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Products of topological spaces and families of filters

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Abstract. We show that, under suitably general formulations, covering properties, accumulation properties and filter convergence are all equivalent notions. This general correspondence is exemplified in the study of products.

We prove that a product is Lindelöf if and only if all subproducts by $\leq \omega_1$ factors are Lindelöf. Parallel results are obtained for final ω_n-compactness, $[\lambda, \mu]$-compactness, the Menger and the Rothberger properties.

Keywords: filter convergence; ultrafilter; product; subproduct; sequential compactness; sequencewise $\mathcal{P}$-compactness; Lindelöf property; final λ-compactness; $[\mu, \lambda]$-compactness; Menger property; Rothberger property

Classification: 54A20, 54B10, 54D20

1. Introduction

All sections of the paper are mostly self-contained. The reader interested only in products of Lindelöf spaces or, more generally, finally ω_n-compact or $[\mu, \lambda]$-compact spaces might skip to Section 4 and turn back when needed. Results about the Menger and Rothberger properties are presented in Section 5. In Section 2 we provide a characterization theorem which shows that, under suitably general formulations, covering properties, accumulation properties and filter convergence are all sides of the same coin.

The main theme of the present note is the following: given a property P of topological spaces, find some cardinal κ such that a product satisfies P if and only if all subproducts by $\leq \kappa$ factors satisfy P. We believe that the problem is best seen in terms of the general context of compactness with respect to a set of filters, as introduced in [15], [17], though in many particular cases we get better results by direct means. We are going to briefly review the general notion here.

The notion of filter and ultrafilter convergence plays a key role in the study of products of topological spaces; see the surveys by R. M. Stephenson in [24],

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J. E. Vaughan in [27], S. García-Ferreira and L. Kočinac in [9], and further references there and in [16], [17]. In [13] A. P. Kombarov introduced a local notion of ultrafilter convergence, where, by "local", we mean that the ultrafilter depends on the sequence intended to converge, rather than being fixed in advance. A. P. Kombarov put in a general setting the idea by J. Ginsburg and V. Saks in [11] that countable compactness has an equivalent formulation in this "local" fashion, but cannot be defined by ultrafilter convergence in a "strict" sense (that is, in terms of a single fixed ultrafilter).

In [17] we extended Kombarov notion to filters, and showed that this is a proper generalization, since, for example, sequential compactness can be characterized in terms of such a "local" filter convergence, but all the filters involved, in this case, are necessarily not maximal, see [17, Section 5]. We called this general notion *sequencewise $\mathcal{P}$-compactness* (here and in what follows $\mathcal{P}$ is always a family of filters over some set I) and in [17] we mentioned that sequencewise $\mathcal{P}$-compactness incorporates many compactness, covering and convergence properties, including sequential compactness, countable compactness, initial κ-compactness, $[\lambda, \mu]$-compactness and the Menger and Rothberger properties.

In fact, the definition of sequencewise $\mathcal{P}$-compactness already appeared hidden and in different terminology in a remark contained in [15], together with equivalent formulations. In Section 2, after recalling the relevant definitions together with some examples, we state the result in full as a theorem, with some details of the proof. Then in Section 3 we show that a product of topological spaces is sequencewise $\mathcal{P}$-compact if and only if so is any subproduct with $\leq |\mathcal{P}|$ factors (Theorem 3.1). This unifies many former results by W. W. Comfort, J. Ginsburg, V. Saks, C. T. Scarborough, A. H. Stone and possibly others.

In the subsequent sections we apply Theorem 3.1 to many particular cases, usually getting better bounds by additional methods. In details, in Section 4 we find optimal values in the case both of final ω_n-compactness and of $[\omega_n, \lambda]$-compactness for λ singular strong limit of cofinality ω_n. See Theorems 4.1 and 4.3. On the other hand, the values obtained for the general case of $[\mu, \lambda]$-compactness are essentially those given by Theorem 3.1. See Corollary 4.10. There are obstacles to extending, say, Theorem 4.1, which deals with final ω_n-compactness, to cardinals $\geq \omega_\omega$; this is briefly hinted at the end of the section.

Section 5 is concerned with the Menger and Rothberger properties and their versions for countable covers.

Throughout the paper we shall assume no separation axiom. In order to avoid trivial exceptions, all topological spaces under consideration are assumed to be nonempty. In all the theorems concerning products, if not otherwise mentioned, repetitions are allowed, that is, the same space might occur multiple times as a factor (in other words, we are dealing with products of sequences, not with products of sets).

2. Equivalents of sequencewise $\mathcal{P}$-compactness

We first recall the basic definitions. We refer to [15], [17] for further motivations, examples and references.

If X is a topological space, I is a set, $(x_i)_{i\in I}$ is an I-indexed sequence of elements of X and F is a filter over I, a point $x \in X$ is an *F-limit point* of the sequence $(x_i)_{i\in I}$ if $\{i \in I \colon x_i \in U\} \in F$ for every open neighborhood U of x. If this is the case, we shall also say that $(x_i)_{i\in I}$ *F-converges* to x. Notice that, in general, unless the Hausdorff separation axiom is assumed, such an x is not necessarily unique. Space X is *F-compact* if every I-indexed sequence of elements of X F-converges to some point of X.

Definition 2.1. If $\mathcal{P}$ is a family of filters over the same set I, a topological space X is *sequencewise $\mathcal{P}$-compact* if for every I-indexed sequence of elements of X, there is $F \in \mathcal{P}$ such that the sequence has an F-limit point.

As observed in [17], sequencewise $\mathcal{P}$-compactness generalizes former notions introduced by A. P. Kombarov in [13] and S. García-Ferreira in [8] under different names. Considering filters which are not necessarily ultra provides a substantial generalization, as shown in the next remark.

Remark 2.2. A sequence $(x_n)_{n\in\omega}$ converges if and only if it has an F-limit point for the Fréchet filter F over ω. This shows that sequential compactness is equivalent to sequencewise $\mathcal{P}$-compactness for an appropriate $\mathcal{P}$. Just take $\mathcal{P} = \{F_Z \colon Z \in [\omega]^\omega\}$, where $F_Z = \{W \subseteq \omega \colon Z \setminus W \text{ is finite}\}$ and $[\omega]^\omega$ denotes the set of all infinite subsets of ω. See [17] for more details.

We are now going to see that sequencewise $\mathcal{P}$-compactness admits equivalent formulations (a result implicit in [15]), but first we need some definitions.

Definition 2.3. Let A be a set, and $B, G \subseteq \mathbf{P}(A)$, where $\mathbf{P}(A)$ denotes the set of all subsets of A. A topological space X is *$[B,G]$-compact* if, whenever $(O_a)_{a\in A}$ is a sequence of open sets of X such that $(O_a)_{a\in K}$ is a cover of X for every $K \in G$, then there is $H \in B$ such that $(O_a)_{a\in H}$ is a cover of X.

Covering properties like compactness and countable compactness, which involve just one "starting" cover, can be expressed as particular cases of Definition 2.3 by taking $G = \{A\}$. For example, to get *countable compactness* take A countable, $G = \{A\}$ and $B = [A]^{<\omega}$, the set of all finite subsets of A.

It is useful to consider the general case in which G contains more than one set. The reason is that in this way we can also get covering properties which involve simultaneously many “starting” covers, as is the case for the Menger and Rothberger properties. For example, take $A = \omega$, G a partition of ω into infinitely many infinite classes, and B the family of those sets that intersect each member of G in a finite (one-element, respectively) set. In this case we get the *Menger* (*Rothberger*, respectively) *property for countable covers*. More generally, if λ, μ are cardinals, take $A = \lambda \cdot \mu$, G a partition of A into λ-many pieces of cardinality μ, and B the family of those sets that intersect each member of G in a set of cardinality $< \kappa$. Then we get the property that any λ-sequence of open covers of size $\leq \mu$ admits a $<\kappa$-selection, a property denoted by $R(\lambda, \mu; <\kappa)$ in [16]. Clearly, the Menger and Rothberger properties can be obtained from $R(\omega, \lambda; <\omega)$, $R(\omega, \lambda; <2)$, respectively, by letting λ be arbitrarily large.

Definition 2.4. Suppose that I is a set, $E \subseteq \mathbf{P}(I)$, $\mathcal{E}$ is a set of subsets of $\mathbf{P}(I)$ and X is a topological space.

If $(x_i)_{i \in I}$ is a sequence of elements of X, we say that $x \in X$ is an *E-accumulation point* of $(x_i)_{i \in I}$ if $\{i \in I \colon x_i \in U\} \in E$ for every open neighborhood U of x in X.

We say that $x \in X$ is an *$\mathcal{E}$-accumulation point* of $(x_i)_{i \in I}$ if and only if there is $E \in \mathcal{E}$ such that x is an E-accumulation point of $(x_i)_{i \in I}$.

We say that X satisfies the *$\mathcal{E}$-accumulation property* (the *E-accumulation property*) if every I-indexed sequence of elements of X has some $\mathcal{E}$-accumulation point (some E-accumulation point).

Remark 2.5. In the particular case when E is a filter, E-accumulation points are exactly E-limit points; hence in this case E-compactness is the same as the E-accumulation property. So, if each member of $\mathcal{E}$ is a filter, then the $\mathcal{E}$-accumulation property is the same as sequencewise $\mathcal{E}$-compactness.

Remark 2.6. Countable compactness is another motivating example for our definition of the $\mathcal{E}$-accumulation property. Indeed, a topological space is countably compact if and only if it satisfies the E-accumulation property, with E being the set of all infinite subsets of ω. But countable compactness is also equivalent to sequencewise $\mathcal{P}$-compactness, for the family $\mathcal{P}$ of all uniform ultrafilters over ω. The equivalent formulations of countable compactness can be seen as a prototypical example of the general equivalence given by Theorem 2.7 below. See [15, Remark 2.5] for a full discussion.

The next theorem is implicit in [15]. We give details for the reader's convenience.

Theorem 2.7. *For every class $\mathcal{K}$ of topological spaces, the following conditions are equivalent:*

(i) $\mathcal{K}$ *is the class of all* $[B,G]$*-compact spaces for some set* A *and sets* $B,G \subseteq \mathbf{P}(A)$.

(ii) $\mathcal{K}$ *is the class of all the spaces satisfying the* $\mathcal{E}$*-accumulation property for some set* I *and some family* $\mathcal{E}$ *of subsets of* $\mathbf{P}(I)$ *such that each member of* $\mathcal{E}$ *is closed under supersets.*

(iii) $\mathcal{K}$ *is the class of all sequencewise* $\mathcal{P}$*-compact spaces for some* $\mathcal{P}$.

Given a class $\mathcal{K}$ *and* B, G *satisfying* (i), *there is* $\mathcal{E}$ *such that* $|\mathcal{E}| \leq |G|$ *and* (ii) *is satisfied. Conversely, if* (ii) *is satisfied for some* $\mathcal{K}$ *and* $\mathcal{E}$, *there are* B *and* G *such that* (i) *is satisfied and* $|G| \leq |\mathcal{E}|$. *On the other hand, there are a class* $\mathcal{K}$ *and some* $\mathcal{E}$ *such that* (ii) *holds, but for any* $\mathcal{P}$ *satisfying* (iii) *we have* $|\mathcal{P}| > |\mathcal{E}|$.

Before proving Theorem 2.7 we need some lemmas which may be of independent interest.

Lemma 2.8 ([15, page 300]). *A space* X *is* $[B,G]$*-compact if and only if for every sequence* $(C_a)_{a\in A}$ *of closed sets of* X, *if* $\bigcap_{a\in H} C_a \neq \emptyset$, *for every* $H \in B$, *then there is* $K \in G$ *such that* $\bigcap_{a\in K} C_a \neq \emptyset$.

Proof: By stating the implication in the definition of $[B,G]$-compactness in contrapositive form and taking complements. □

Lemma 2.9. *Given* A,B,G *as in the definition of* $[B,G]$*-compactness, let* $I = B$ *and* $\mathcal{E} = \{E_K \colon K \in G\}$ *where for* $K \in G$ *we set* $E_K = \{Z \subseteq B\colon$ *for every* $a \in K$, *there is* $H \in Z$ *such that* $a \in H\} = \{Z \subseteq B\colon \bigcup_{H\in Z} H \supseteq K\}$.

Under the above definitions, $[B,G]$*-compactness is equivalent to the* $\mathcal{E}$*-accumulation property.*

Proof: Assume that X satisfies the $\mathcal{E}$-accumulation property, for $\mathcal{E}$ as in the statement of the lemma, and assume that $(C_a)_{a\in A}$ is a sequence of closed sets such that $\bigcap_{a\in H} C_a \neq \emptyset$ for every $H \in B$. For every $H \in B$ pick $x_H \in \bigcap_{a\in H} C_a$. By the $\mathcal{E}$-accumulation property, the sequence $(x_H)_{H\in B}$ has an E_K-accumulation point x for some $K \in G$ (recall that $I = B$). Thus, for every neighborhood U of x, $\{H \in B\colon x_H \in U\} \in E_K$, that is for every $a \in K$, there is some H such that $x_H \in U$ and $a \in H$. If $a \in H$, then $x_H \in C_a$, by construction; thus every neighborhood of x intersects C_a, hence $x \in C_a$, since C_a is closed. This holds for every $a \in K$, hence $x \in \bigcap_{a\in K} C_a$, thus $\bigcap_{a\in K} C_a \neq \emptyset$. This implies $[B,G]$-compactness, by Lemma 2.8.

Conversely, assume that X is $[B,G]$-compact and let $(x_H)_{H\in B}$ be a sequence of elements in X. For $a \in A$, let $C_a = \overline{\{x_H : a \in H\}}$, thus $x_H \in \bigcap_{a\in H} C_a$ for every $H \in B$. In particular, $\bigcap_{a\in H} C_a \neq \emptyset$. By Lemma 2.8, $\bigcap_{a\in K} C_a \neq \emptyset$ for some $K \in G$. Let $x \in \bigcap_{a\in K} C_a$. Since $C_a = \overline{\{x_H : a \in H\}}$, then for every $a \in K$ and every neighborhood U of x, there is some H such that $a \in H$ and $x_H \in U$. This means that for every neighborhood U of x, $\bigcup\{H\colon x_H \in U\} \supseteq K$, that is, x is an E_K-accumulation point of $(x_H)_{H\in B}$, in particular, an $\mathcal{E}$-accumulation point.

Compare the above proof with [15, Theorem 5.8 (1) $\Rightarrow$ (5)]. □

If $E \subseteq \mathbf{P}(I)$, we say that E is *closed under supersets (in I)* if whenever $e \in E$ and $e \subseteq f \subseteq I$, then $f \in E$. We let $E^+_I = \{a \subseteq I\colon a \cap e \neq \emptyset$ for every $e \in E\}$. Usually, the set I will be clear from the context and reference to it shall be dropped. Notice that, in case E is a filter, then E^+ is the complement in $\mathcal{P}(I)$ of the dual ideal of E. This observation justifies the notation. Symmetrically, if $\mathcal{P}(I) \setminus E$ is an ideal, then E^+ is the filter dual to this ideal.

Lemma 2.10. *For every $E \subseteq \mathbf{P}(I)$, we have that E^+ is closed under supersets. Moreover, $E^{++} = E$ if and only if E is closed under supersets.*

Proof: The first statement is immediate from the definition. In particular, E^{++} is closed under supersets, hence if $E^{++} = E$, then E is closed under supersets.

To prove the converse, $E^{++} \supseteq E$ is immediate from the definition. Suppose by contradiction that $E^{++} \supsetneq E$ and E is closed under supersets, thus there is $f \in E^{++} \setminus E$ such that $f \cap a \neq \emptyset$ for every $a \in E^+$. Since E is closed under supersets and $f \notin E$, then for every $e \in E$, there is some $i_e \in e \setminus f$. Then, by construction, $a = \{i_e\colon e \in E\} \in E^+$, but $a \cap f = \emptyset$, a contradiction. □

Lemma 2.11 ([15, Proposition 3.11])**.** *Suppose that X is a topological space, $x \in X$, I is a set, and $(x_i)_{i\in I}$ is a sequence of elements of X. Suppose that $K \subseteq \mathcal{P}(I)$, $E = K^+$, and for $a \in K$, put $D_a = \overline{\{x_i\colon i \in a\}}$.*

Then the following conditions are equivalent:

(1) *x is an E-accumulation point of $(x_i)_{i\in I}$.*
(2) *$x \in \bigcap_{a\in K} D_a$.*

Proof: If (1) holds and $a \in K$, then for every neighborhood U of x, $e_U = \{i \in I\colon x_i \in U\} \in E$, thus $a \cap e_U \neq \emptyset$, by the definition of E. If $i \in a \cap e_U$, then $x_i \in D_a \cap U$, hence $D_a \cap U \neq \emptyset$. Since D_a is closed, and $D_a \cap U \neq \emptyset$ for every neighborhood U of x, then $x \in D_a$. Since a was arbitrary in the above argument, we have $x \in \bigcap_{a\in K} D_a$.

If (2) holds and U is a neighborhood of x, let $e_U = \{i \in I\colon x_i \in U\}$. For every $a \in K$, by (2), $x \in D_a$ and, by the definition of D_a, there is $i \in a$ such that $x_i \in U$. By the definition of e_U, $i \in e_U$, thus $i \in e_U \cap a \neq \emptyset$. Thus $e_U \in E = K^+$ for every neighborhood of x, and this means that x is an E-accumulation point of $(x_i)_{i\in I}$. □

Lemma 2.12. *Suppose that I is a set, $\mathcal{E}$ is a set of subsets of $\mathbf{P}(I)$ and every $E \in \mathcal{E}$ is closed under supersets. Let $A = \mathbf{P}(I)$, $G = \{E^+ \colon E \in \mathcal{E}\}$ and $B = \{i^< \colon i \in I\}$, where for $i \in I$, $i^< = \{a \in A \colon i \in a\}$.*

Then, for every topological space X, the following conditions are equivalent:

(1) *X satisfies the $\mathcal{E}$-accumulation property.*
(2) *X is $[B, G]$-compact.*

Proof: (1) $\Rightarrow$ (2) Using Lemma 2.8, suppose that $(C_a)_{a\in A}$ are closed sets and $\bigcap_{a\in H} C_a \neq \emptyset$ for every $H \in B$. Because of the definition of B, this means $\bigcap\{C_a \colon i \in a\} \neq \emptyset$ for every $i \in I$. For each $i \in I$, choose $x_i \in \bigcap\{C_a \colon i \in a\}$. By the $\mathcal{E}$-accumulation property, there are $E \in \mathcal{E}$ and $x \in X$ such that x is an E-accumulation point of $(x_i)_{i\in I}$. If $K = E^+$, then $E = E^{++} = K^+$, by Lemma 2.10 and since E is closed under supersets, by assumption. If $D_a = \overline{\{x_i \colon i \in a\}}$ for $a \in K$, then by Lemma 2.11 $x \in \bigcap_{a\in K} D_a \subseteq \bigcap_{a\in K} C_a$. Since $K = E^+ \in G$, this shows that X is $[B, G]$-compact.

(2) $\Rightarrow$ (1) Suppose that $(x_i)_{i\in I}$ is a sequence, and set $C_a = \overline{\{x_i \colon i \in a\}}$ for $a \in A$. If $H \in B$, say, $H = i^<$, then $x_i \in \bigcap_{a\in H} C_a$, hence, by $[B, G]$-compactness and Lemma 2.8, there is $K \in G$ such that $\bigcap_{a\in K} C_a \neq \emptyset$. By the definition of G, $K = E^+$ for some $E \in \mathcal{E}$, hence $E = E^{++} = K^+$, by Lemma 2.10, since E is closed under supersets. Then $(x_i)_{i\in I}$ has an E-accumulation point, by Lemma 2.11. This proves the $\mathcal{E}$-accumulation property. □

Proof of Theorem 2.7: (i) $\Rightarrow$ (ii) follows from Lemma 2.9, noticing that the E_Ks defined there are closed under supersets.

(ii) $\Rightarrow$ (i) follows from Lemma 2.12.

(ii) $\Rightarrow$ (iii) If E is closed under supersets, x is an E-accumulation point of the sequence $(x_i)_{i\in I}$ and for every neighborhood U of x we set $x_U = \{i \in I \colon x_i \in U\}$, then the family $\{x_U \colon U$ a neighborhood of $x\}$ generates a filter E' which is contained in E, and x is an E'-accumulation point of the sequence $(x_i)_{i\in I}$.

Thus if each $E \in \mathcal{E}$ is closed under supersets, then the $\mathcal{E}$-accumulation property is equivalent to the $\mathcal{E}'$-accumulation property, where $\mathcal{E}'$ is the set of all filters which are contained in some element of $\mathcal{E}$. Then $\mathcal{E}'$ contains only filters, and the $\mathcal{E}'$-accumulation property is the same as sequencewise $\mathcal{E}'$-compactness, by Remark 2.5. The implication (iii) $\Rightarrow$ (ii) is trivial from the same remark.

Again by Lemmas 2.9 and 2.12, the sets $\mathcal{E}$ and G can be chosen to satisfy the cardinality requirements. On the other hand, as mentioned, countable compactness can be characterized as the $\mathcal{E}$-accumulation property for some one-element $\mathcal{E}$, but countable compactness is *not* equivalent to sequencewise $\mathcal{P}$-compactness for any one-element $\mathcal{P}$. Indeed, this would mean F-compactness for the single filter F belonging to $\mathcal{P}$, but it is well known that F-compactness is preserved under products, while countable compactness is not. ☐

Corollary 2.13 ([15, Corollaries 3.5 and 3.10]). *For every class $\mathcal{K}$ of topological spaces, the following conditions are equivalent:*

(i) *$\mathcal{K}$ is the class of all $[B, \{A\}]$-compact spaces for some A and $B \subseteq \mathbf{P}(A)$.*
(ii) *$\mathcal{K}$ is the class of all the spaces satisfying the E-accumulation property for some set I and some $E \subseteq \mathbf{P}(I)$ closed under supersets.*

At first sight one could be tempted to believe that condition (iii) in Theorem 2.7 is always preferable to condition (ii), since $\mathcal{P}$ in (iii) contains only filters, which are surely more manageable subsets of $\mathbf{P}(I)$ than the members of $\mathcal{E}$ in (ii), which are only supposed to be closed under supersets. However the last statement in Theorem 2.7 shows that there are cases in which the $\mathcal{E}$ in (ii) has the advantage of having much smaller cardinality than $\mathcal{P}$.

The proof of Theorem 2.7 gives explicit constructions, which are of some use even in particular cases. For example, the proof of Theorem 2.7 (i) $\Rightarrow$ (ii) can be used to express the Menger properties as some kind of accumulation properties, as we explicitly worked out in [16, Lemma 2.2 (3)]. See also [15, Corollary 5.13], which, however, is stated in nonstandard terminology: there we used the expressions "Menger property" ("Rothberger property", respectively), in place of their versions for countable covers, that is, $R(\omega, \omega; {<}\omega)$ ($R(\omega, \omega; {<}2)$, respectively). Then in [16, Theorem 2.3] the Menger properties are explicitly described as sequencewise $\mathcal{P}$-compactness for some appropriate $\mathcal{P}$ consisting only of ultrafilters. That the Menger properties can be described as sequencewise $\mathcal{P}$-compactness for some $\mathcal{P}$, follows directly from Theorem 2.7; the main point in [16] is that the members of $\mathcal{P}$ can be chosen to be ultrafilters; this follows also abstractly from [17, Corollary 5.3].

In the other direction, the proof of Theorem 2.7 (ii) $\Rightarrow$ (i) can be used to provide alternative formulations in terms of open covers both of D-compactness [15, Proposition 1.3], stated here as Proposition 4.5, and of sequential compactness [15, Corollary 5.12]. See [15, Corollaries 2.6, 3.14 and 5.15] for further results of this kind and [15, Section 4 and Theorems 5.9 and 5.11] for further theorems dealing with pseudocompact-like generalizations.

We do not know whether the technical assumption that the members of $\mathcal{E}$ are closed under supersets is necessary in condition (ii) in Theorem 2.7, namely, whether, for every $\mathcal{E}$, there is some $\mathcal{E}'$ such that the $\mathcal{E}$-accumulation property is equivalent to the $\mathcal{E}'$-accumulation property and all members of $\mathcal{E}'$ are closed under supersets.

3. Checking compactness by means of subproducts

Recall the definition of sequencewise $\mathcal{P}$-compactness from Definition 2.1. If $\prod_{j\in J} X_j$ is a product of topological spaces, a *subproduct* is a space of the form $\prod_{j\in K} X_j$ for some $K \subseteq J$. Formally, if $K = \emptyset$, the corresponding subproduct is a one-element space (hence it satisfies all reasonable compactness properties). Otherwise, the reader might always exclude the case of subproducts with respect to an empty index set.

Theorem 3.1. *Let $\mathcal{P}$ be a nonempty family of filters over some set I. A product of topological spaces is sequencewise $\mathcal{P}$-compact if and only if so is any subproduct with $\leq |\mathcal{P}|$ factors.*

Proof: The only if part is immediate from the observation that sequencewise $\mathcal{P}$-compactness is preserved under continuous surjective images.

For the other direction, by contraposition, suppose that $X = \prod_{j\in J} X_j$ is not sequencewise $\mathcal{P}$-compact, thus there is a sequence $(x_i)_{i\in I}$ of elements of X such that for no $F \in \mathcal{P}$, $(x_i)_{i\in I}$ F-converges in X. Notice that a sequence in a product $\prod_{j\in J} X_j$ of topological spaces F-converges if and only if for every $j \in J$ the projection of the sequence into X_j F-converges in X_j. Hence, for every $F \in \mathcal{P}$ there is some $j_F \in J$ such that the projection of $(x_i)_{i\in I}$ into X_{j_F} does not F-converge in X_{j_F}. Choose one such j_F for each $F \in \mathcal{P}$, and let $K = \{j_F : F \in \mathcal{P}\}$, thus $|K| \leq |\mathcal{P}|$.

Let $X' = \prod_{j\in K} X_j$, and let $(x'_i)_{i\in I}$ be the natural projection of $(x_i)_{i\in I}$ into X'. We claim that the sequence $(x'_i)_{i\in I}$ witnesses that X' is not sequencewise $\mathcal{P}$-compact. Indeed, for every $F \in \mathcal{P}$ we have that $(x'_i)_{i\in I}$ does not F-converge in X', since the projection of $(x'_i)_{i\in I}$ into X_{j_F} (which is the same as the projection of $(x_i)_{i\in I}$ into X_{j_F}) does not F-converge in X_{j_F}. Thus we have found a subproduct with $\leq |\mathcal{P}|$ factors which is not sequencewise $\mathcal{P}$-compact. □

Remark 3.2. Notice that the particular case $\mathcal{P} = \{F\}$ of Theorem 3.1 states that a product is F-compact if and only if each factor is F-compact (however, this does not follow from Theorem 3.1, since it is used in the proof). Thus Theorem 3.1 incorporates Tychonoff theorem, since a topological space is compact if and only if it is D-compact for every ultrafilter D.

Apparently, besides Tychonoff theorem, the first result of the form of Theorem 3.1 has been proved by C. T. Scarborough and A. H. Stone in [23, Theorem 5.6], asserting that a product is countably compact, provided that all subproducts by at most $2^{2^{\mathfrak{c}}}$ factors are countably compact. C. T. Scarborough and A. H. Stone in [23, Corollary 5.7] also obtained the improved value $2^{2^{\omega}}$ for the particular case of first countable factors. J. Ginsburg and V. Saks in [11, Theorem 2.6] then obtained the improved bound $2^{2^{\omega}}$ for powers of a single space, and

W. W. Comfort in [6] and V. Saks in [22] observed that the methods from [11] give the result for arbitrary factors, a result which is a particular case of Theorem 3.1, by Remark 2.6 and since there are 2^{2^ω} nonprincipal ultrafilters over ω.

V. Saks in [22, Theorem 2.3] also proved that a product satisfies CAP_λ if and only if each subproduct by $\leq 2^{2^\lambda}$ factors satisfies it; actually, he stated the result in terms of an interval of cardinals and in different terminology. Recall that a topological space is said to satisfy CAP_λ if every subset Y of cardinality λ has a *complete accumulation point*, that is, a point each neighborhood of which intersects Y in a set of cardinality λ. Saks' result, too, can be obtained as a consequence of Theorem 3.1, but some care should be taken of the case when λ is singular.

Concerning a related property, X. Caicedo in [5, Section 3] essentially gave, in the present terminology, a characterization of $[\mu, \lambda]$-compactness as sequencewise $\mathcal{P}$-compactness for an appropriate $\mathcal{P}$. This will be recalled in Theorem 4.4 below. Theorem 3.1 can then be applied in order to provide a characterization of those products which are $[\mu, \lambda]$-compact. We shall work this out in Corollary 4.10. In the particular cases of final ω_n-compactness and of $[\omega_n, \lambda]$-compactness for λ singular strong limit, better results can be obtained using further arguments, as we will show in Theorems 4.1 and 4.3.

Other possible examples of applications of Theorem 3.1 deal with the Menger, the Rothberger and the related properties mentioned after Definition 2.3. However, in this case, too, best results about these properties are obtained by direct means: see [16] and also Section 5 here. A similar situation occurs with regard to sequential compactness. See [18] and Section 6 below.

Notice that the equivalence of conditions (i) and (ii) in [17, Theorem 2.1] can be obtained as an immediate consequence of Theorem 3.1.

Let us remark that Theorem 3.1 stresses the importance of studying the problem when sequencewise $\mathcal{P}$-compactness is equivalent to sequencewise $\mathcal{P}'$-compactness for various sets $\mathcal{P}$ and $\mathcal{P}'$, as already mentioned in [17]. In particular, given $\mathcal{P}$, Theorem 3.1 implies that it is useful to characterize the minimal cardinality of some $\mathcal{P}'$ such that the above equivalence holds. The cardinality of such a "minimal" $\mathcal{P}'$ is also connected with some other invariants. See Section 7 in [19], an unpublished manuscript from which the present work has been extracted.

Let F be the trivial filter over κ, that is, $F = \{\kappa\}$. Then a topological space X is F-compact if and only if for every subset Y of X of cardinality $\leq \kappa$, there is $x \in X$ such that every neighborhood of x contains *the whole of* Y. Such spaces are called κ^+*-filtered.* See [4] for further details and characterizations. Trivially, if $\mathcal{P}$ is a nonempty family of filters over κ, then any κ^+-filtered space is sequencewise $\mathcal{P}$-compact. The next lemma is trivial, but it has some use, see the proof of Proposition 5.1.

Lemma 3.3. *If $\mathcal{P}$ is a family of filters over κ and X_1 is a κ^+-filtered topological space, then a product $X_1 \times X_2$ is sequencewise $\mathcal{P}$-compact if and only if X_2 is sequencewise $\mathcal{P}$-compact.*

Proof: The "only if" part is trivial.

For the other direction, suppose that X_1 is κ^+-filtered and X_2 is sequencewise $\mathcal{P}$-compact. Let $(x_\alpha)_{\alpha\in\kappa}$ be a sequence of elements of $X_1 \times X_2$. Since X_2 is sequencewise $\mathcal{P}$-compact, there is $F \in \mathcal{P}$ such that the second projection of $(x_\alpha)_{\alpha\in\kappa}$ F-converges in X_2. Since X_1 is κ^+-filtered, the first projection of $(x_\alpha)_{\alpha\in\kappa}$ F-converges in X_1, hence $(x_\alpha)_{\alpha\in\kappa}$ F-converges in $X_1 \times X_2$. □

A more significant result shall be proved in Corollary 4.7, where the assumption of being κ^+-filtered shall be replaced by initial 2^κ-compactness, provided that all members of $\mathcal{P}$ are ultrafilters.

4. Final μ-compactness and $[\mu, \lambda]$-compactness

Final ω_n-compactness. In this section we present some generalizations of the following theorem, which can be obtained as consequence of results from [14] (it just needs a small elaboration besides [14, Corollary 33]). Recall that a topological space is *finally μ-compact* if every open cover has a subcover of cardinality $< \mu$.

In what follows we shall freely use the categorical properties of products and, in case there is no risk of confusion, we shall identify, say, $\prod_{j\in J} Y_j$ with $\prod_{j\in H} Y_j \times \prod_{j\in J\setminus H} Y_j$ for $H \subseteq J$.

Theorem 4.1. *If X is a product of topological spaces, then the following conditions are equivalent:*

(i) *X is finally ω_n-compact.*
(ii) *All subproducts of X by $\le \omega_n$ factors are finally ω_n-compact.*
(iii) *All but $< \omega_n$ factors of X are compact, and the product of the non compact factors if any is finally ω_n-compact.*
(iv) *The product of the non compact factors (if any) is finally ω_n-compact.*

Proof: (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are trivial.

(ii) $\Rightarrow$ (iii) If $n = 0$, this is immediate since compactness is final ω_0 compactness. If $n > 0$, suppose by contradiction that there are (at least) ω_n factors which are not compact. Theorem 2 in [14] (in contrapositive form) asserts that their product is not finally ω_n-compact, contradicting (ii). Hence all but $< \omega_n$ factors of X are compact, and the product of the remaining factors is finally ω_n-compact, by (ii).

(iv) $\Rightarrow$ (i) Letting apart the trivial improper cases, group together the compact factors, on one hand, and the non compact factors, on the other hand. Then we get by Tychonoff theorem that X is (homeomorphic to) a product of a compact space with a finally ω_n-compact space, and a standard argument shows that any such product is finally ω_n-compact (anyway, a more general result shall be proved in Corollary 4.8 below). $\square$

Since Lindelöfness is the same as final ω_1-compactness, we get the following corollary which might be known, though we know no reference for it.

Corollary 4.2. *A product is (linearly) Lindelöf if and only if all subproducts by $\leq \omega_1$ factors are (linearly) Lindelöf if and only if all but countably many factors are compact and the product of the non compact factors, if any, is (linearly) Lindelöf.*

Recall that a topological space is *linearly Lindelöf* if every open cover which is linearly ordered by inclusion has a countable subcover (some authors use the term *chain-Lindelöf*). The linear Lindelöf case of Corollary 4.2 follows from [14, Theorem 3], arguing as in the proof of Theorem 4.1.

$[\omega_n, \lambda]$-compactness. We now combine the arguments in Theorem 4.1 with some classical methods from R. M. Stephenson and J. E. Vaughan in [25] in order to get a similar characterization of $[\omega_n, \lambda]$-compact products for λ a singular strong limit cardinal having cofinality $\geq \omega_n$. Recall that a topological space X is $[\mu, \lambda]$-*compact* if every open cover by at most λ sets has a subcover of cardinality $< \mu$. *Initial λ-compactness* is $[\omega, \lambda]$-compactness.

Theorem 4.3. *Suppose that $n \in \omega$, λ is a singular strong limit cardinal, and $\operatorname{cf} \lambda \geq \omega_n$. If X is a product of topological spaces, then the following conditions are equivalent:*

- (i) *X is $[\omega_n, \lambda]$-compact.*
- (ii) *Every subproduct of X by $\leq \omega_n$ factors is $[\omega_n, \lambda]$-compact.*
- (iii) *All but $< \omega_n$ factors of X are initially λ-compact, and the product of the non initially λ-compact factors (if any) is $[\omega_n, \lambda]$-compact.*
- (iv) *The product of the non initially λ-compact factors (if any) is $[\omega_n, \lambda]$-compact.*

Some auxiliary results are needed before we can give the proof of Theorem 4.3. By $[\lambda]^{<\mu}$ we denote the set of all subsets of λ of cardinality $< \mu$. This is now a quite standard notation, but notice that some authors (including the present one) sometimes used alternative notations for this, such as $S_\mu(\lambda)$, $\mathcal{P}_{<\mu}(\lambda)$ and other. We say that an ultrafilter D over $[\lambda]^{<\mu}$ *covers* λ in case $\{Z \in [\lambda]^{<\mu} : \alpha \in Z\} \in D$ for every $\alpha \in \lambda$.

Theorem 4.4 (X. Caicedo [5]). *A topological space is $[\mu,\lambda]$-compact if and only if it is sequencewise $\mathcal{P}$-compact for the family $\mathcal{P}$ of the ultrafilters over $[\lambda]^{<\mu}$ which cover λ.*

If λ is regular, then a topological space is $[\lambda,\lambda]$-compact if and only if it is sequencewise $\mathcal{P}$-compact for the family $\mathcal{P}$ of the uniform ultrafilters over λ.

Theorem 4.4 is essentially proved in [5, Section 3]. Full details for the first statement can be found in [16, Theorem 2.3], considering the particular case $\lambda = 1$ therein: see the remark at the bottom of [16, page 2509]. The second statement is much simpler; actually, it is a reformulation of some remarks from [22], in particular, (i) on pages 80–81 therein. Notice, that for λ regular $[\lambda,\lambda]$-compactness is equivalent to CAP_λ, $C[\lambda,\lambda]$ in Saks' notation.

Proposition 4.5. *If D is an ultrafilter over I, then a topological space X is D-compact if and only if for every open cover $(O_Z)_{Z\in D}$ of X, there is some $i \in I$ such that $(O_Z)_{i\in Z\in D}$ is a cover of X.*

See [15, Proposition 1.3 and Remark 3.12] for a proof of Proposition 4.5.

Corollary 4.6. *If X is an initially λ-compact topological space and $2^\kappa \leq \lambda$, then X is D-compact for every ultrafilter D over some set of cardinality $\leq \kappa$.*

Proof: We use Proposition 4.5. Let $(O_Z)_{Z\in D}$ be an open cover of X. Since $|D| \leq 2^\kappa \leq \lambda$, then by initial λ-compactness $(O_Z)_{Z\in D}$ has a finite subcover, say $O_{Z_1},\dots,O_{Z_m}$. Since D is (in particular) a filter, $Z_1 \cap \dots \cap Z_m \neq \emptyset$. If $i \in Z_1 \cap \dots \cap Z_m$, then $(O_Z)_{i\in Z\in D}$ is a cover of X. By Proposition 4.5, X is D-compact. □

Corollary 4.6 can also be obtained as a consequence of implications (8) and (5) in [24, Diagram 3.6], exchanging λ and κ.

Corollary 4.7. *Suppose that $2^\kappa \leq \lambda$ and $\mathcal{P}$ is a family of ultrafilters over some set I of cardinality $\leq \kappa$. Then the product of an initially λ-compact and of a sequencewise $\mathcal{P}$-compact topological space is sequencewise $\mathcal{P}$-compact.*

Proof: Let X_1 be initially λ-compact, X_2 be sequencewise $\mathcal{P}$-compact, and let $(x_i)_{i\in I}$ be a sequence in $X_1 \times X_2$. By the sequencewise $\mathcal{P}$-compactness of X_2, there is some $D \in \mathcal{P}$ such that the second projection of $(x_i)_{i\in I}$ D-converges in X_2. Since $2^\kappa \leq \lambda$, the first projection of $(x_i)_{i\in I}$ D-converges in X_1, by Corollary 4.6. Hence $(x_i)_{i\in I}$ D-converges in $X_1 \times X_2$, thus $X_1 \times X_2$ is sequencewise $\mathcal{P}$-compact. □

By $\nu^{<\mu}$ we denote $\sup_{\mu'<\mu} \nu^{\mu'}$. Notice that $[\nu]^{<\mu}$ has cardinality $\nu^{<\mu}$.

Corollary 4.8. *If $2^{\nu^{<\mu}} \leq \lambda$, then the product $X_1 \times X_2$ of a $[\mu, \nu]$-compact space X_1 and an initially λ-compact space X_2 is $[\mu, \nu]$-compact.*

If the interval $[\mu, \nu]$ consists only of regular cardinals, the assumption $2^{\nu^{<\mu}} \leq \lambda$ above can be relaxed to $2^\nu \leq \lambda$.

Proof: By Theorem 4.4, $[\mu, \nu]$-compactness is equivalent to sequencewise $\mathcal{P}$-compactness for a family $\mathcal{P}$ of ultrafilters over $[\nu]^{<\mu}$, a set of cardinality $\nu^{<\mu}$. Hence the first statement is immediate from Corollary 4.7 with $\kappa = \nu^{<\mu}$.

To prove the last statement, recall that $[\mu, \nu]$-compactness is equivalent to $[\mu', \mu']$-compactness for every μ' such that $\mu \leq \mu' \leq \nu$. From the second statement in Theorem 4.4, and applying again Corollary 4.7, we get that $X_1 \times X_2$ is $[\mu', \mu']$-compact for every μ' as above, since $2^{\mu'} \leq 2^\nu \leq \lambda$. Hence $X_1 \times X_2$ is $[\mu, \nu]$-compact. □

Proof of Theorem 4.3: (i) ⇒ (ii) and (iii) ⇒ (iv) are trivial.

(ii) ⇒ (iii) The case $n = 0$ is immediate, since $[\omega_0, \lambda]$-compactness is the same as initial λ-compactness, hence, assuming (ii), all factors are initial λ-compact. If $n > 0$, suppose by contradiction that there are $\geq \omega_n$ factors which are not initially λ-compact. By (ii), each such factor is $[\omega_n, \lambda]$-compact, hence not initially ω_{n-1}-compact, otherwise it would be initially λ-compact. Hence we have at least ω_n factors which are not initially ω_{n-1}-compact, and, by [14, Theorem 6], their product is not $[\omega_n, \omega_n]$-compact, hence not $[\omega_n, \lambda]$-compact, contradicting (ii).

Hence the set of factors which are not initially λ-compact has cardinality $< \omega_n$, and their product is $[\omega_n, \lambda]$-compact by (ii).

(iv) ⇒ (i) By Stephenson and Vaughan's theorem, see [25, Theorem 1.1], the product of the initially λ-compact factors, if any, is still initially λ-compact. By (iv), the product of the non initially λ-compact factors, if any, is $[\omega_n, \lambda]$-compact. Hence, excluding the improper cases, X is (homeomorphic to) the product of an initially λ-compact space with an $[\omega_n, \lambda]$-compact one. By Corollary 4.8, and since λ is strong limit, then for every $\nu < \lambda$, X is $[\omega_n, \nu]$-compact. Since $\lambda > \operatorname{cf} \lambda \geq \omega_n$, then X is $[\operatorname{cf} \lambda, \operatorname{cf} \lambda]$-compact. Then X is $[\omega_n, \lambda]$-compact, by the well-known fact that $[\omega_n, \nu]$-compactness for every $\nu < \lambda$, together with $[\operatorname{cf} \lambda, \operatorname{cf} \lambda]$-compactness imply $[\omega_n, \lambda]$-compactness. □

$[\mu, \lambda]$**-compactness.**

Remark 4.9. Certain values obtained in Theorems 4.1 and 4.3 are much better than the values which could be obtained by a simple direct application of Theorems 3.1 and 4.4. For example, if λ is a singular strong limit cardinal, and $\operatorname{cf} \lambda \geq \omega_n$, then there are $\kappa = 2^{2^\lambda}$ ultrafilters over $\lambda = \lambda^{\omega_n}$. Then Theorems 3.1

and 4.4 imply that some product X is $[\omega_n, \lambda]$-compact if and only if all subproducts of X by $\leq \kappa$ factors are finally $[\omega_n, \lambda]$-compact. However, Theorem 4.3 shows that the value of κ can be improved to ω_n. See the next subsection for related comments.

In the more general case of arbitrary μ and λ, we have the following corollary of Theorem 4.4, a corollary in which we essentially get the values given by Theorem 3.1, sometimes with minor improvements.

Corollary 4.10. *A product of topological spaces is $[\mu, \lambda]$-compact if and only if so is any subproduct by $\leq 2^{2^\kappa}$ factors, where $\kappa = \lambda^{<\mu}$. The value of κ can be improved to $\kappa = \lambda$ in case the interval $[\mu, \lambda]$ contains only regular cardinals.*

More generally, a product is $[\mu, \lambda]$-compact if and only if so is any subproduct by $< \theta$ factors, where θ is the smallest cardinal such that both

(a) $\theta > 2^{2^\nu}$ *for every regular ν such that $\mu \leq \nu \leq \lambda$, and*

(b) $\theta > 2^{2^{\nu^{<\mu}}}$ *for every singular ν of cofinality $< \mu$ such that $\mu \leq \nu \leq \lambda$.*

Proof: The first two statements are immediate from Theorems 3.1 and 4.4, since there are 2^{2^κ} ultrafilters over $[\lambda]^{<\mu}$, 2^{2^ν}, respectively, ultrafilters over ν. Here ν varies among the cardinals such that $\mu \leq \nu \leq \lambda$, and we are using again the mentioned fact that $[\mu, \lambda]$-compactness is equivalent to $[\nu, \nu]$-compactness for every ν such that $\mu \leq \nu \leq \lambda$.

In order to prove the last statement, recall that for every ν, $[\mathrm{cf}\,\nu, \mathrm{cf}\,\nu]$-compactness implies $[\nu, \nu]$-compactness. Using this property, together with the fact mentioned at the end of the previous paragraph, it is easy to see that $[\mu, \lambda]$-compactness is equivalent to the conjunction of

(i) $[\nu, \nu]$-compactness for every regular ν with $\mu \leq \nu \leq \lambda$, and

(ii) $[\mu, \nu]$-compactness for every singular ν of cofinality $< \mu$ and such that $\mu \leq \nu \leq \lambda$.

Now we get the result by applying for each ν, the corresponding (and already proved) statements in the first paragraph of the corollary (with ν in place of λ). $\square$

Remark 4.11. Corollary 4.10 complements [22, Theorem 2.3], which asserts that a product satisfies CAP_ν for every $\nu \in [\mu, \lambda]$ if and only if so does every subproduct by $\leq 2^{2^\lambda}$ factors. Notice that if the interval $[\mu, \lambda]$ contains only regular cardinals, then [22, Theorem 2.3] and Corollary 4.10 overlap, since it is well-known that, if ν is a regular infinite cardinal, then CAP_ν and $[\nu, \nu]$-compactness are equivalent notions.

Short remarks about final μ-compactness for arbitrary μ. Under special set-theoretical assumptions, we know improvements of all the results proved in the present section. However, we cannot go exceedingly far. Of course, the equivalence of (i) and (iv) both in Theorem 4.1 and in Theorem 4.3 holds for every infinite cardinal in place of ω_n. However, the other equivalences do not necessarily remain true, when ω_n is replaced by some larger cardinal.

For example, if κ is a strongly compact cardinal, then every power of ω with the discrete topology is finally κ-compact. This is a consequence of a classical result by I. Mycielski in [20], asserting that if κ is strongly compact, then every product of finally κ-compact spaces is still finally κ-compact. This can be obtained also from Theorem 4.4 together with the ultrafilter characterization of strong compactness. Thus if κ is strongly compact, then the analogue of Theorem 4.1 (i) $\Rightarrow$ (iii) with κ in place of ω_n badly fails, since every power of ω is finally κ-compact, but ω is not compact.

Concerning condition 4.1 (ii), first define for every infinite cardinal μ the cardinal $\mathfrak{s}(P_\mu)$ as the smallest cardinal, if it exists, such that some product is finally μ-compact if and only if so is every subproduct by $< \mathfrak{s}(P_\mu)$ factors. Here P_μ is intended to be the property of being finally μ-compact, as we want the notation to be consistent with the general one we have introduced in [19, Section 7]. With this terminology, clearly $\mathfrak{s}(P_\omega) = 2$, as a reformulation of Tychonoff theorem. Moreover, Theorem 4.1 (i) $\Leftrightarrow$ (ii) implies that if $n > 0$, then $\mathfrak{s}(P_{\omega_n}) = \omega_{n+1}$. Indeed, $\omega_{n-1}^{\omega_{n-1}}$ is finally ω_n-compact, but not every power of ω_{n-1} is, hence the value given by Theorem 4.1 cannot be improved. Contrary to the case of ω_n, we know examples in which, under certain set theoretical constraints, $\mathfrak{s}(P_\mu)$ is far larger than μ. Full details shall be presented elsewhere, since they involve deep set theoretical problems. On the other hand, Mycielski's theorem mentioned above implies that if κ is a strongly compact cardinal, then $\mathfrak{s}(P_\kappa) = 2$.

We also remark that a characterization of Lindelöf products in terms of *factors*, rather than *subproducts* must necessarily involve deep structural properties of the factors. Even the product of two Lindelöf spaces may turn out to be very incompact. Moreover, it is consistent that there are three regular Lindelöf spaces whose product has very large Lindelöf number, while every pairwise product of two of them is still Lindelöf, see [26]. On the other hand, under a weak set-theoretical assumption, sequentially compact products can be characterized in terms of factors, see Theorem 6.1 (ii) below.

Notice that, if we apply Theorem 4.4 to final μ-compactness, we get a proper class $\mathcal{P}$, since final μ-compactness is equivalent to $[\mu, \lambda]$-compactness for every $\lambda \geq \mu$, alternatively, equivalent to $[\lambda, \lambda]$-compactness for every $\lambda \geq \mu$. However, we can take good advantage of the theorem by Mycielski mentioned above, in order to find bounds for $\mathfrak{s}(P_\mu)$, when μ is smaller than some strongly compact cardinal.

Proposition 4.12. *Suppose that μ is an infinite cardinal, $\mu \leq \theta$ and θ is strongly compact. If X is a product of topological spaces, then the following conditions are equivalent:*

(i) *X is finally μ-compact.*

(ii) *All subproducts of X by $< \theta$ factors are finally μ-compact.*

Proof: Suppose that (ii) holds; in particular, all factors are finally θ-compact, since $\mu \leq \theta$. By Mycielski theorem, X is finally θ-compact. By Corollary 4.10 for every λ with $\mu \leq \lambda < \theta$, X is $[\lambda, \lambda]$-compact, since strongly compact cardinals are inaccessible. Hence X is finally μ-compact. $\square$

5. Menger and Rothberger

Recall that a topological space X satisfies the *Rothberger property* (the *Rothberger property for countable covers*, respectively) if given a countable family of open covers (of countable open covers, respectively) of X, one can obtain another cover of X by selecting an open set from each one of the given covers. We get the *Menger property* when we allow to select a finite number of open sets from each cover. Recall that we are not assuming any separation axiom.

Recall that a topological space is *κ-filtered* if for every subset Y of X of cardinality $< \kappa$, there is $x \in X$ such that every neighborhood of x contains Y. In the terminology from [4], a topological space is *supercompact* if it is κ-filtered for all κ. Equivalently, a space X is supercompact if and only if there is a point whose only neighborhood is the whole of X if and only if X is $[2, \infty]$-compact. Here *$[2, \infty]$-compact* is a shorthand for $[2, \lambda]$-compact for every cardinal λ.

Notice that if a product of T_1 spaces is Rothberger, then all but finitely many spaces are one-element. Indeed, a T_1 space with more than one element contains a closed copy of the two-element discrete topological space $\mathbf{2}$, and $\mathbf{2}^\omega$ is not Rothberger. Hence most results in the present section are significant only in the (quite exotic) context of spaces satisfying little or no separation axiom. We present the results since the proofs need very little special efforts and, on the other hand, they might be of some interest due to renewed interest in spaces satisfying few separation axioms, for example, in connection with the specialization (pre)order, which becomes trivial for T_1 spaces, and because of significant applications to theoretical computer science. See, e.g., [10] and [12]. Compare also [28]. See [21] for an interesting recent manifesto in support of the study of spaces satisfying lower separation axioms from a purely topological point of view.

Proposition 5.1. *If X is a product of topological spaces, then the following conditions are equivalent:*

(i) *X satisfies the Rothberger property.*
(ii) *Every subproduct of X by countably many factors satisfies the Rothberger property.*
(iii) *All but a finite number of factors of X are supercompact, and the product of the non supercompact factors (if any) satisfies the Rothberger property.*
(iv) *The product of the non supercompact factors of X (if any) satisfies the Rothberger property.*

Proof: (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are trivial (as will be the case for all the corresponding implications throughout the present section).

(ii) $\Rightarrow$ (iii) If by contradiction there is an infinite number of factors which are not supercompact, i.e., not $[2, \infty]$-compact, then their product is not Rothberger, by [16, Proposition 3.1]. Hence the number of factors which are not supercompact is finite, and their product is Rothberger by (ii).

(iv) $\Rightarrow$ (i) As a particular case of Theorem 2.7 (i) $\Leftrightarrow$ (iii) we have that for every λ the Rothberger property for covers of cardinality $\leq \lambda$ is equivalent to sequencewise $\mathcal{P}$-compactness for some $\mathcal{P}$ (an explicit description of such a $\mathcal{P}$ can be found in [16, Proposition 4.1]). Thus, by Lemma 3.3, and since any product of supercompact spaces is supercompact, we get for every λ that X satisfies the Rothberger property for covers of cardinality $\leq \lambda$. This means exactly that X satisfies the Rothberger property. □

Proposition 5.2. *If X is a product of topological spaces, then the following conditions are equivalent:*

(i) *X satisfies the Rothberger property for countable covers.*
(ii) *Every subproduct of X by countably many factors satisfies the Rothberger property for countable covers.*
(iii) *In every factor of X every sequence converges, except possibly for a finite number of factors, and the product of such factors (if any) satisfies the Rothberger property for countable covers.*
(iv) *The product of the factors of X (if any) in which there exists a nonconverging sequence satisfies the Rothberger property for countable covers.*

Proof: (ii) $\Rightarrow$ (iii) It is enough to show that if we are given an infinite number of topological spaces, each with a nonconverging sequence, then their product does not satisfy the Rothberger property for countable covers. By [17, Lemma 4.1 (iv) $\Rightarrow$ (i)], if some topological space Y has a nonconvergent sequence, then Y is not $[2, \omega]$-compact, hence an infinite product of such spaces does not satisfy the Rothberger property for countable covers, by [16, Proposition 3.1].

(iv) $\Rightarrow$ (i) The property that every sequence converges is preserved under products. Hence X is the product of a space in which every sequence converges and of a space satisfying the Rothberger property for countable covers. By Theorem 2.7 (i) $\Leftrightarrow$ (iii), the Rothberger property for countable covers can be characterized as sequencewise $\mathcal{P}$-compactness for some $\mathcal{P}$, and $\mathcal{P}$ can be chosen to consist of filters over ω, by [16, Proposition 4.1], taking $\kappa = 2$ and $\lambda = \mu = \omega$ there. On the other hand, a space is ω_1-filtered if and only if in it every sequence converges. This is proved by S. Brandhorst in [3] or S. Brandhorst and M. Erné in [4, Lemma 5.1], and can be also proved by P. Lipparini in [17, Lemma 4.1]. Since X is the product of a space in which every sequence converges and of a space satisfying the Rothberger property for countable covers, then Lemma 3.3 shows that X satisfies the Rothberger property for countable covers. □

Given any infinite cardinal λ, Proposition 5.2 can be generalized to deal with the *Rothberger property for covers of cardinality* $\leq \lambda$. We leave the generalization to the reader.

Corollary 5.3. *If X is a product of topological spaces, then the following conditions are equivalent:*

(i) *X satisfies the Menger property.*
(ii) *Every subproduct of X by countably many factors satisfies the Menger property.*
(iii) *All but a finite number of factors of X are compact, and the product of the non compact factors (if any) satisfies the Menger property.*
(iv) *The product of the non compact factors of X (if any) satisfies the Menger property.*

Proof: (ii) $\Rightarrow$ (iii) By [16, Proposition 3.1], a product of infinitely many non compact spaces is not Menger, thus if (ii) holds, then there is only a finite number of non compact spaces, and their product is Menger.

(iv) $\Rightarrow$ (i) The product of the compact factors is compact, hence X is the product of a compact space with a Menger space, and any such product is Menger. □

Results related to the present section appear in [16], e.g., Proposition 3.3 and Corollaries 2.5, 3.4 and 4.2 there. We do not know whether results similar to the ones presented in this section can be proved for the Menger property for countable covers. However, it follows from [16, Corollary 2.5] that a product satisfies the Menger property for countable covers if and only if so does any subproduct by $\leq 2^{2^{2^\omega}}$ factors. Again, we do not know whether this is the best possible value. Notice that a better value does work in the case of powers of a single space, or, more generally, in case we consider all possible products of spaces in a given family, see [16, Corollary 3.2].

6. Further remarks

In [18] we give a proof of the following theorem.

Theorem 6.1.

(1) *A product of topological spaces is sequentially compact if and only if all subproducts by $\leq \mathfrak{s}$ factors are sequentially compact.*

(2) *Assume that $\mathfrak{h} = \mathfrak{s}$. If X is a product of topological spaces, then the following conditions are equivalent:*

  (i) *X is sequentially compact.*

  (ii) *All factors of X are sequentially compact, and the set of factors with a nonconverging sequence has cardinality $< \mathfrak{s}$.*

  (iii) *All factors of X are sequentially compact, and all but at most $< \mathfrak{s}$ factors are ultraconnected.*

Recall that a space X is called *ultraconnected* if no pair of nonempty closed sets of X is disjoint. Recall that $\mathfrak{s}$ denotes the *splitting number* and $\mathfrak{h}$ the *distributivity number*, see [1].

The proof of Theorem 6.1 is direct and does not use Theorem 3.1. However, it is interesting to discuss the connections between Theorems 6.1 and 3.1.

The value $\mathfrak{s}$ in Theorem 6.1 (1) is the best possible value, since $\mathfrak{s}$ is the smallest cardinal such that $\mathbf{2}^{\mathfrak{s}}$ is not sequentially compact, see [2], [7]. Here $\mathbf{2}$ is the two-element discrete space.

If in Theorem 6.1 (1) we replace $\mathfrak{s}$ by the rougher estimate $\mathfrak{c}$, then the theorem is indeed a consequence of Theorem 3.1, since, by Remark 2.2, there is some $\mathcal{P}$ of cardinality $\mathfrak{c}$ such that sequential compactness is equivalent to sequencewise $\mathcal{P}$-compactness. As we mentioned in [17, Problem 4.4], we do not know the value of the smallest cardinal $\mathfrak{ms}$ such that sequential compactness is equivalent to sequencewise $\mathcal{P}$-compactness for some $\mathcal{P}$ with $|\mathcal{P}| = \mathfrak{ms}$. Of course, if $\mathfrak{ms}$ were equal to $\mathfrak{s}$, then Theorem 6.1 (1) would be a direct consequence of Theorem 3.1.

It follows from Remark 2.2 that $\mathfrak{ms} \leq \mathfrak{c}$. Moreover, $\mathfrak{ms} \geq \mathfrak{s}$. If, to the contrary, $\mathfrak{ms} < \mathfrak{s}$, then by Theorem 3.1 we could prove Theorem 6.1 (1) for the improved value $\mathfrak{ms}$ in place of $\mathfrak{s}$. However, as we mentioned above, $\mathfrak{s}$ is the best possible value. Also the comment after [17, Problem 4.4] shows, in different terminology, that $\mathfrak{ms} \geq \mathfrak{s}$.

See [18], [19] for further comments and for the definitions of invariants related to $\mathfrak{s}$ and $\mathfrak{h}$ in such a general context as a partial infinitary semigroup with a specified subclass.

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