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MULTI-TYPE SYNCHRONIZATION OF IMPULSIVE COUPLED OSCILLATORS VIA TOPOLOGY DEGREE

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Abstract. The existence of synchronization is an important issue in complex dynamical networks. In this paper, we study the synchronization of impulsive coupled oscillator networks with the aid of rotating periodic solutions of impulsive system. The type of synchronization is closely related to the rotating matrix, which gives an insight for finding various types of synchronization in a united way. We transform the synchronization of impulsive coupled oscillators into the existence of rotating periodic solutions in a relevant impulsive system. Some existence theorems about rotating periodic solutions for a nonhomogeneous linear impulsive system and a nonlinear perturbation system are established by topology degree theory. Finally, we give two examples to show synchronization behaviors in impulsive coupled oscillator networks.

Keywords: synchronization; impulsive coupled oscillator; rotating periodic solution; impulsive system

MSC 2020: 34A37, 34C25, 34D06

1. Introduction

Complex dynamical networks(CDNs) are widely used to describe various phenomena and analyze internal mechanisms in a vast number of natural and artificial systems such as biological systems, communication networks, electric power grids etc. [5], [19], [11], [7]. The research of synchronization can be traced back to the phase synchronization of the pendulum discovered by Huygens [8]. Nowadays, the footprints of synchronization have spread across all fields of science as a result of its wide availability and various applications. Moreover, the research on synchronization phenomena is an important branch in complex dynamical network theory.

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At present, researchers have defined and investigated several types of synchronization in CDNs as needed, including, for example, complete synchronization [3], phase synchronization [17], anti-phase synchronization [9], cluster synchronization [32], lag synchronization [10], [29], and so on. However, united methods or results for various types of synchronization are few. In this paper, we touch this by the means of affine periodic solutions.

Affine periodic solution is a kind of generalized periodic solution of the form $x(t+T) = Ax(t)$, $A \in GL(n, \mathbb{R})$. In nature, many physical phenomena exhibit not only periodicity in time, but also affine symmetry in space such as spiral waves [2]. Many types of periodic solutions belong to affine periodic solutions. For example,

1) $A = I$, I is the identity matrix, affine periodic solutions are pure periodic;
2) $A = -I$, affine periodic solutions are anti-periodic;
3) $A = Q$, Q is an orthogonal matrix, affine periodic solutions are rotating periodic;
4) $A = Q$, $Q^k = I$, k is a positive integer, affine periodic solutions are subharmonic.

In recent years, affine periodic solutions have been concerned and studied extensively because of generality and applicability. The relevant research can be classified roughly by the type of systems: perturbed system [26], [4], discrete system [14], dissipative system [30], impulsive system [24], Hamiltonian system [12], [27], Newtonian system [28], time scale system [20] and references therein.

The research on affine periodic solutions in CDNs composed of coupled oscillators is an interesting topic. Particularly, rotating waves, which can be described by rotating periodic solutions, are usually used to describe the synchronization in coupled oscillator systems. In our previous works [21], [22], [25], some existence theorems of synchronous solutions in coupled oscillator networks were given. Based on the optimization method, we proposed the Gauss-Newton method for finding rotating periodic solutions in coupled oscillator networks [23]. In [31], rotational symmetry in the spatio-temporal structure could be maintained in the reconstruction process by reservoir computing.

In this paper, we consider the synchronization of a dynamical network consisting of N impulsive coupled identical oscillators. The ith node can be described by the coupled differential equation

$$\begin{aligned} (1) \qquad & \dot{x_i} = ax_i + f(x_i) + \varepsilon_1 g_i(x_1, x_2, \dots, x_N), \quad t \neq t_k,\ t \in \mathbb{R}, \\ & \Delta x_i = x_i(t_k^+) - x_i(t_k) = \varepsilon_2 \sum_{j=1, j\neq i}^{N} b_{ij}(t_k)(x_j - x_i) + c_i x_i + h_i(t_k), \\ & \qquad\qquad t = t_k,\ k \in \mathbb{Z},\ i = 1, 2, \dots, N, \end{aligned}$$

where x_i is the state of the ith node, $\dot{x_i} = ax_i + f(x_i)$ and $\Delta x_i = c_i x_i + h_i(t_k)$ are used to describe the dynamics of ith oscillator. The continuous function $g_i(x_1, x_2, \dots, x_N)$

represents the coupling added on the ith node from other nodes at the continuous time. Further, ε_1 is the continuous time coupling strength; ε_2 is the discrete time coupling strength. Here b_{ij} represents the linear connection between nodes at the discrete time, which is defined as follows if there is a connection from node j to node i ($j \neq i$), then $b_{ij} > 0$; otherwise $b_{ij} = 0$. In this impulsive system, the continuous time network connection can be different from the discrete time network connection. The quantity x_i can be a vector or a scalar, here we choose it as a scalar for simplicity. We rewrite the above system as

$$\begin{aligned} \dot{X} &= AX + F(X) + G(X), \quad t \neq t_k,\ t \in \mathbb{R}, \\ \Delta X &= B_k X + h_k, \quad t = t_k,\ k \in \mathbb{Z}. \end{aligned} \tag{2}$$

Here $X{=}(x_1, x_2, \ldots, x_N)^\top$, $A{=}\mathrm{diag}(a, a, \ldots, a)$, $F(X){=}(f(x_1), f(x_2), \ldots, f(x_N))^\top$, $G(X) = (\varepsilon_1 g_1(X), \varepsilon_1 g_2(X), \ldots, \varepsilon_1 g_N(X))^\top$, $B_k = \varepsilon_2(\widehat{b}_{ij}(t_k))_{N\times N}$ (here $\widehat{b}_{ij} = b_{ij}$ if $i \neq j$, $\widehat{b}_{ii} = -\sum\limits_{j=1,j\neq i}^{N} b_{ij} + c_i/\varepsilon_2$), $h_k = (h_1(t_k), h_2(t_k), \ldots, h_N(t_k))^\top$.

In the following, we build the relationship between the type of synchronization and rotating periodic solutions. Some existence theorems of rotating periodic solutions are given for the impulsive differential equation. In addition, on the account of the spatio-temporal structure contained in rotating periodic solutions, the rotating periodic condition is essential for the system (2). Precisely, if the system is described by $\dot{x} = f(t, x)$, $f(t + T, x) = Qf(t, Q^{-1}x)$ is the rotating periodic condition and $Q \in O(n)$ is called the rotating matrix. In coupled oscillator networks, the rotating matrix is usually taken from the elements of the symmetric group S_N. That is, Q is usually an orthogonal matrix that maintains the network invariance after the network exchanges the positions of multiple oscillators.

This paper is structured as follows: Section 2 outlines the relationship between different synchronous solutions and rotating periodic solutions. Section 3 presents an existence theorem and the Massera theorem for rotating periodic solutions in linear non-homogeneous systems. Section 4 uses fundamental matrix and topological degree theory to establish the existence of rotating periodic solutions in corresponding impulsive nonlinear perturbation systems. Finally, we provide examples to illustrate our findings in the last section.

2. THE RELATIONSHIP BETWEEN SYNCHRONIZATION AND ROTATING PERIODIC SOLUTION

In this part, we discuss the relationship between various types of synchronous solutions and rotating periodic solutions. Assuming that the system (2) has periodic synchronous solutions, that is, all oscillators have the same periodic trajectory and the same phase difference (this type of synchronous solution can be easily found in unidirectionally connected ring networks), then there must exist t_1 so that

$$X(t+t_1) = QX(t), \quad \text{where } Q = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

In fact, $Q^N = I$, so we have

$$X(t) = X(t + Nt_1)$$

which means the system has a Nt_1-periodic solution. The equation $X(t + t_1) = QX(t)$ gives one (Q, t_1)-rotating-periodic solution. Since the oscillators are identical, there may be many types of permutation matrices Q corresponding to the periodic synchronous solution of the system. All these permutation matrices belong to the same conjugate class of the symmetric group S_N. We know all these permutation matrices in the same conjugate class are similar, so we can judge the type of the synchronous solution according to their eigenvalues. Similarly, we can obtain the relationship between various types of synchronization and rotating periodic solutions. In fact, when all oscillators are identical, the network is undirected and the coupling strength is the same, then the system (2) has the following property.

Proposition 1. *Assuming that the system* (2) *has a rotating periodic solution* $x(t + T) = Qx(t)$ *for some orthogonal matrix* Q *and the oscillator network remains topology invariant under the action of* Q, *the following two conditions should be imposed:*

1) $G(X){=}QG(Q^{-1}X)$,
2) *for* $k \in \mathbb{Z}$, *there exists a* $p \in \mathbb{N}$ *such that* $B_{k+p} = QB_kQ^{-1}$, $h_{k+p} = Qh_k$ *and* $t_{k+p} = t_k + T$.

Suppose the matrix Q *has the form*

$$\widetilde{Q} = \operatorname{diag}(q_1, q_2, q_3, \dots, q_N)$$

after diagonalization. Then we have:

1) *If* $q_1 = q_2 = \dots = q_N$, *the impulsive coupled oscillator system has completely synchronous solutions.*
2) *If* N *is even*, $q_1 = q_2 = \dots = q_{N/2} = -q_{N/2+1} = -q_{N/2+2} = \dots = -q_N$, *the impulsive coupled oscillator system has anti-phase synchronous solutions;*
3) *If* $q_j = \exp(2j\pi\mathrm{i}/n)$ $(j = 1, 2, \dots, N;$ i *is the imaginary unit), the impulsive coupled oscillator system has periodic synchronous solutions;*

4) *If* q_1,q_2,...,q_N *are divided into several groups, and in the same group they satisfy* $q_{j1} = q_{j2} = \ldots = q_{jm}$ *(here the label* jm *represents the* m*th oscillator in the* j*th group), the system has cluster synchronous solutions.*

Remark 1. The above results concern only the existence of synchronous solutions, not the stability of synchronous solutions. Since all the oscillators are identical, the symmetry of the network directly determines the type of synchronous solutions, and the rotating matrix Q can represent the symmetry of the network. The symmetry of the network determines that every permutation which keeps the network invariance forms some permutation subgroups of the symmetric group S_N. By the conjugate classes of these groups and the diagonalization method, all kinds of synchronous solutions of finite identical oscillator systems can be obtained. More details can be found in [22], [25], [16].

Based on the aforementioned property, the investigation of diverse synchronization issues in impulsive systems can be reframed as the quest for corresponding rotating periodic solutions. Consequently, the primary objective of this paper is to explore the existence of rotating periodic solutions for impulsive systems in a general sense. It should be noted that rotating periodic solutions have broad applications in studying synchronization issues in networks [25], as well as the existence of periodic, anti-periodic, and quasi-periodic solutions in neural networks [6], [13], [15].

The existence of the synchronous solution of the system (2) we try to study can be summarized as the existence of the rotating periodic solution of the nonlinear perturbation system

$$\begin{aligned} \dot{x} &= A(t)x + g(t,x), \quad t \neq t_k,\ t \in \mathbb{R}, \\ \Delta x &= B_k x + h_k, \quad t = t_k,\ k \in \mathbb{Z}. \end{aligned} \tag{3}$$

According to Proposition 1, the system (3) in this paper should satisfy the hypotheses H1:

1) $A(\cdot) \in C(\mathbb{R}, \mathbb{R}^{n\times n})$, $g(\cdot) \in C(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^n)$, $A(t+T) = QA(t)Q^{-1}$, $g(t+T,x) = Qg(t,Q^{-1}x)$ $(t \in \mathbb{R})$ for some $G \in SO_n(\mathbb{R})$.
2) $B_k \in \mathbb{R}^{n\times n}$, $\det(I + B_k) \neq 0$, $h_k \in \mathbb{R}^n$, $t_k < t_{k+1}$ $(k \in \mathbb{Z})$.
3) For $k \in \mathbb{Z}$, there exists a $p \in \mathbb{N}$ such that $B_{k+p} = QB_kQ^{-1}$, $h_{k+p} = Qh_k$ and $t_{k+p} = t_k + T$.

We present the following lemma and it can be seen that the existence of the rotating periodic solution is equivalent to the rotating boundary value problem.

Lemma 1. *The existence of* (Q,T)*-rotating-periodic solutions of the equation* (3) *is equivalent to the existence of the solutions for* (3) *with* $x(T) = Qx(0)$.

Proof. Let the solution $x(t)$ of the equation (3) be defined for $t \in [0,T]$. Then

$$u(t)=\begin{cases} x(t), & t \in (0,T],\\ Q^j x(t-jT), & t \in (jT,(j+1)T],\ j=1,2,\dots, \end{cases}$$

is a (Q,T)-rotating-periodic solution of (3). Indeed, if $t \in (jT, jT+T]$, then $t-jT \in (0,T]$ and

$$\begin{aligned}\frac{\mathrm{d}u(t)}{\mathrm{d}t} &= Q^j\frac{\mathrm{d}x(t-jT)}{\mathrm{d}t} = Q^j[A(t-jT)x(t-jT)+g(t-jT,x(t-jT))]\\ &= Q^j\cdot Q^{-j}A(t)Q^jx(t-jT)+Q^j\cdot Q^{-j}g(t,Q^jx(t-jT))\\ &= A(t)u(t)+g(t,u(t)).\end{aligned}$$

And if $t_k \in (jT, jT+T]$, then $t_{k-jq} = t_k - jT \in (0,T]$ and

$$\begin{aligned}\Delta u(t_k) = Q^j\Delta x(t_k-jT) &= Q^j[B_{k-jq}x(t_k-jT)+h_{k-jq}]\\ &= B_kQ^jx(t_k-jT)+h_k\\ &= B_ku(t_k)+h_k.\end{aligned}\tag{4}$$

Then we have completed the proof. □

3. ROTATING PERIODIC SOLUTIONS OF NON-HOMOGENEOUS LINEAR IMPULSIVE SYSTEMS

In this part, we give some existence results for rotating periodic solutions in non-homogeneous linear impulsive systems. We consider the system

$$\begin{aligned}\dot{x} &= A(t)x+g(t), \quad t \neq t_k,\ t\in\mathbb{R},\\ \Delta x &= B_kx+h_k, \quad t=t_k,\ k\in\mathbb{Z}.\end{aligned}\tag{5}$$

The system satisfies the hypotheses H2:

1) $A(\cdot) \in C(\mathbb{R},\mathbb{R}^{n\times n})$, $A(t+T)=QA(t)Q^{-1}$, $g(t+T)=Qg(t)$ $(t\in\mathbb{R})$ for some $Q\in SO_n(\mathbb{R})$.
2) $B_k \in \mathbb{R}^{n\times n}$, $\det(I+B_k)\neq 0$, $h_k\in\mathbb{R}^n$, $t_k<t_{k+1}$ $(k\in\mathbb{Z})$.
3) For $k\in\mathbb{Z}$, there exists a $p\in\mathbb{N}$ such that $B_{k+p}=QB_kQ^{-1}$, $h_{k+p}=Qh_k$ and $t_{k+p}=t_k+T$.

We investigate the existence of (Q,T)-rotating-periodic solutions of (5). We first introduce the fundamental matrix $X(t)$ of homogeneous linear impulsive differential equations, which can be found in [1], [18]. We assume that $U(t,\tau)$ is the state transition matrix of $\dot{x}=A(t)x$, then the fundamental matrix of the linear homogeneous system is

$$X(t)=U(t,t_j)\prod_{i=j}^{k}(I+B_i)U(t_i,t_{i-1})U(t_k,\tau),\quad t_j<t\leqslant t_{j+1},\ t_{k-1}<\tau\leqslant t_k. \tag{6}$$

Then from the variation of parameters formula, the solution of (5) has the form

$$x(t)=X(t)x(0)+\int_0^t X(t)X^{-1}(s)g(s)\,\mathrm{d}s+\sum_{0\leqslant t_k<t}X(t)X^{-1}(t_k^+)h_k,$$

where $X(t)=W(t,0)$ is the normalized (at $t=0$) fundamental matrix of the equation (5).

The solution $x(t)$ of (5) is (Q,T)-rotating-periodic if $x(T)=Qx(0)$ (see Lemma 1), or if

$$(Q-X(T))x(0)=\int_0^T X(T)X^{-1}(s)g(s)\,\mathrm{d}s+\sum_{0\leqslant t_k<T}X(T)X^{-1}(t_k^+)h_k. \tag{7}$$

3.1. Non-critial case.

$$\det(Q-X(T))\neq 0.$$

At this time the equation (5) has the unique (Q,T)-rotating-periodic solution

$$\begin{aligned}\widetilde{x}(t)&=X(t)[Q-X(T)]^{-1}\Bigl[\int_0^T X(T)X^{-1}(s)g(s)\,\mathrm{d}s+\sum_{0\leqslant t_k<T}X(T)X^{-1}(t_k^+)h_k\Bigr]\\&\quad+\int_0^t X(t)X^{-1}(s)g(s)\,\mathrm{d}s+\sum_{0\leqslant t_k<t}X(t)X^{-1}(t_k^+)h_k.\end{aligned} \tag{8}$$

Remark 2. Here we can define Green's function for (Q,T)-rotating-periodic solutions of the equation (5)

$$G(t,s)=\begin{cases}[X(t)(Q-X(T))^{-1}X(T)X^{-1}(t)+I]X(t)X^{-1}(s), & 0<s<t\leqslant T,\\ X(t)[Q-X(T)]^{-1}X(T)X^{-1}(s), & 0<t\leqslant s\leqslant T,\\ G(t-kT,s-jT),\quad kT<t\leqslant kT+T, & \\ \qquad\qquad jT<s\leqslant jT+T,\ k\in\mathbb{Z},\ j\in\mathbb{Z} & \end{cases}$$

and rewrite $\widetilde{x}(t)$ as

$$\widetilde{x}(t)=\int_0^T G(t,s)g(s)\,\mathrm{d}s+\sum_{0\leqslant t_k<T}G(t,t_k^+)h_k.$$

Let the conditions (H2) and $\det(Q - X(T)) \neq 0$ hold, then the non-homogeneous equation (5) has a unique (Q, T)-rotating-periodic solution $x(t)$ for which the representation (8) holds.

3.2. Critial case.

$$\det(Q - X(T)) = 0.$$

Then the equation (7) may have no solution. Consider the following adjoint equations with respect to the homogeneous linear equation of the system (5):

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= -A^*(t)x, \quad t \neq t_k,\ t \in \mathbb{R}, \\ \Delta x &= -(I + B_k^*)^{-1} B_k^* x, \quad t = t_k,\ k \in \mathbb{Z}, \end{aligned} \tag{9}$$

where $A^*(t)$ and B_k^* are the transposes of $A(t)$ and B_k, respectively. We can give the following existence theorem of (5).

Theorem 1. *Let the conditions* (H2) *hold and let the corresponding homogeneous equation has* m *linearly independent* (Q, T)*-rotating-periodic solutions* $\varphi_1(t), \ldots, \varphi_m(t)$ $(1 < m < n)$*. Then:*

1) *The adjoint equation* (9) *also has* m *linearly independent* (Q^{-1}, T)*-rotating-periodic solutions* $\varphi_1(t), \ldots, \varphi_m(t)$*.*
2) *The equation* (5) *has a* (Q, T)*-rotating-periodic solution if and only if the following conditions are satisfied:*

$$\int_0^T \varphi_j^*(t) g(t)\,\mathrm{d}t + \sum_{0 \leqslant t_k < T} \varphi_j^*(t_k^+) h_k = 0, \quad j = 1, \ldots, m. \tag{10}$$

3) *If the condition* (10) *is satisfied, then each* (Q, T)*-rotating-periodic solution of the equation* (5) *has the form*

$$x(t) = c_1\varphi_1(t) + \cdots + c_m\varphi_m(t) + x_0(t),$$

where $x_0(t)$ *is a particular* (Q, T)*-rotating-periodic solution of* (5)*.*

4) *If the condition* (10) *is satisfied, then the equation* (5) *has a unique* (Q, T)*-rotating-periodic solution* $\widetilde{x}(t)$ *such that*

$$\varphi_i^*(0)\widetilde{x}(0) = 0, \quad i = 1, \ldots, m. \tag{11}$$

The proof of Theorem 1 is similar to the proofs of the corresponding theorem in [27], [21], [22] and the details are omitted.

The existence of rotating periodic solutions of a linear non-homogeneous rotating periodic system is closely related to the existence of bounded solutions. This relationship is established by the following theorem, which generalizes a theorem of Massera for rotating periodic impulsive system.

Theorem 2. *Let the conditions* (H2) *hold. If Q is an orthogonal matrix and the linear non-homogeneous (Q,T)-rotating-periodic equation* (5) *has a bounded solution for $t \geqslant 0$, then this equation has a (Q,T)-rotating-periodic solution.*

Proof. Let $\widetilde{y}(t)$ be a bounded solution of (5) for $t \geqslant 0$. Then for $t \in \mathbb{R}_+$ we have

$$\widetilde{y}(t) = W(t,0)\widetilde{y}(0) + \int_0^t W(t,s)g(s)\,\mathrm{d}s + \sum_{0 \leqslant t_k < t} W(t,t_k^+)h_k,$$

where $W(t,s)$ is the Cauchy matrix for the corresponding homogeneous equation. Hence,

$$\widetilde{y}((n+1)T) = X(T)\widetilde{y}(nT) + b,$$

where $X(T) = W(T,0)$ is a monodromy matrix for the corresponding homogeneous equation and

$$b = \int_0^T W(T,s)g(s)\,\mathrm{d}s + \sum_{0 \leqslant t_k < T} W(T,t_k^+)h_k.$$

Suppose that the equation (5) has no (Q,T)-rotating-periodic solution. Then the linear algebraic equation

$$(Q - X(T))y = b$$

has no solution and there exists $u^* \in \mathbb{R}^n$ such that

$$u^*(Q - X(T)) = 0$$

and

$$u^*b \neq 0.$$

Hence $u^*Q = u^*X(T)$, and we have the equation

$$\begin{aligned} u^*\widetilde{y}((n+1)T) &= u^*X(T)\widetilde{y}(nT) + u^*b \\ &= u^*(X(T) - Q)\widetilde{y}(nT) + u^*Q\widetilde{y}(nT) + u^*b \\ &= u^*Q\widetilde{y}(nT) + u^*b \\ &\;\vdots \\ &= u^*Q^{n+1}\widetilde{y}(0) + (n+1)u^*b. \end{aligned} \tag{12}$$

Since Q is an orthogonal matrix, the equation (12) is a contradiction with the boundedness of $\widetilde{y}(t)$. Hence, the assumption is not true and the equation (5) has at least one (Q,T)-rotating-periodic solution. □

4. ROTATING PERIODIC SOLUTIONS OF NONLINEAR PERTURBATION IMPULSIVE SYSTEMS

To investigate the existence of rotating periodic solutions of the system (3), we consider an auxiliary equation

$$\begin{aligned} \dot{x} &= A(t)x + \lambda g(t,x), \quad t \neq t_k,\ t \in \mathbb{R}, \\ \Delta x &= B_k x + \lambda h_k, \quad t = t_k,\ k \in \mathbb{Z}, \end{aligned} \tag{13}$$

where $\lambda \in [0,1]$.

Then we give an existence theorem for (Q,T)-rotating-periodic solutions by the method of homotopy and degree theory.

Theorem 3. *Let $D \subset \mathbb{R}^n$ be a bounded open set. Assume the following hypotheses hold for the system* (13).

(H$_1$) *For any $\lambda \in [0,1]$, every possible (Q,T)-rotating-periodic solution $x(t)$ of the system* (13) *satisfies*

$$x(t) \notin \partial D \quad \forall\, t.$$

(H$_2$) *The Brouwer degree,*

$$\deg(f, D \cap \operatorname{Ker}(Q - X(T)), 0) \neq 0 \text{ if } \operatorname{Ker}(Q - X(T)) \neq 0,$$

where

$$f(a) = \frac{1}{T} P\biggl[\int_0^T W(t,s) g(s,a)\,\mathrm{d}s + \sum_{0 \leqslant t_k < t} W(t, t_k^+) h_k\biggr],$$

$P\colon \mathbb{R}^n \to \operatorname{Ker}(Q - X(T))$ is an orthogonal projection; $X(t) = W(t,0)$ is the normalized (at $t = 0$) fundamental matrix of the equation (5).

(H$_3$) $\operatorname{Ker}(Q - X(T)) \bigoplus \operatorname{Im}(Q - X(T)) = \mathbb{R}^n$.

Then the system (3) *has at least one (Q,T)-rotating-periodic solution $x_*(t) \in D$ for all t.*

Proof. Consider the auxiliary equation (13) with the boundary value condition $x(T) = Qx(0)$, where $\lambda \in [0,1]$. Let $x(t)$ be any solution of (13) with $x(T) = Qx(0)$. Rewriting (13) in the form of an equivalent integral equation, we obtain

$$\begin{aligned} x(T) &= X(T)x(0) + \lambda\biggl[\int_0^T X(T)X^{-1}(s) g(s, x(s))\,\mathrm{d}s + \sum_{0 \leqslant t_k < T} X(T) X^{-1}(t_k^+) h_k\biggr] \\ &= Qx(0). \end{aligned}$$

Denote $x(0)$ by x_0. Then

$$(Q-X(T))x_0=\lambda\biggl[\int_0^T W(T,s)g(s,x(s))\,\mathrm{d}s+\sum_{0\leqslant t_k<T}W(T,t_k^+)h_k\biggr]. \tag{14}$$

Consider (14) in two parts.

(I) $\operatorname{Ker}(Q-X(T))\neq\{0\}$. In this case, $(Q-X(T))^{-1}$ does not exist. Since $\operatorname{Ker}(Q-X(T))\bigoplus\operatorname{Im}(Q-X(T))=\mathbb{R}^n$, without loss of generality we can just let

$$Q=\begin{pmatrix}Q_1 & 0\\ 0 & Q_2\end{pmatrix}\quad\text{and}\quad X(T)=\begin{pmatrix}Q_1 & 0\\ 0 & X(T)_1\end{pmatrix},$$

where $(Q_2-X(T)_1)^{-1}$ exists.

Let $P\colon\ \mathbb{R}^n\to\operatorname{Ker}(Q-X(T))$ be the orthogonal projection. Then:

$$\begin{aligned}(Q-X(T))x_0&=(Q-X(T))(x^0_{\mathrm{ker}}+x^0_{\perp})\\ &=\lambda\biggl[\int_0^T W(T,s)g(s,x(s))\,\mathrm{d}s+\sum_{0\leqslant t_k<T}W(T,t_k^+)h_k\biggr]\\ &=\lambda P\biggl[\int_0^T W(T,s)g(s,x(s))\,\mathrm{d}s+\sum_{0\leqslant t_k<T}W(T,t_k^+)h_k\biggr]\\ &\quad+\lambda(I-P)\biggl[\int_0^T W(T,s)g(s,x(s))\,\mathrm{d}s+\sum_{0\leqslant t_k<T}W(T,t_k^+)h_k\biggr],\end{aligned}\tag{15}$$

where $x^0_{\mathrm{ker}}\in\operatorname{Ker}(Q-X(T))$, $x^0_{\perp}\in\operatorname{Im}(Q-X(T))$ and $x_0=x^0_{\mathrm{ker}}+x^0_{\perp}$.

Let $L_p=(Q-X(T))\mid_{\operatorname{Im}(Q-X(T))}$. It is easy to see that L_p^{-1} exists. Thus, the equation (15) is equivalent to:

$$\begin{aligned}(Q-X(T))x^0_{\mathrm{ker}}&=\lambda P\biggl[\int_0^T W(T,s)g(s,x(s))\,\mathrm{d}s+\sum_{0\leqslant t_k<T}W(T,t_k^+)h_k\biggr]=0,\\ (Q-X(T))x^0_{\perp}&=\lambda(I-P)\biggl[\int_0^T W(T,s)g(s,x(s))\,\mathrm{d}s+\sum_{0\leqslant t_k<T}W(T,t_k^+)h_k\biggr].\end{aligned}$$

Thus, we have

$$x^0_{\perp}=\lambda L_p^{-1}(I-P)\biggl[\int_0^T W(T,s)g(s,x(s))\,\mathrm{d}s+\sum_{0\leqslant t_k<T}W(T,t_k^+)h_k\biggr].$$

Let

$$\mathrm{X}=\{x:[0,T]\to\mathbb{R}^n\colon\ x(t)\text{ is continuous on }[0,T]\},$$

and define the norm as $\|x\|=\sup\limits_{t\in[0,T]}|x(t)|$. It is easy to see that X is a Banach space with the norm $\|x\|$. We also define the norm of the fundamental matrix $\|X(t)\|=\max\limits_{i=1,2,\dots,n}\|x_i\|$.

For $x \in \mathrm{X}$ which satisfies that $x(t) \in \overline{D}$ for all $t \in [0,T]$, we define an operator $\mathtt{T}(x^0_{\mathrm{ker}}, x, \lambda)$ by

$$\begin{aligned}
(16)\qquad & \mathtt{T}(x^0_{\mathrm{ker}}, x, \lambda) \\
&= \begin{pmatrix}
x^0_{\mathrm{ker}} + \dfrac{1}{T} P\Big[\displaystyle\int_0^T W(T,s) g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<T} W(T,t_k^+) h_k\Big] \\
X(t)x^0_{\mathrm{ker}} + \lambda X(t) L_p^{-1}(I-P)\Big[\displaystyle\int_0^T W(T,s) g(s,x(s))\,\mathrm{d}s \\
\qquad + \displaystyle\sum_{0\leqslant t_k<T} W(T,t_k^+) h_k\Big] \\
\qquad + \lambda\Big[\displaystyle\int_0^t W(t,s) g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<t} W(t,t_k^+) h_k\Big]
\end{pmatrix},
\end{aligned}$$

where $\lambda \in [0,1]$. We claim that each fixed point x of $\mathtt{T}$ in X is a solution of (13) with $x(T) = Qx(0)$.

In fact, if x is a fixed point of $\mathtt{T}$, we have

$$\begin{pmatrix} x^0_{\mathrm{ker}} \\ \\ x(t) \end{pmatrix} = \begin{pmatrix}
x^0_{\mathrm{ker}} + \dfrac{1}{T} P\Big[\displaystyle\int_0^T W(T,s) g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<T} W(T,t_k^+) h_k\Big] \\
X(t)x^0_{\mathrm{ker}} + \lambda X(t) L_p^{-1}(I-P)\Big[\displaystyle\int_0^T W(T,s) g(s,x(s))\,\mathrm{d}s \\
\qquad + \displaystyle\sum_{0\leqslant t_k<T} W(T,t_k^+) h_k\Big] \\
\qquad + \lambda\Big[\displaystyle\int_0^t W(t,s) g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<t} W(t,t_k^+) h_k\Big]
\end{pmatrix}.$$

Thus,

$$(17)\qquad \frac{1}{T} P\Big[\int_0^T W(T,s) g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<T} W(T,t_k^+) h_k\Big] = 0,$$

$$\begin{aligned}
(18)\qquad x(t) = X(t)x^0_{\mathrm{ker}} &+ \lambda X(t) L_p^{-1}(I-P)\Big[\int_0^T W(T,s) g(s,x(s))\,\mathrm{d}s \\
&+ \sum_{0\leqslant t_k<T} W(T,t_k^+) h_k\Big] \\
&+ \lambda\Big[\int_0^t W(t,s) g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<t} W(t,t_k^+) h_k\Big].
\end{aligned}$$

By the equation (18), we know that

$$x_0 = x_{\mathrm{ker}}^0 + \lambda L_p^{-1}(I-P)\biggl[\int_0^T W(T,s)g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<T} W(T,t_k^+)h_k\biggr].$$

According to $(Q - X(T))x_{\mathrm{ker}}^0 = 0$, we have

$$\begin{aligned}
Qx_0 &= Qx_{\mathrm{ker}}^0 + \lambda QL_p^{-1}(I-P)\biggl[\int_0^T W(T,s)g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<T} W(T,t_k^+)h_k\biggr] \\
&= X(T)x_{\mathrm{ker}}^0 + \lambda QL_p^{-1}(I-P)\biggl[\int_0^T W(T,s)g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<T} W(T,t_k^+)h_k\biggr].
\end{aligned}$$

Since the equation (17) holds, we have

$$\begin{aligned}
(Q - X(T))L_p^{-1}(I-P)&\biggl[\int_0^T W(T,s)g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<T} W(T,t_k^+)h_k\biggr] \\
&= (I-P)\biggl[\int_0^T W(T,s)g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<T} W(T,t_k^+)h_k\biggr] \\
&= (I-P)\biggl[\int_0^T W(T,s)g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<T} W(T,t_k^+)h_k\biggr] \\
&\quad + P\biggl[\int_0^T W(T,s)g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<T} W(T,t_k^+)h_k\biggr] \\
&= \int_0^T W(T,s)g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<T} W(T,t_k^+)h_k.
\end{aligned}$$

Thus

$$\begin{aligned}
\lambda QL_p^{-1}(I-P)&\biggl[\int_0^T W(T,s)g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<T} W(T,t_k^+)h_k\biggr] \\
&= \lambda X(T)L_p^{-1}(I-P)\biggl[\int_0^T W(T,s)g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<T} W(T,t_k^+)h_k\biggr] \\
&\quad + \lambda\biggl[\int_0^T W(T,s)g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<T} W(T,t_k^+)h_k\biggr].
\end{aligned}$$

Then,

(19)

$$\begin{aligned}Qx_0 &= X(T)x^0_{\mathrm{ker}} + \lambda Q L_p^{-1}(I-P)\Big[\int_0^T W(T,s)g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<T} W(T,t_k^+)h_k\Big]\\ &= X(T)x^0_{\mathrm{ker}} + \lambda X(T) L_p^{-1}(I-P)\Big[\int_0^T W(T,s)g(s,x(s))\,\mathrm{d}s\\ &\qquad + \sum_{0\leqslant t_k<T} W(T,t_k^+)h_k\Big]\\ &\qquad + \lambda\Big[\int_0^T W(T,s)g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<T} W(T,t_k^+)h_k\Big]\\ &= x(T).\end{aligned}$$

By equations (18) and (19), the equation (14) holds. Thus,

$$x^0_{\perp} = \lambda L_p^{-1}(I-P)\Big[\int_0^T W(T,s)g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<T} W(T,t_k^+)h_k\Big].$$

Then,

$$\begin{aligned}x(t) &= X(t)x^0_{\mathrm{ker}} + \lambda X(t) L_p^{-1}(I-P)\Big[\int_0^T W(T,s)g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<T} W(T,t_k^+)h_k\Big]\\ &\quad + \lambda\Big[\int_0^t W(t,s)g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<T} W(t,t_k^+)h_k\Big]\\ &= X(t)x^0_{\mathrm{ker}} + X(t)x^0_{\perp} + \lambda\Big[\int_0^t W(t,s)g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<T} W(t,t_k^+)h_k\Big]\\ &= X(t)x_0 + \lambda\Big[\int_0^t W(t,s)g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<t} W(t,t_k^+)h_k\Big].\end{aligned}$$

This means that the fixed point x is a solution of (13) with $x(T) = Qx(0)$.

Now, we need to prove the existence of the fixed point of $\mathtt{T}$. Take a constant M such that

$$\begin{aligned}M > \max\Big\{ &\sup_{t\in[0,T]}\|A(t)\|,\ \sup_{t\in[0,T]}\|X(t)\|,\ \sup_{\substack{t\in[0,T]\\ x\in\overline{D}}}|X^{-1}(t)g(t,x(t))|,\\ &\max_{k=1,2\ldots,q}|X^{-1}(t_k^+)h_k|\Big\},\end{aligned}$$

and let

$$\mathrm{X}_\lambda = \Big\{x\in \mathrm{X}\colon \Big|\frac{x(t)-x(r)}{t-r}\Big| \leqslant M^3 + \lambda[M^2 + (q+T)M^3]\ \forall\, t,r\in(t_k,t_{k+1}], t\neq r\Big\}.$$

Then, it is easy to introduce a retraction $\alpha_\lambda\colon \mathrm{X}\to\mathrm{X}_\lambda$.

Define an operator $\widehat{\mathtt{T}}(x^0_{\ker}, x, \lambda)$ by

(20)

$$\widehat{\mathtt{T}}(x^0_{\ker}, x, \lambda) = \begin{pmatrix} x^0_{\ker} + \dfrac{1}{T}\Big[\displaystyle\int_0^T Pf(s, \alpha_\lambda \circ x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<T} PI(x(t_k^+))\Big] \\ \alpha_\lambda \circ X(t)x^0_{\ker} + \lambda X(t)L_p^{-1}(I-P)\Big[\displaystyle\int_0^T W(T,s)g(s,\alpha_\lambda \circ x(s))\,\mathrm{d}s \\ \qquad + \displaystyle\sum_{0\leqslant t_k<T} W(T,t_k^+)h_k\Big] \\ \qquad + \lambda\Big[\displaystyle\int_0^t W(t,s)g(s,\alpha_\lambda \circ x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<t} W(t,t_k^+)h_k\Big] \end{pmatrix}.$$

Since $P\colon \mathbb{R}^n \to \operatorname{Ker}(Q - X(T))$, it is easy to see that

$$\frac{1}{T}P\Big[\int_0^T W(T,s)g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<T} W(T,t_k^+)h_k\Big] \in \operatorname{Ker}(Q - X(T)).$$

Also,

$$\frac{1}{T}P\Big[\int_0^T W(T,s)g(s,\alpha_\lambda \circ x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<T} W(T,t_k^+)h_k\Big] \in \operatorname{Ker}(Q - X(T)).$$

Let us consider the homotopy:

$$H(x^0_{\ker}, x, \lambda) = \widehat{\mathtt{T}}(x^0_{\ker}, x, \lambda), \tag{21}$$

$$(x^0_{\ker}, x, \lambda) \in (D \cap \operatorname{Ker}(Q - X(T)) \times \widetilde{D} \times [0,1]), \tag{22}$$

where $\widetilde{D} = \{x \in X : x(t) \in \overline{D}\ \forall\, t \in [0,T]\}$.

We claim that

$$0 \notin (\mathrm{id} - H)(\partial(D \cap \operatorname{Ker}(Q - X(T)) \times \widetilde{D}) \times [0,1]). \tag{23}$$

Suppose, on the contrary, that there exists $(\widehat{x}^0_{\ker}, \widehat{x}, \widehat{\lambda}) \in \partial(D \cap \operatorname{Ker}(Q - X(T)) \times \widetilde{D}) \times [0,1]$, such that $(\mathrm{id} - H)(\widehat{x}^0_{\ker}, \widehat{x}, \widehat{\lambda}) = 0$. As $\widehat{x}^0_{\ker} \in \partial D$ is contradictory to (H_1), and $\partial(D \cap \operatorname{Ker}(Q - X(T))) \subset \partial D$, we have that $\widehat{x}^0_{\ker} \notin \partial(D \cap \operatorname{Ker}(Q - X(T)))$. In other words, $\widehat{x} \in \partial D$. Then the equation (23) can be proved as follows.

When $\widehat{\lambda} \in [0,1]$, as $0 = (\mathrm{id} - H)(\widehat{x}^0_{\mathrm{ker}}, \widehat{x}, \widehat{\lambda})$, we have

$$\begin{pmatrix} \widehat{x}^0_{\mathrm{ker}} \\ \\ \widehat{x}(t) \end{pmatrix} = \begin{pmatrix} \widehat{x}^0_{\mathrm{ker}} + \dfrac{1}{T} P\Big[\displaystyle\int_0^T W(T,s) g(s, \alpha_{\widehat{\lambda}} \circ \widehat{x}(s))\,\mathrm{d}s + \sum_{0 \leqslant t_k < T} W(T, t_k^+) h_k\Big] \\ \alpha_{\widehat{\lambda}} \circ X(t)\widehat{x}^0_{\mathrm{ker}} + \lambda X(t) L_p^{-1}(I-P)\Big[\displaystyle\int_0^T W(T,s) g(s, \alpha_{\widehat{\lambda}} \circ \widehat{x}(s))\,\mathrm{d}s \\ \qquad + \displaystyle\sum_{0 \leqslant t_k < T} W(T, t_k^+) h_k\Big] \\ \qquad + \lambda\Big[\displaystyle\int_0^t W(t,s) g(s, \alpha_{\widehat{\lambda}} \circ \widehat{x}(s))\,\mathrm{d}s + \sum_{0 \leqslant t_k < t} W(t, t_k^+) h_k\Big] \end{pmatrix}.$$

Thus,

$$\frac{1}{T} P\Big[\int_0^T W(T,s) g(s, \alpha_{\widehat{\lambda}} \circ \widehat{x}(s))\,\mathrm{d}s + \sum_{0 \leqslant t_k < T} W(T, t_k^+) h_k\Big] = 0$$

and

(24)

$$\begin{aligned} \widehat{x}(t) = {} & \alpha_{\widehat{\lambda}} \circ X(t)\widehat{x}^0_{\mathrm{ker}} \\ & + \widehat{\lambda} X(t) L_p^{-1}(I-P)\Big[\int_0^T W(T,s) g(s, \alpha_{\widehat{\lambda}} \circ \widehat{x}(s))\,\mathrm{d}s + \sum_{0 \leqslant t_k < T} W(T, t_k^+) h_k\Big] \\ & + \widehat{\lambda}\Big[\int_0^t W(t,s) g(s, \alpha_{\widehat{\lambda}} \circ \widehat{x}(s))\,\mathrm{d}s + \sum_{0 \leqslant t_k < t} W(t, t_k^+) h_k\Big]. \end{aligned}$$

Notice that

$$\begin{aligned} \Big|\frac{x(t) - x(r)}{t - r}\Big| = \frac{1}{|t-r|}\Big| & \alpha_{\widehat{\lambda}} \circ (X(t) - X(r))\widehat{x}^0_{\mathrm{ker}} + \widehat{\lambda}(X(t) - X(r)) L_p^{-1}(I-P) \\ & \times \Big[\int_0^T W(T,s) g(s, \alpha_{\widehat{\lambda}} \circ \widehat{x}(s))\,\mathrm{d}s + \sum_{0 \leqslant t_k < T} W(T, t_k^+) h_k\Big] \\ & + \widehat{\lambda}\Big[\int_0^t W(t,s) g(s, \alpha_{\widehat{\lambda}} \circ \widehat{x}(s))\,\mathrm{d}s + \sum_{0 \leqslant t_k < t} W(t, t_k^+) h_k\Big] \\ & - \widehat{\lambda}\Big[\int_0^r W(r,s) g(s, \alpha_{\widehat{\lambda}} \circ \widehat{x}(s))\,\mathrm{d}s + \sum_{0 \leqslant t_k < r} W(r, t_k^+) h_k\Big]\Big|. \end{aligned}$$

The fundamental matrix of the linear homogeneous system is $W(t,0) = X(t) = U(t,t_k)\prod_{i=1}^{k}(I + B_i)U(t_i, t_{i-1})U(t_1, 0)$ and $t, r \in (t_k, t_{k+1}] \subset [0,T]$, then

$$\|X(t) - X(r)\| = \|[U(t,t_k) - U(r,t_k)]X(t_k^+)\| \leqslant \Big\|\int_r^t A(s)\,\mathrm{d}s\Big\| \cdot M \leqslant M^2|t-r|.$$

Note that $x_{\perp}^{0} = \lambda L_p^{-1}(I-P)\Big[\int_0^T W(T,s)g(s,x(s))\,\mathrm{d}s + \sum\limits_{0\leqslant t_k<T} W(T,t_k^+)h_k\Big]$. Then

$$
\begin{aligned}
\Big|\frac{x(t)-x(r)}{t-r}\Big| &= \frac{1}{|t-r|}\bigg|\alpha_{\widehat{\lambda}}\circ(X(t)-X(r))\widehat{x}_{\mathrm{ker}}^{0} + \alpha_{\widehat{\lambda}}\circ(X(t)-X(r))\widehat{x}_{\perp}^{0} \\
&\quad + \widehat{\lambda}\bigg[\int_0^t W(t,s)g(s,\alpha_{\widehat{\lambda}}\circ\widehat{x}(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<t} W(t,t_k^+)h_k\bigg] \\
&\quad - \widehat{\lambda}\bigg[\int_0^r W(r,s)g(s,\alpha_{\widehat{\lambda}}\circ\widehat{x}(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<r} W(r,t_k^+)h_k\bigg]\bigg| \\
&= \frac{1}{|t-r|}\bigg|\alpha_{\widehat{\lambda}}\circ(X(t)-X(r))\widehat{x}_0 \\
&\quad + \widehat{\lambda}\bigg[\int_0^t W(t,s)g(s,\alpha_{\widehat{\lambda}}\circ\widehat{x}(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<t} W(t,t_k^+)h_k\bigg] \\
&\quad - \widehat{\lambda}\bigg[\int_0^r W(r,s)g(s,\alpha_{\widehat{\lambda}}\circ\widehat{x}(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<r} W(r,t_k^+)h_k\bigg]\bigg| \\
&\leqslant M^3 + \frac{\widehat{\lambda}}{|t-r|}\bigg|\bigg[X(t)\int_0^t X^{-1}(s)g(s,x(s))\,\mathrm{d}s - X(r)\int_0^r X^{-1}(s)g(s,x(s))\,\mathrm{d}s\bigg] \\
&\quad + \bigg[X(t)\sum_{0\leqslant t_k<t} X^{-1}(t_k^+)h_k\bigg] - \bigg[X(r)\sum_{0\leqslant t_k<r} X^{-1}(t_k^+)h_k\bigg]\bigg|.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\bigg|X(t)\int_0^t X^{-1}(s)g(s,x(s))\,\mathrm{d}s - X(r)\int_0^r X^{-1}(s)g(s,x(s))\,\mathrm{d}s\bigg| \\
&\quad = \bigg|X(t)\int_0^t X^{-1}(s)g(s,x(s))\,\mathrm{d}s - X(t)\int_0^r X^{-1}(s)g(s,x(s))\,\mathrm{d}s \\
&\qquad + X(t)\int_0^r X^{-1}(s)g(s,x(s))\,\mathrm{d}s - X(r)\int_0^r X^{-1}(s)g(s,x(s))\,\mathrm{d}s\bigg| \\
&\quad \leqslant \bigg|X(t)\int_r^t X^{-1}(s)g(s,x(s))\,\mathrm{d}s + (X(t)-X(r))\int_0^r X^{-1}(s)g(s,x(s))\,\mathrm{d}s\bigg| \\
&\quad \leqslant |t-r|M^2 + |t-r|M^3T, \\
&\bigg|X(t)\sum_{0\leqslant t_k<t} X^{-1}(t_k^+)h_k - X(r)\sum_{0\leqslant t_k<r} X^{-1}(t_k^+)h_k\bigg| \\
&\quad = \bigg|(X(t)-X(r))\sum_{0\leqslant t_k<r} X^{-1}(t_k^+)h_k\bigg| \\
&\quad \leqslant \|X(t)-X(r)\|\cdot q\cdot \max_{k=1,2,\ldots,q}|X^{-1}(t_k^+)h_k| \leqslant |t-r|qM^3,
\end{aligned}
$$

we can prove

$$
\Big|\frac{x(t)-x(r)}{t-r}\Big| \leqslant M^3 + \widehat{\lambda}[M^2 + (q+T)M^3].
$$

By the definition of X_λ, we obtain $\widehat{x} \in \mathrm{X}_{\widehat{\lambda}}$, which means that $\alpha_{\widehat{\lambda}} \circ \widehat{x} = \widehat{x}$. Now we can rewrite the equation (24) as

$$\widehat{x}(t) = X(t)\widehat{x}^0_{\mathrm{ker}} + \widehat{\lambda}X(t)L_p^{-1}(I-P)\Bigg[\int_0^T W(T,s)g(s,\widehat{x}(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<T} W(T,t_k^+)h_k\Bigg]$$
$$+ \widehat{\lambda}\Bigg[\int_0^t W(t,s)g(s,\widehat{x}(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<t} W(t,t_k^+)h_k\Bigg].$$

By a similar discussion to the equation (18), we can prove that $\widehat{x}(t)$ is a solution of the equation (13). By the hypothesis (H_1), we know that $\widehat{x}(t) \notin \partial\widetilde{D}$ for any $t \in [0,T]$. This is a contradiction to $\widehat{x} \in \partial\widetilde{D}$.

We obtain that

$$0 \notin (\mathrm{id} - H)(\partial((D \cap \mathrm{Ker}(Q - X(T))) \times \widetilde{D}) \times [0,1]).$$

Therefore, by the homotopy invariance and the Brouwer degree, we have

$$\begin{aligned}\deg\,&(\mathrm{id} - H(x^0_{\mathrm{ker}},\cdot,1),(D\cap\mathrm{Ker}(Q-X(T)))\times\widetilde{D},0)\\ &= \deg(\mathrm{id} - H(x^0_{\mathrm{ker}},\cdot,0),(D\cap\mathrm{Ker}(Q-X(T)))\times\widetilde{D},0)\\ &= \deg(f, D\cap\mathrm{Ker}(Q-X(T)),0) \neq 0.\end{aligned}$$

This means that there exists $\widehat{x}_* \in \widetilde{D}$, such that

$$\begin{pmatrix}\widehat{x}^0_{*\mathrm{ker}}\\ \widehat{x}_*(t)\end{pmatrix} = \widehat{\mathtt{T}}(\widehat{x}^0_{*\mathrm{ker}},\widehat{x}_*(t),1). \tag{25}$$

Similarly to the proof in (ii), we get $\widehat{x}_* \in \mathrm{X}_\lambda$. Then

$$\widehat{\mathtt{T}}(\widehat{x}^0_{*\mathrm{ker}},\widehat{x}_*(t),1) = \mathtt{T}(\widehat{x}^0_{*\mathrm{ker}},\widehat{x}_*(t),1). \tag{26}$$

By the equations (25) and (26), $\widehat{x}_*$ is a fixed point of $\mathtt{T}$ in X. Thus, $\widehat{x}_*$ is a solution of the system (3) with the boundary value condition $x(T) = Qx(0)$.

(II) $\mathrm{Ker}(Q - X(T)) = \{0\}$. In this case, $(Q - X(T))^{-1}$ exists. Then

$$x_0 = \lambda L_p^{-1}(I-P)\Bigg[\int_0^T W(T,s)g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<T} W(T,t_k^+)h_k\Bigg].$$

We no longer need the hypothesis (H_2). Consider the homotopy

$$H(x,\lambda) = \lambda X(T)L_p^{-1}(I-P)\Bigg[\int_0^T W(T,s)g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<T} W(T,t_k^+)h_k\Bigg]$$
$$+ \lambda\Bigg[\int_0^t W(t,s)g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<t} W(t,t_k^+)h_k\Bigg].$$

Similarly to the proof of $\operatorname{Ker}(Q - X(T)) \neq 0$, we have $0 \notin (\mathrm{id} - H)(\partial\widetilde{D} \times [0,1])$. Hence,

$$\deg(\mathrm{id} - H(\cdot,1), \widetilde{D}, 0) = \deg(\mathrm{id} - H(\cdot,0), \widetilde{D}, 0) = \deg(\mathrm{id}, \widetilde{D}, 0) = 1.$$

This means that there exists $\widehat{x}_*$ with $\widehat{x}_*(t) \in D$ for all $t \in [0,T]$, such that

$$\widehat{x}_*(t) = X(t)\widehat{x}_*(0) + \int_0^t W(t,s)g(s,x(s))\,\mathrm{d}s + \sum_{0\leqslant t_k<t} W(t,t_k^+)h_k.$$

By Lemma 1 and the proofs in (I) and (II), it is easy to see that the system (5) has a rotating periodic solution $x_*(t)$, which is an extension of $\widehat{x}_*(t)$ on $\mathbb{R}$. Since $x_*(t) \in D$ for $t \in [0,T]$ and $x_*(t)$ is piecewise continuous, by hypothesis (H_1) we know that $x_*(t) \in D$ for all t. Then the proof is complete. $\square$

5. Numerical simulation

In this part, we give two examples to show the existence of synchronization.

Consider the system

$$\begin{aligned}(27)\qquad \dot{x} = {} & -x - |x|^2x \\ & + (\sin t, \cos t, \sin t, \cos t, \sin 2\pi t, \cos 2\pi t, \sin 2\pi t, \cos 2\pi t)^\top, \quad t \neq N, \\ \Delta x = {} & \Big(\frac{1}{\mathrm{e}} - 1\Big)x, \quad t = N.\end{aligned}$$

Set

$$Q = \begin{pmatrix} \cos(2\pi-1) & -\sin(2\pi-1) & 0 & 0 & 0 & 0 & 0 & 0 \\ \sin(2\pi-1) & \cos(2\pi-1) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(2\pi-1) & -\sin(2\pi-1) & 0 & 0 & 0 & 0 \\ 0 & 0 & \sin(2\pi-1) & \cos(2\pi-1) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and $T = 1$. According to Q, it is easy to know that the oscillator network is cluster synchronous. (See Fig. 1.)

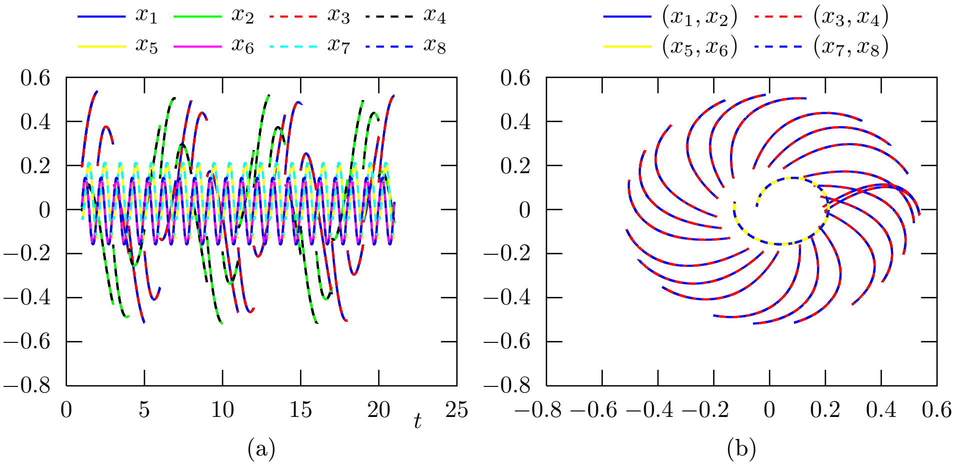

Figure 1. Cluster synchronous solutions of (27): (a) If each equation in (27) is one oscillator, then the oscillators x_1 and x_3, x_2 and x_4, x_5 and x_7, x_6 and x_8, respectively, are synchronous. (b) If each two equations in (27) represent one oscillator, then the oscillators (x_1, x_2) and (x_3, x_4), (x_5, x_6) and (x_7, x_8), respectively, are synchronous.

Consider the equations

$$\begin{cases} \dot{x}_1 = -0.01x_1 - x_1^3 + \cos(2t) - \dfrac{\cos(2t)}{3}(x_1 - x_2), \\ \dot{x}_2 = -0.01x_2 - x_2^3 + \cos(2t) - \dfrac{\cos(2t)}{3}(x_2 - x_3), \\ \dot{x}_3 = -0.01x_3 - x_3^3 + \cos(2t) - \dfrac{\cos(2t)}{3}(x_3 - x_1), \\ \Delta x_1 = -\dfrac{1}{4}x_2, \quad t = \dfrac{N\pi}{10}, \\ \Delta x_2 = -\dfrac{1}{4}x_3, \quad t = \dfrac{N\pi}{10}, \\ \Delta x_3 = -\dfrac{1}{4}x_1, \quad t = \dfrac{N\pi}{10}. \end{cases} \tag{28}$$

Here we set

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and $T = \pi$. Numerical simulation in Fig. 2 shows that this system has a periodic synchronous solution.

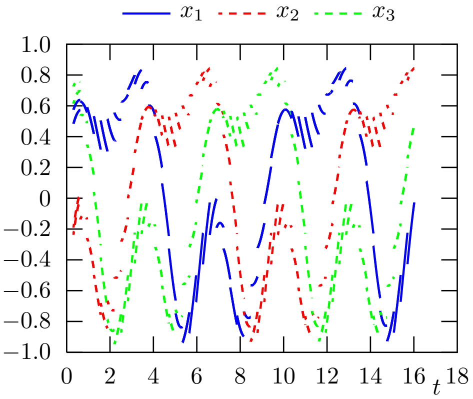

Figure 2. The periodic synchronous solution of the system (28).

6. Conclusion

The research on synchronization in complex dynamical networks is always an important issue for its ubiquitousness and applications. In this paper, we study the existence of synchronization phenomena in complex dynamical networks consisting of impulsive coupled oscillators. With the aid of rotating periodic solutions, we may identify the type of synchronization. Concretely, the type of synchronization corresponds to the conjugate class consisting of rotating matrices satisfying the rotating periodic condition. As mentioned above, the existence theorems of rotating periodic solutions of a non-homogeneous linear impulsive system and nonlinear perturbation impulsive system are presented. Finally, some examples are given to show the synchronization phenomena in impulsive coupled oscillator networks.

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