = \bigcap_{i=1}^{k} M(p_i) = M(q).\end{aligned}$$

This shows (12). Since $\Theta(a) \subset \Theta(a)$, by (12) we have (13). □

Corollary 3.6. *If $a \in \mathbb{N}_2$, then $M(a)$ is closed in $(\mathbb{N}, \tau_G)$ if and only if a is square-free.*

Proof: Let $a = \prod_{i=1}^{k} p_i^{\alpha_i}$ be the standard prime decomposition of a and $q = p_1 p_2 \cdots p_k$. Clearly q is square-free. If $M(a)$ is closed in $(\mathbb{N}, \tau_G)$ then by (13),

$$P(a,a) = M(a) = \mathrm{cl}_{(\mathbb{N},\tau_G)}(M(a)) = M(q) = P(q,q).$$

Hence, by the second part of Theorem 2.1, $a = q$. This shows that a is square-free.

Now assume that a is square-free. Then $a = q$ and by (13), $\mathrm{cl}_{(\mathbb{N},\tau_G)}(M(a)) = M(q) = M(a)$. Hence, $M(a)$ is closed in $(\mathbb{N}, \tau_G)$. □

Since $M(1) = P(1,1) = \mathbb{N}$ is closed in $(\mathbb{N}, \tau_G)$, by this and Corollary 3.6, the set $M(a)$ is closed in $(\mathbb{N}, \tau_G)$ if and only if either $a = 1$ or a is square-free.

A subset A of a topological space X is *nowhere dense in* X if $\mathrm{int}_X(\mathrm{cl}_X(A)) = \emptyset$.

Theorem 3.7. *If $a \in \mathbb{N}_2$ and $b \in \mathbb{N}$ are such that $\Theta(a) \subset \Theta(b)$, then $P(a,b)$ and $M(a)$ are nowhere dense in $\mathbb{N}$.*

Proof: Let $a = \prod_{i=1}^{k} p_i^{\alpha_i}$ be the standard prime decomposition of a, and $q = p_1 p_2 \cdots p_k$. Note that $q \in \mathbb{N}_2$. By (12) and Theorem 3.3 we have

$$\mathrm{int}_{(\mathbb{N},\tau_G)}\big(\mathrm{cl}_{(\mathbb{N},\tau_G)}(P(a,b))\big) = \mathrm{int}_{(\mathbb{N},\tau_G)}(M(q)) = \emptyset.$$

This shows that $P(a,b)$ is nowhere dense in $\mathbb{N}$. The rest of the proof follows from this and the fact that $\Theta(a) \subset \Theta(a)$. □

Question 3.8. Let $a \in \mathbb{N}_2$ and $b \in \mathbb{N}$ be such that $P(a,b)$ is nowhere dense in $\mathbb{N}$. Does it follows that $\Theta(a) \subset \Theta(b)$?

The next theorem and its corollary are proved in [1, Theorem 5.12 and Corollary 5.14]. The equivalence between assertions (3) and (4) of the next result was first shown in [9, Theorem 3.3].

Theorem 3.9. *For $a,b \in \mathbb{N}$, the following assertions are equivalent:*

(1) *$P(a,b)$ is totally Brown in $(\mathbb{N}, \tau_G)$;*
(2) *$P(a,b)$ is Brown in $(\mathbb{N}, \tau_G)$;*
(3) *$P(a,b)$ is connected in $(\mathbb{N}, \tau_G)$;*
(4) *$\Theta(a) \subset \Theta(b)$.*

Corollary 3.10. *If $a,b \in \mathbb{N}$, then $P(a,b)$ is totally separated if and only if $\Theta(a) \not\subset \Theta(b)$.*

Now, as the main theorem of this section, we characterize all the arithmetic progressions in $\mathbb{N}$ which are aposyndetic in $(\mathbb{N}, \tau_G)$. It turns out that those are precisely the ones which are either connected, or Brown, or totally Brown.

Theorem 3.11. *For every $a,b \in \mathbb{N}$ the arithmetic progression $P(a,b)$ is aposyndetic if and only if $\Theta(a) \subset \Theta(b)$.*

Proof: Assume first that $P(a,b)$ is aposyndetic. If $\Theta(a) \not\subset \Theta(b)$ then, by Corollary 3.10, $P(a,b)$ is totally separated in $(\mathbb{N}, \tau_G)$. In particular, $P(a,b)$ is hereditarily disconnected, so it does not contain nondegenerate connected subsets. However, since $P(a,b)$ is aposyndetic, there exists a closed and connected subset M of $P(a,b)$ such that $b \in \mathrm{int}_{P(a,b)}(M)$ and $a+b \notin M$. Take $c \in \mathbb{N}$ so that $\langle b,c \rangle = 1$ and

$$b \in P(c,b) \cap P(a,b) \subset \mathrm{int}_{P(a,b)}(M) \subset M \subset P(a,b).$$

By (4), $P(c,b) \cap P(a,b) = P([c,a],b)$, so the connected set M is infinite. This implies that $P(a,b)$ contains a nondegenerate connected subset, a contradiction. Hence, $\Theta(a) \subset \Theta(b)$.

Now assume that $\Theta(a) \subset \Theta(b)$. By the equivalence between (1) and (4) of Theorem 3.9, $P(a,b)$ is totally Brown in $(\mathbb{N}, \tau_G)$. Since $(\mathbb{N}, \tau_G)$ is T_2 and this property is hereditary, $P(a,b)$ is totally Brown and T_2 so, by Theorem 1.3, $P(a,b)$ is aposyndetic. □

The argument presented in the first part of the proof of Theorem 3.11, can be used to show the following result.

Theorem 3.12. *Let $a,b \in \mathbb{N}$. If $P(a,b)$ is aposyndetic, then it contains no isolated points.*

PROOF: Take $c \in P(a,b)$. Since $P(a,b)$ is infinite, there exists $d \in P(a,b)$ with $c \neq d$. Using that $P(a,b)$ is aposyndetic, there is a closed and connected subset M of $P(a,b)$ such that $c \in \mathrm{int}_{P(a,b)}(M)$ and $d \notin M$. Take $e \in \mathbb{N}$ so that $\langle e, c\rangle = 1$ and

$$c \in P(e,c) \cap P(a,b) \subset \mathrm{int}_{P(a,b)}(M).$$

By (5), $P(e,c) \cap P(a,b) = P([e,a], z)$ for $z = \min(P(e,c) \cap P(a,b))$. This implies that the open set $P(e,c) \cap P(a,b)$ in $P(a,b)$ that contains c, is infinite. Hence c is not an isolated point of $P(a,b)$. □

By Theorems 3.9 and 3.11, for an arithmetic progression $P(a,b)$ with $\Theta(a) \subset \Theta(b)$, being totally Brown, being Brown, being connected, and being aposyndetic are all equivalent. Hence we have the following result, that generalizes Theorem 3.9.

Theorem 3.13. *For $a, b \in \mathbb{N}$, the following assertions are equivalent:*

(1) *$P(a,b)$ is totally Brown in $(\mathbb{N}, \tau_G)$;*
(2) *$P(a,b)$ is Brown in $(\mathbb{N}, \tau_G)$;*
(3) *$P(a,b)$ is connected in $(\mathbb{N}, \tau_G)$;*
(4) *$P(a,b)$ is aposyndetic in $(\mathbb{N}, \tau_G)$;*
(5) *$\Theta(a) \subset \Theta(b)$.*

4. The Kirch space

In [7], A. M. Kirch considered the family

$$\mathcal{B}_K = \{P(a,b) \in \mathcal{B}_G \colon a \text{ is square-free}\},$$

which is a base of a topology τ_K on $\mathbb{N}$ so that $\tau_K \subset \tau_G$, and showed that the topological space $(\mathbb{N}, \tau_K)$ is connected, locally connected and T_2. In [8, page 82, Examples 60 and 61] this topology is called the prime integer topology. However we call τ_K the *Kirch topology* and refer to the topological space $(\mathbb{N}, \tau_K)$ as the *Kirch space.* Clearly

$$\tau_K = \{\emptyset\} \cup \{U \subset \mathbb{N} \colon \text{for each } b \in U \text{ there exists } a \in \mathbb{N}_2 \text{ such that } P(a,b) \in \mathcal{B}_K \text{ and } P(a,b) \subset U\}.$$

In this section we present new properties of the Kirch space, that complement the ones presented in [1, Section 6]. The following result is proved in [1, Theorem 6.2].

Theorem 4.1. *If $b \in \mathbb{N}$ and $p \in \mathbb{P}$, then*

$$\mathrm{cl}_{(\mathbb{N},\tau_K)}(P(p^n,b)) = M(p) \cup [P_F(p,b) \cap \mathbb{N}] \qquad \text{for each } n \in \mathbb{N}. \tag{14}$$

In particular,

$$\mathrm{cl}_{(\mathbb{N},\tau_K)}(P(p^n,b)) = \mathrm{cl}_{(\mathbb{N},\tau_K)}(P(p,b)) \qquad \text{for every } n \in \mathbb{N}. \tag{15}$$

Recall that for each $p \in \mathbb{P}$ the set $M(p)$ is closed in $(\mathbb{N},\tau_G)$. We show that in $(\mathbb{N},\tau_K)$ this result as well as Theorem 3.3 remain true in $(\mathbb{N},\tau_K)$.

Theorem 4.2. *For each $p \in \mathbb{P}$ the set $M(p)$ is closed in $(\mathbb{N},\tau_K)$ and, for every $a \in \mathbb{N}_2$, the set $M(a)$ has empty interior in $(\mathbb{N},\tau_K)$.*

Proof: Let $p \in \mathbb{P}$ and $a \in \mathbb{N}_2$. By (14) we have

$$\begin{aligned}\mathrm{cl}_{(\mathbb{N},\tau_K)}(M(p)) &= \mathrm{cl}_{(\mathbb{N},\tau_K)}(P(p,p)) = M(p) \cup [P_F(p,p) \cap \mathbb{N}] \\ &= M(p) \cup P(p,p) = M(p).\end{aligned}$$

This shows that $M(p)$ is closed in $(\mathbb{N},\tau_K)$. The same proof of Theorem 3.3 can be applied to show that $M(a)$ has empty interior in $(\mathbb{N},\tau_K)$. $\square$

Now we show that (10) is valid in $(\mathbb{N},\tau_K)$.

Theorem 4.3. *If $p \in \mathbb{P}$, then*

$$\mathrm{cl}_{(\mathbb{N},\tau_K)}(M(p^n)) = M(p) \qquad \text{for each } n \in \mathbb{N}. \tag{16}$$

Proof: Let $p \in \mathbb{P}$ and $n \in \mathbb{N}$. By (14), (15) and the fact that $M(p)$ is closed in $(\mathbb{N},\tau_K)$, we have

$$\begin{aligned}\mathrm{cl}_{(\mathbb{N},\tau_K)}(M(p^n)) &= \mathrm{cl}_{(\mathbb{N},\tau_K)}(P(p^n,p^n)) = \mathrm{cl}_{(\mathbb{N},\tau_K)}(P(p,p)) \\ &= \mathrm{cl}_{(\mathbb{N},\tau_K)}(M(p)) = M(p).\end{aligned}$$

$\square$

The following result is proved in [1, Theorem 6.3].

Theorem 4.4. *If $a_1, b_1, a_2, b_2, \ldots, a_k, b_k \in \mathbb{N}$ satisfy that $\bigcap_{i=1}^{k} P(a_i,b_i) \neq \emptyset$, then*

$$\mathrm{cl}_{(\mathbb{N},\tau_K)}\left(\bigcap_{i=1}^{k} P(a_i,b_i)\right) = \bigcap_{i=1}^{k} \mathrm{cl}_{(\mathbb{N},\tau_K)}(P(a_i,b_i)). \tag{17}$$

Equality (17) also holds in $(\mathbb{N},\tau_G)$ as proved in [1, Theorem 5.9]. Now we show that (13) is valid in $(\mathbb{N},\tau_K)$.

Theorem 4.5. *Let $a \in \mathbb{N}_2$ and assume that $a = \prod_{i=1}^{k} p_i^{\alpha_i}$ is the standard prime decomposition of a. If $q = p_1 p_2 \cdots p_k$, then $\mathrm{cl}_{(\mathbb{N},\tau_K)}(M(a)) = M(q)$.*

Proof: By (6), (16), and (17) we have

$$\begin{aligned}\mathrm{cl}_{(\mathbb{N},\tau_K)}(M(a)) &= \mathrm{cl}_{(\mathbb{N},\tau_K)}\left(\bigcap_{i=1}^{k} M(p_i^{\alpha_i})\right) = \bigcap_{i=1}^{k} \mathrm{cl}_{(\mathbb{N},\tau_K)}(M(p_i^{\alpha_i})) \\ &= \bigcap_{i=1}^{k} M(p_i) = M(q).\end{aligned}$$

□

By Theorem 4.5 it follows that Corollary 3.6 is valid in $(\mathbb{N}, \tau_K)$ and its proof is the same. Hence we have the following result.

Corollary 4.6. *If $a \in \mathbb{N}_2$, then $M(a)$ is closed in $(\mathbb{N}, \tau_K)$ if and only if a is square-free.*

Since $M(1) = P(1,1) = \mathbb{N}$ is closed in $(\mathbb{N}, \tau_K)$, by this and Corollary 4.6, the set $M(a)$ is closed in $(\mathbb{N}, \tau_K)$ if and only if either $a = 1$ or a is square-free.

Now we show that Theorem 3.5 is valid in $(\mathbb{N}, \tau_K)$.

Theorem 4.7. *Let $a \in \mathbb{N}_2$ and $b \in \mathbb{N}$ be such that $\Theta(a) \subset \Theta(b)$. If $a = \prod_{i=1}^{k} p_i^{\alpha_i}$ is the standard prime decomposition of a and $q = p_1 p_2 \cdots p_k$, then*

$$\mathrm{cl}_{(\mathbb{N},\tau_K)}(P(a,b)) = M(q). \tag{18}$$

Proof: Since q is square-free, by Corollary 4.6 the set $M(q)$ is closed in $(\mathbb{N}, \tau_K)$. The inclusion $\Theta(a) \subset \Theta(b)$ implies that $q|a$ and $q|b$, so $P(a,b) \subset M(q)$. Taking closures in $(\mathbb{N}, \tau_K)$, we have

$$\mathrm{cl}_{(\mathbb{N},\tau_K)}(P(a,b)) \subset \mathrm{cl}_{(\mathbb{N},\tau_K)}(M(q)) = M(q).$$

Now, by (12) and the fact that $\tau_K \subset \tau_G$, we have

$$M(q) = \mathrm{cl}_{(\mathbb{N},\tau_G)}(P(a,b)) \subset \mathrm{cl}_{(\mathbb{N},\tau_K)}(P(a,b)).$$

This shows (18). □

The same proof presented in Theorem 3.7 can be applied in $(\mathbb{N}, \tau_K)$, to obtain the following result.

Theorem 4.8. *If $a \in \mathbb{N}_2$ and $b \in \mathbb{N}$ are such that $\Theta(a) \subset \Theta(b)$, then $P(a,b)$ and $M(a)$ are nowhere dense in $\mathbb{N}$.*

The next result is proved in [1, Theorem 6.9].

Theorem 4.9. *If $a, b \in \mathbb{N}$, then $P(a,b)$ is totally Brown in $(\mathbb{N}, \tau_K)$.*

We end this paper with the following theorem.

Theorem 4.10. *If $a, b \in \mathbb{N}$, then $P(a,b)$ is aposyndetic in $(\mathbb{N}, \tau_K)$.*

Proof: By Theorem 4.9, $P(a,b)$ is totally Brown in $(\mathbb{N}, \tau_K)$. Since $(\mathbb{N}, \tau_K)$ is T_2 and this property is hereditary, $P(a,b)$ is totally Brown and T_2 so, by Theorem 1.3, $P(a,b)$ is aposyndetic. □

Since $\mathbb{N} = M(1) = P(1,1)$, by Theorems 3.11 and 4.10 both $(\mathbb{N}, \tau_G)$ and $(\mathbb{N}, \tau_K)$ are aposyndetic, as it was pointed in [1, Corollary 5.13] and [1, page 214], respectively.

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References

[1] Alberto-Domínguez J. D. C., Acosta G., Delgadillo-Piñón G., *Totally Brown subsets of the Golomb space and the Kirch space*, Comment. Math. Univ. Carolin. **63** (2022), no. 2, 189–219.

[2] Alberto-Domínguez J. D. C., Acosta G., Madriz-Mendoza M., *The common division topology on* $\mathbb{N}$, Comment. Math. Univ. Carolin. **63** (2022), no. 3, 329–349.

[3] Banakh T., Mioduszewski J., Turek S., *On continuous self-maps and homeomorphisms of the Golomb space*, Comment. Math. Univ. Carolin. **59** (2018), no. 4, 423–442.

[4] Engelking R., *General Topology*, Sigma Ser. Pure Math., 6, Heldermann Verlag, Berlin, 1989.

[5] Golomb S. W., *A connected topology for the integers*, Amer. Math. Monthly **66** (1959), 663–665.

[6] Golomb S. W., *Arithmetica topologica*, General Topology and Its Relations to Modern Analysis and Algebra, Proc. Sympos., Prague, 1961, Academic Press, New York, 1961, pages 179–186 (Italian).

[7] Kirch A. M., *A countable, connected, locally connected Hausdorff space*, Amer. Math. Monthly **76** (1969), 169–171.

[8] Steen L. A., Seebach J. A., Jr., *Counterexamples in Topology*, Dover Publications, Mineola, New York, 1995.

[9] Szczuka P., *The connectedness of arithmetic progressions in Furstenberg's, Golomb's, and Kirch's topologies*, Demonstratio Math. **43** (2010), no. 4, 899–909.

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