Chein-Shan Liu; Botong Li

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THE EIGENVALUES OF SYMMETRIC STURM-LIOUVILLE PROBLEM AND INVERSE POTENTIAL PROBLEM, BASED ON SPECIAL MATRIX AND PRODUCT FORMULA

Chein-Shan Liu, Keelung, Botong Li, Beijing

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Abstract. The Sturm-Liouville eigenvalue problem is symmetric if the coefficients are even functions and the boundary conditions are symmetric. The eigenfunction is expressed in terms of orthonormal bases, which are constructed in a linear space of trial functions by using the Gram-Schmidt orthonormalization technique. Then an n-dimensional matrix eigenvalue problem is derived with a special matrix $\mathbf{A} := [a_{ij}]$, that is, $a_{ij} = 0$ if $i + j$ is odd.

Based on the product formula, an integration method with a fictitious time, namely the fictitious time integration method (FTIM), is developed to obtain the higher-index eigenvalues. Also, we recover the symmetric potential function $q(x)$ in the Sturm-Liouville operator by specifying a few lower-index eigenvalues, based on the product formula and the Newton iterative method.

Keywords: symmetric Sturm-Liouville problem; inverse potential problem; special matrix eigenvalue problem; product formula; fictitious time integration method

MSC 2020: 34B24, 34A55

1. Introduction

The Sturm-Liouville eigenvalue problem has a considerable physical interest and is important in many fields, including the longitudinal free vibration of beam and rod. Except for some limited cases, there are in general no exact solutions.

The Rayleigh minimal principle is usually applied to solve the Sturm-Liouville eigenvalue problem, and is effective only for the first few lower-index eigenvalues [6], [8], [11]. A growing interest in the numerical solutions of the Sturm-Liouville eigenvalue problem is to reduce the computational time and cost [1], [3], [4], [5], [13], [23].

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In the paper we first propose a sequentially closed-form solutions of the following Sturm-Liouville eigenvalue problem:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\Big[p(x)\frac{\mathrm{d}y(x)}{\mathrm{d}x}\Big]+q(x)y(x)=\lambda s(x)y(x),\quad 0<x<l, \tag{1}$$

where the coefficients $p(x)$, $q(x)$ and $s(x)$ are symmetric functions with respect to the midpoint of the interval, i.e., $p(x)=p(l-x)$, $q(x)=q(l-x)$, $s(x)=s(l-x)$, and the boundary conditions are symmetric [7]. Recently, Liu [15] using the fictitious time integration method (FTIM), a half-interval method and a derivative-free iterative scheme can achieve high precision eigenvalues of (1). Liu et al. [18] proposed a boundary shape function method for computing eigenvalues and eigenfunctions of (1).

The symmetric Sturm-Liouville eigenvalue problem may happen. For example, consider the longitudinal frequency of a one-dimensional rod

$$-\frac{\mathrm{d}}{\mathrm{d}x}\Big(E(x)A(x)\frac{\mathrm{d}y(x)}{\mathrm{d}x}\Big)=\omega^2\varrho(x)A(x)y(x), \tag{2}$$

where Young's modulus is $E(x)$ and cross-sectional area is $A(x)$; $\varrho(x)$ is the variable mass-density, $y(x)$ is the displacement and ω is the vibration frequency. Usually, the rod is made to be symmetric such that $p(x)=E(x)A(x)$ and $s(x)=\varrho(x)A(x)$ are even functions with $p(l-x)=p(x)$ and $s(l-x)=s(x)$. Liu and Atluri [16] have applied the fictitious time integration method (FTIM) to recover $q(x)$ under some specified eigenvalues. After developing a powerful product formula for the symmetric Sturm-Liouville eigenvalue problem, we are going to use the FTIM to find higher-index eigenvalues. Liu and Li [20] used an energetic boundary function iterative method to reconstruct the Sturm-Liouville operator. On the other hand, we are going to use the product formula and the Newton iterative method to recover $q(x)$ in (1) with a few eigenvalues given.

2. Transformation of a matrix eigenvalue problem

Let

$$y(x)=\sum_{k=1}^{n}x_k\varphi_k(x) \tag{3}$$

be the eigenfunction, where x_k are unknowns, and $\varphi_k(x)$ are defined as trial functions satisfying boundary conditions. When $y(0)=y(l)=0$,

$$B_k(x)=\sin\frac{k\pi x}{l}. \tag{4}$$

Then we can construct an $s(x)$-orthogonal system from $\{B_k(x),\ k = 1, \dots, n\}$ by using the Gram-Schmidt orthogonalization technique:

$$\widetilde{\varphi}_1(x) = B_1(x), \quad \widetilde{\varphi}_k(x) = B_k(x) - \sum_{j=1}^{k-1} \beta_{kj}\widetilde{\varphi}_j(x), \tag{5}$$

$$\beta_{kj} = \langle B_k(x), \widetilde{\varphi}_j(x)\rangle = \int_0^l s(x)B_k(x)\widetilde{\varphi}_j(x)\,\mathrm{d}x.$$

Therefore,

$$\varphi_j(x) := \frac{\widetilde{\varphi}_j(x)}{\int_0^l s(x)\widetilde{\varphi}_j(x)\widetilde{\varphi}_j(x)\,\mathrm{d}x}, \quad j = 1, \dots, n, \tag{6}$$

form an $s(x)$-orthonormal system, i.e.,

$$\langle \varphi_j(x), \varphi_k(x)\rangle := \int_0^l s(x)\varphi_j(x)\varphi_k(x)\,\mathrm{d}x = \delta_{jk}, \tag{7}$$

where δ_{jk} is the Kronecker delta.

From (1), one can further derive

$$\int_0^l [p(x)y'(x)\varphi_j'(x) + q(x)y(x)\varphi_j(x)]\,\mathrm{d}x = \lambda \int_0^l s(x)y(x)\varphi_j(x)\,\mathrm{d}x. \tag{8}$$

Inserting (3) into (8), and employing (7), one obtains:

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}, \tag{9}$$

where $\mathbf{x} = (x_1, \dots, x_n)^\top$ and $a_{jk} = a_{kj}$ of $\mathbf{A}$ are given by

$$a_{jk} := \int_0^l [p(x)\varphi_j'(x)\varphi_k'(x) + q(x)\varphi_j(x)\varphi_k(x)]\,\mathrm{d}x, \quad j, k = 1, \dots, n. \tag{10}$$

Solving the matrix eigenvalue problem (9), we can obtain $\lambda_j,\ j = 1, 2, \dots, n$ to approximate the first n eigenvalues of the Sturm-Liouville eigenvalue problem (1).

3. Special matrix with $a_{jk} = 0$ if $j + k$ is odd

Before giving a mathematical proof of the special structure of the coefficient matrix $\mathbf{A}$ in (9), we begin with the following preliminaries.

Definition 1. A function $f(x)$, $x \in [0, l]$ is an even function if $f(x) = f(l - x)$, and an odd function if $f(x) = -f(l - x)$.

The following lemmas are easily proved.

Lemma 1. *The differential $f'(x)$ of an odd function $f(x)$ is even, and conversely, the differential $f'(x)$ of an even function is odd. While the integral of an odd function $f(x)$ from $x = 0$ to $x = l$ is zero, i.e., $\int_0^l f(x) = 0$, the integral of a nonzero even function is nonzero.*

Lemma 2. *The functions $B_j(x)$ in (4) have the following properties:*

$$B_{2j}(x) \quad \textit{is odd}, \tag{11}$$

$$B_{2j-1}(x) \quad \textit{is even}. \tag{12}$$

P r o o f. It follows from (4) that

$$\begin{aligned}
B_{2j}(l-x) &= \sin\frac{2j\pi(l-x)}{l} = \sin\Big(2j\pi - \frac{2j\pi x}{l}\Big) = -\sin\frac{2j\pi x}{l} = -B_{2j}(x),\\
B_{2j-1}(l-x) &= \sin\frac{(2j-1)\pi(l-x)}{l} = \sin\Big((2j-1)\pi - \frac{(2j-1)\pi x}{l}\Big)\\
&= \sin\frac{(2j-1)\pi x}{l} = B_{2j-1}(x).
\end{aligned}$$

This ends the proof. □

Lemma 3. *The product of $B_j(x)$ and $B_k(x)$ has the following properties:*

$$B_j(x)B_k(x) \quad \textit{is odd if } j+k \textit{ is odd}, \tag{13}$$

$$B_j(x)B_k(x) \quad \textit{is even if } j+k \textit{ is even}. \tag{14}$$

P r o o f. From (4) it follows that

$$B_j(x)B_k(x) = \frac{1}{2}\Big(\cos\frac{(j-k)\pi x}{l} - \cos\frac{(j+k)\pi x}{l}\Big). \tag{15}$$

Then from

$$\begin{aligned}
B_j(l-x)B_k(l-x) &= \frac{1}{2}\Big(\cos\frac{(j-k)\pi(l-x)}{l} - \cos\frac{(j+k)\pi(l-x)}{l}\Big)\\
&= \frac{1}{2}\Big(\cos(j-k)\pi\cos\frac{(j-k)\pi x}{l} - \cos(j+k)\pi\cos\frac{(j+k)\pi x}{l}\Big),
\end{aligned}$$

it follows that $B_j(l-x)B_k(l-x) = -B_j(x)B_k(x)$ if $j+k$ is an odd integer (hence, $|j-k|$ is an odd integer), and that $B_j(l-x)B_k(l-x) = B_j(x)B_k(x)$ if $j+k$ is an even integer (hence, $|j-k|$ is an even integer). □

From (5) and (6) we can derive the following linear relation between $\varphi_j(x)$ and $B_k(x)$, see [19]:

$$\varphi_j(x) = \sum_{k=1}^{n} c_{jk} B_k(x), \tag{16}$$

where c_{jk} is generated by the following processes:

▷ DO 1 $j = 1, n$
▷ DO 2 $k = 1,\ j-1$
▷ $\varphi_k(x) = \sum\limits_{i=1}^{k} c_{ki} B_i(x)$
▷ $c_{jk} = -\dfrac{\langle B_j(x), \widetilde{\varphi}_k(x)\rangle}{\langle \widetilde{\varphi}_k(x), \widetilde{\varphi}_k(x)\rangle}$
▷ 2 CONTINUE
▷ DO 3 $L = 1,\ j-2$
▷ DO 3 $M = L+1,\ j-1$
▷ $c_{jL} = c_{jL} + c_{jM} c_{ML}$
▷ 3 CONTINUE
▷ 1 CONTINUE

$\mathbf{C} := [c_{jk}]$ is a lower triangular matrix, and $c_{jk} = 1$ if $j = k$. By further using the scalar product in (5), where $s(x)$ is an even function, we can prove the following result.

Lemma 4. *The coefficient matrix* $\mathbf{C} := [c_{jk}]$ *in* (16) *has the following properties:*

$$c_{jk} = 0 \quad \text{if } j+k \text{ is odd}, \tag{17}$$

$$c_{jk} \neq 0 \quad \text{if } j+k \text{ is even}. \tag{18}$$

Proof. In order to prove this lemma, it suffices to prove

$$\beta_{jk} = 0 \quad \text{if } j+k \text{ is odd}, \tag{19}$$

$$\beta_{jk} \neq 0 \quad \text{if } j+k \text{ is even}, \tag{20}$$

because c_{jk} is obtained by dividing β_{jk} by some nonzero constants in the normalization procedures. The following results follow from Lemmas 1 and 3:

$$\langle B_j(x), B_k(x)\rangle = \int_0^l s(x) B_j(x) B_k(x)\,\mathrm{d}x = 0 \quad \text{if } j+k \text{ odd}, \tag{21}$$

$$\langle B_j(x), B_k(x)\rangle = \int_0^l s(x) B_j(x) B_k(x)\,\mathrm{d}x \neq 0 \quad \text{if } j+k \text{ even}, \tag{22}$$

because $s(x)$ is an even function.

By (21) it is straightforward to write

$$
\begin{aligned}
&\widetilde{\varphi}_1(x) = B_1(x), \quad \beta_{11} = 1,\\
&\widetilde{\varphi}_2(x) = B_2(x) - \langle B_2(x), \widetilde{\varphi}_1(x)\rangle \widetilde{\varphi}_1(x) = B_2(x), \quad \beta_{21} = 0,\ \beta_{22} = 1,\\
&\widetilde{\varphi}_3(x) = B_3(x) - \langle B_3(x), \widetilde{\varphi}_1(x)\rangle \widetilde{\varphi}_1(x), \quad \beta_{32} = 0,\ \beta_{33} = 1,\\
&\widetilde{\varphi}_4(x) = B_4(x) - \langle B_4(x), \widetilde{\varphi}_2(x)\rangle \widetilde{\varphi}_2(x), \quad \beta_{41} = 0,\ \beta_{43} = 0,\ \beta_{44} = 1,\\
&\qquad\vdots\\
&\widetilde{\varphi}_j(x) = B_j(x) - \langle B_j(x), \widetilde{\varphi}_2(x)\rangle \widetilde{\varphi}_2(x) - \langle B_j(x), \widetilde{\varphi}_4(x)\rangle \widetilde{\varphi}_4(x) - \ldots, \quad j \text{ is even},\\
&\widetilde{\varphi}_k(x) = B_k(x) - \langle B_k(x), \widetilde{\varphi}_1(x)\rangle \widetilde{\varphi}_1(x) - \langle B_k(x), \widetilde{\varphi}_3(x)\rangle \widetilde{\varphi}_3(x) - \ldots, \quad k \text{ is odd}.
\end{aligned}
$$

Then, by Lemma 2 we can conclude that $\widetilde{\varphi}_j(x)$ is an odd function if j is even; conversely, $\widetilde{\varphi}_k(x)$ is an even function if k is odd. Then from (5) and Lemmas 1 and 2, we can prove (19) and (20). □

Theorem 1. *For the symmetric Sturm-Liouville eigenvalue problem* (1), *the coefficient matrix* $\mathbf{A} := [a_{jk}]$ *in* (9) *has the following properties:*

$$a_{jk} = 0 \quad \text{if } j+k \text{ is odd}, \tag{23}$$

$$a_{jk} \neq 0 \quad \text{if } j+k \text{ is even}. \tag{24}$$

P r o o f. When $j+k$ is odd and if j is an even integer, k must be an odd integer. From (16) and Lemma 4 it follows that

$$\varphi_j(x) = c_{j2}B_2(x) + c_{j4}B_4(x) + \ldots,\ j \text{ is even}, \tag{25}$$

$$\varphi_k(x) = c_{k1}B_1(x) + c_{k3}B_3(x) + \ldots,\ k \text{ is odd}. \tag{26}$$

Then by Lemma 2 we have

$$\varphi_j(x) \quad \text{is an odd function},\ j \text{ is even}, \tag{27}$$

$$\varphi_k(x) \quad \text{is an even function},\ k \text{ is odd}. \tag{28}$$

Similarly, by Lemma 1 we have

$$\varphi_j'(x) \quad \text{is an even function},\ j \text{ is even}, \tag{29}$$

$$\varphi_k'(x) \quad \text{is an odd function},\ k \text{ is odd}. \tag{30}$$

Hence, from (10) and Lemma 1 we can prove (23) and (24), upon knowing that $p(x)$ and $q(x)$ are even functions. □

4. Demonstrating the special matrix

In order to reveal the special structure for the matrix $\mathbf{A}$ and show its eigenvalues λ, let us consider the following Sturm-Liouville eigenvalue problem as a demonstrative case.

Example 1.

$$-y''(x) = \lambda(2+\cos(2\pi x))y(x), \;\; y(0) = y(1) = 0. \tag{31}$$

For this example, λ does not have closed-form.

For $n = 3$ we can obtain a matrix eigenvalue problem with

$$\mathbf{A} = \begin{bmatrix} 6.57974 & 1.26843\times10^{-14} & 1.98387 \\ 1.26843\times10^{-14} & 19.7392 & 4.35763\times10^{-14} \\ 1.98387 & 4.35763\times10^{-14} & 49.0489 \end{bmatrix}. \tag{32}$$

The small numbers with order 10^{-14} appear, which is due to the round-off error of computer machine and the numerical integration error. Actually, $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} 6.57974 & 0 & 1.98387 \\ 0 & 19.7392 & 0 \\ 1.98387 & 0 & 49.0489 \end{bmatrix}. \tag{33}$$

Denote the matrix in (33) by

$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{13} & 0 & a_{33} \end{bmatrix}, \tag{34}$$

and the characteristic equation follows:

(35)

$$\mathrm{Det}(\mathbf{A}-\lambda\mathbf{I}_3) = \begin{vmatrix} a_{11}-\lambda & 0 & a_{13} \\ 0 & a_{22}-\lambda & 0 \\ a_{13} & 0 & a_{33}-\lambda \end{vmatrix} = (a_{22}-\lambda)\begin{vmatrix} a_{11}-\lambda & a_{13} \\ a_{13} & a_{33}-\lambda \end{vmatrix} = 0,$$

whose solutions of λ can be obtained immediately. We list the first three eigenvalues in Table 1.

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
$n=3$	6.48727	19.7392	49.1414	$\times$	$\times$	$\times$	$\times$
$n=4$	6.48727	19.3468	49.1414	85.9290	$\times$	$\times$	$\times$
$n=5$	6.48721	19.3468	47.1565	85.9290	137.971	$\times$	$\times$
$n=6$	6.48721	19.3458	47.1565	81.3731	137.971	201.009	$\times$
$n=7$	6.48721	19.3458	47.1565	81.3731	128.376	201.009	278.561

Table 1. For Example 1, comparing the eigenvalues obtained from the closed-form solutions of the matrix eigenvalue problem.

Equation (35) indicates that we can employ the special matrix in (34) to obtain the eigenvalues, which can greatly simplify the work. When n is increased, we can obtain more eigenvalues with index lower than n; however, seeking the solutions of λ becomes more difficult when the size of the matrix $\mathbf{A}$ is increased with n. Fortunately, thank to the special structure of $\mathbf{A}$, in addition to (35) for $n=3$ we have the following product formulae for $n=4,5,6$:

$$\begin{vmatrix} a_{11}-\lambda & 0 & a_{13} & 0 \\ 0 & a_{22}-\lambda & 0 & a_{24} \\ a_{13} & 0 & a_{33}-\lambda & 0 \\ 0 & a_{24} & 0 & a_{44}-\lambda \end{vmatrix} = \begin{vmatrix} a_{11}-\lambda & a_{13} \\ a_{13} & a_{33}-\lambda \end{vmatrix} \begin{vmatrix} a_{22}-\lambda & a_{24} \\ a_{24} & a_{44}-\lambda \end{vmatrix} = 0, \tag{36}$$

$$\begin{vmatrix} a_{11}-\lambda & 0 & a_{13} & 0 & a_{15} \\ 0 & a_{22}-\lambda & 0 & a_{24} & 0 \\ a_{13} & 0 & a_{33}-\lambda & 0 & a_{35} \\ 0 & a_{24} & 0 & a_{44}-\lambda & 0 \\ a_{15} & 0 & a_{35} & 0 & a_{55}-\lambda \end{vmatrix} = \begin{vmatrix} a_{22}-\lambda & a_{24} \\ a_{24} & a_{44}-\lambda \end{vmatrix} \begin{vmatrix} a_{11}-\lambda & a_{13} & a_{15} \\ a_{13} & a_{33}-\lambda & a_{35} \\ a_{15} & a_{35} & a_{55}-\lambda \end{vmatrix} = 0, \tag{37}$$

$$\begin{vmatrix} a_{11}-\lambda & 0 & a_{13} & 0 & a_{15} & 0 \\ 0 & a_{22}-\lambda & 0 & a_{24} & 0 & a_{26} \\ a_{13} & 0 & a_{33}-\lambda & 0 & a_{35} & 0 \\ 0 & a_{24} & 0 & a_{44}-\lambda & 0 & a_{46} \\ a_{15} & 0 & a_{35} & 0 & a_{55}-\lambda & 0 \\ 0 & a_{26} & 0 & a_{46} & 0 & a_{66}-\lambda \end{vmatrix} = \begin{vmatrix} a_{11}-\lambda & a_{13} & a_{15} \\ a_{13} & a_{33}-\lambda & a_{35} \\ a_{15} & a_{35} & a_{55}-\lambda \end{vmatrix} \begin{vmatrix} a_{22}-\lambda & a_{24} & a_{26} \\ a_{24} & a_{44}-\lambda & a_{46} \\ a_{26} & a_{46} & a_{66}-\lambda \end{vmatrix} = 0. \tag{38}$$

Therefore, we only need to solve quadratic equations and cubic equations to obtain the closed-form solutions of eigenvalues. Even for $n = 7$ we have

$$
(39)\quad \begin{vmatrix}
a_{11}-\lambda & 0 & a_{13} & 0 & a_{15} & 0 & a_{17} \\
0 & a_{22}-\lambda & 0 & a_{24} & 0 & a_{26} & 0 \\
a_{13} & 0 & a_{33}-\lambda & 0 & a_{35} & 0 & a_{37} \\
0 & a_{24} & 0 & a_{44}-\lambda & 0 & a_{46} & 0 \\
a_{15} & 0 & a_{35} & 0 & a_{55}-\lambda & 0 & a_{57} \\
0 & a_{26} & 0 & a_{46} & 0 & a_{66}-\lambda & 0 \\
a_{17} & 0 & a_{37} & 0 & a_{57} & 0 & a_{77}-\lambda
\end{vmatrix}
$$

$$
= \begin{vmatrix}
a_{22}-\lambda & a_{24} & a_{26} \\
a_{24} & a_{44}-\lambda & a_{46} \\
a_{26} & a_{46} & a_{66}-\lambda
\end{vmatrix}
\begin{vmatrix}
a_{11}-\lambda & a_{13} & a_{15} & a_{17} \\
a_{13} & a_{33}-\lambda & a_{35} & a_{37} \\
a_{15} & a_{35} & a_{55}-\lambda & a_{57} \\
a_{17} & a_{37} & a_{57} & a_{77}-\lambda
\end{vmatrix} = 0.
$$

Hence, we encounter a problem to solve λ in a quartic equation. Because $\lambda_1 = 6.48721$ and $\lambda_3 = 47.1565$ are already computed in Table 1 under the item $n = 6$, the second eigenvalue problem in (39) is decomposed as

$$
(40)\quad \mathrm{Det}(\mathbf{B}_o - \lambda \mathbf{I}_4) = \begin{vmatrix}
a_{11}-\lambda & a_{13} & a_{15} & a_{17} \\
a_{13} & a_{33}-\lambda & a_{35} & a_{37} \\
a_{15} & a_{35} & a_{55}-\lambda & a_{57} \\
a_{17} & a_{37} & a_{57} & a_{77}-\lambda
\end{vmatrix}
= (\lambda - \lambda_1)(\lambda - \lambda_3)(\lambda^2 + a_0\lambda + b_0) = 0,
$$

where $\lambda_1 = 6.48721$, $\lambda_3 = 47.1565$, and

$$
(41)\quad a_0 = \lambda_1 + \lambda_3 - \mathrm{Tr}\,\mathbf{B}_o, \quad b_0 = \frac{\mathrm{Det}\,\mathbf{B}_o}{\lambda_1\lambda_3}.
$$

Again, $\lambda^2 + a_0\lambda + b_0 = 0$ is solved to obtain λ_5 and λ_7. In Example 1, the eigenvalues are listed and compared in Table 1 up to $n = 7$.

By using the product formulae (36)–(39) we can further compute the eigenvalues for $n = 4$ to $n = 7$ in Table 1. As comparing the eigenvalues obtained from $n = 6$ and $n = 7$, λ_2, λ_4 and λ_6 are equal and no further improvement of the accuracy for λ_1 and λ_3 is achieved, while λ_5 for $n = 7$ is more accurate than that in $n = 6$. Increasing n will make the first $n - 2$ eigenvalues more accurate.

5. Product formula

Liu and Li [21] in the study of the free vibration of a symmetric composite beam with nonsmooth interface also found a matrix eigenvalue problem with the coefficient matrix $\mathbf{A} = [a_{jk}]$ having the properties (23) and (24). And they proved the following product formula:

$$\mathrm{Det}(\mathbf{A} - \lambda \mathbf{I}_n) = \mathrm{Det}(\mathbf{B}_o - \lambda \mathbf{I}_{m_1}) \times \mathrm{Det}(\mathbf{B}_e - \lambda \mathbf{I}_{m_2}) = 0, \tag{42}$$

where $m_1 = m_2 = n/2$ if n is even, $m_1 = (n+1)/2$, $m_2 = (n-1)/2$ if n is an odd integer, and $m_1 + m_2 = n$. The elements of the sub-matrix $\mathbf{B}_o$ are taken from $\mathbf{A}$ at the ijth entry when i and j are odd, while of $\mathbf{B}_e$ with i and j being even.

Equation (42) is significant and very useful, which greatly simplifies the work to obtain the eigenvalues of the symmetric Sturm-Liouville eigenvalue problem. The odd numbered and even numbered eigenvalues are, respectively, solved from

$$\mathrm{Det}(\mathbf{B}_o - \lambda \mathbf{I}_{m_1}) = 0, \quad \mathrm{Det}(\mathbf{B}_e - \lambda \mathbf{I}_{m_2}) = 0. \tag{43}$$

By observing (36)–(39) and Table 1 we can further assert the following.

Theorem 2. *If n is an odd number, the odd eigenvalues of this n-dimensional matrix eigenvalue problem are succeeded by the odd eigenvalues in the next $(n+1)$-dimensional matrix eigenvalue problem. Conversely, if n is an even number, the even eigenvalues of this n-dimensional matrix eigenvalue problem are succeeded by the even eigenvalues in the next $(n+1)$-dimensional matrix eigenvalue problem.*

In each problem to find n eigenvalues, four operations exist: succeeding, improving, creating and transmitting. For instance, when $n = 6$, it succeeds the eigenvalues λ_1, λ_3 and λ_5 from $n = 5$ as shown in Table 1, improves the accuracies of λ_2 and λ_4, creates a new eigenvalue λ_6, and then transmits λ_2, λ_4 and λ_6 to the eigenvalue problem with $n = 7$ as shown in Table 1. Continuing this process, we can sequentially obtain higher-index eigenvalues in closed-form.

Besides Example 1 given in (31), we give:

Example 2.

$$-\Big[\frac{y'(x)}{2+\cos(2\pi x)}\Big]' + (x-0.5)^2 y(x) = \lambda(2+\cos(2\pi x))y(x), \ y'(0) = y'(1) = 0, \tag{44}$$

Example 3.

$$-y''(x) + e_0 \cos(2x) y(x) = \lambda y(x), \ y(0) = y(\pi) = 0. \tag{45}$$

For Example 2, we take $B_k(x) = \cos(k\pi x)$, $k \geqslant 0$, and for Example 3, we take $\varphi_k(x) = \sqrt{2}\sin(kx)/\sqrt{\pi}$.

In Table 2 we list the eigenvalues with $n = 8$ for Examples 1–3. The technique to obtain the eigenvalues for $n = 8$ is similar to (40) and (41), but it is applied to $\mathrm{Det}(\mathbf{B}_e - \lambda \mathbf{I}_{m_2}) = 0$. Example 3 is an eigenvalue problem of Mathieu's equation [14], [22]. When e_0 is large, it is hard to obtain the eigenvalues. For $e_0 = 10$ the first eight eigenvalues are given in Table 2.

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
Ex. 1	6.48721	19.3458	47.1565	81.3731	128.376	184.181	278.561	369.422
Ex. 2	0.04162	2.52513	9.98303	22.6672	40.5691	63.8711	139.566	199.730
Ex. 3	−5.78869	2.10058	9.28982	16.6780	25.3022	36.2196	50.1967	65.0018

Table 2. Listing the eigenvalues obtained for Examples 1–3.

Example 3 gives the asymptotic values of higher-index eigenvalues of Mathieu's equation, which are $\lim\limits_{n\to\infty} \lambda_n^{1/2}/n = 1$, see [22].

In order to obtain higher-index eigenvalues we can consider the equations obtained from (43):

$$F(\lambda) := \mathrm{Det}(\mathbf{B}_o - \lambda \mathbf{I}_{m_1}) = \mathrm{Det}(\mathbf{B}_e - \lambda \mathbf{I}_{m_2}) = 0. \tag{46}$$

FTIM [17], separately used to solve these two algebraic equations in (46), is as follows:

(i) Give n, ε, ν, u_0, N, and s_0,

(ii) Do $j = 1, \ldots, n$,

(iii) Do $k = 1, \ldots, N$,

$$t_k = \frac{k-1}{|F(0)|}, \quad u_k = u_{k-1} - (-1)^j \frac{\nu}{(1+t_k)|F(0)|} F(u_{k-1}),$$

if $|F(u_k)| < \varepsilon$, then $\lambda_j = u_k$, set $u_0 = \lambda_j + s_0$ and go to (ii); otherwise go to (iii).

We have applied the FTIM to check the eigenvalues listed in Table 2, which are very close to the ones obtained from the matrix eigenvalue problem. One more example is given as follows:

Example 4.

$$-y''(x) + 4\cos(2x)y(x) = \lambda y(x), \quad y(0) = y(3\pi) = 0, \tag{47}$$

and the first twelve eigenvalues obtained by the FTIM are listed in Table 3.

λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
−1.472	−1.416	−1.385	2.614	3.206	3.675
λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}
6.226	7.708	9.385	11.538	13.817	16.325

Table 3. For Example 4 the first twelve eigenvalues.

6. Numerical algorithm to recover $q(x)$

Now we are going to reconstruct an unknown symmetric potential $q(x)$ of the following Sturm-Liouville problem:

$$
\begin{aligned}
-y''(x) + q(x)y(x) &= \lambda y(x), \quad 0 < x < \pi, \\
y(0) = y(\pi) &= 0,
\end{aligned}
\tag{48}
$$

where $q(x) = q(\pi - x)$ is a symmetric function [7], [10], [9]. Borg [2] has shown that the eigenvalues determine $q(x)$ uniquely provided $q(x) = q(\pi - x)$ for almost all x in $[0, \pi]$.

Before giving a special structure of (48) for $\mathbf{A}$, we begin with the following assumption:

$$
q(x) = q_1 + \sum_{k=2}^{n} 2q_k \cos(2(k-1)x),
\tag{49}
$$

where $q_1, \ldots, q_n$ are unknown constants to be determined from the eigenvalues $\lambda_1, \ldots, \lambda_n$. The symmetry $q(x) = q(\pi - x)$ is automatically satisfied by (49).

Theorem 3. *For the symmetric Sturm-Liouville problem* (48), *the coefficient matrix* $\mathbf{A} = [a_{ij}]$ *is given by*

$$
a_{ij} = a_{ij}^0 + \sum_{k=1}^{n} q_k a_{ij}^k,
\tag{50}
$$

where

$$
a_{ij}^0 = i^2 \delta_{ij}, \quad a_{ij}^k = \begin{cases} 1 & \text{if } i - j = \pm 2(k-1), \\ -1 & \text{if } i + j = 2(k-1), \end{cases}
\tag{51}
$$

and δ_{ij} *is the Kronecker delta. Moreover,* a_{ij} *satisfies properties* (23) *and* (24).

Proof. By inserting (49) for $q(x)$ and (4) for $\varphi_j(x)$ into (10), we can derive (50). Equations (23) and (24) follow from (50) and (51). □

Now we reconstruct the symmetric potential function $q(x)$ by giving some eigenvalues: $\{\lambda_i,\ i = 1, \ldots, n\}$. In order to find the unknown coefficients $\{q_i,\ i = 1, \ldots, n\}$ in (49) we insert n eigenvalues into the characteristic equation, which results in n nonlinear algebraic equations to solve $\{q_i,\ i = 1, \ldots, n\}$:

$$
F_i(\lambda_i) := P_i(\lambda_i) Q_i(\lambda_i) = 0, \quad i = 1, \ldots, n,
\tag{52}
$$

where $P_i(\lambda_i)$ and $Q_i(\lambda_i)$ are, respectively, obtained from $\mathrm{Det}(\mathbf{B}_o - \lambda_i \mathbf{I}_{m_1})$ and $\mathrm{Det}(\mathbf{B}_e - \lambda_i \mathbf{I}_{m_2})$ in (43).

The numerical algorithm is summarized as follows.

(i) Give n, λ_i, $i = 1, \ldots, n$, ε, N, and an initial guess $q_k^0 = 0$,

(ii) Do $j = 0, 1, \ldots, N$,

(iii) Do $k = 1, \ldots, n$,

$$
\begin{aligned}
F_i &= F_i(\lambda_i), \quad i = 1, \ldots, n, \\
J_{ik} &= \partial F_i / \partial q_k, \quad i = 1, \ldots, n, \\
b_{ki} &= [J_{ik}]^{-1}, \\
q_k^{j+1} &= q_k^j - b_{ki} F_i,
\end{aligned}
$$

if $\sqrt{\sum_{i=1}^{n} F_i^2} < \varepsilon$, then stop; otherwise go to (ii).

In the construction of the Jacobian matrix $[J_{ik}]$, we need to compute the differential of a_{ij} with respect to q_k, which is just $\partial a_{ij}/\partial q_k = a_{ij}^k$ by viewing (50). In the appendix we give the explicit form of the Jacobian matrix $[J_{ik}]$.

We consider the following four examples:

▷ Example 1: $q(x) = \cos(2x)$;

▷ Example 2: $q(x) = \dfrac{1}{2 + \cos(6x)}$;

▷ Example 3: $q(x) = \left| x - \dfrac{\pi}{2} \right|$;

▷ Example 4: $q(x) = \begin{cases} 1 & \text{if } \dfrac{\pi}{4} < x < \dfrac{3\pi}{4}, \\ -1 & \text{if } 0 \leqslant x \leqslant \dfrac{\pi}{4},\ \dfrac{3\pi}{4} \leqslant x \leqslant \pi. \end{cases}$

For the above four examples, the numerical algorithm converges within five iterations under $\varepsilon = 10^{-4}$. For Example 1, Figure 1 compares the recovered potential, by specifying $n = 2$ eigenvalues, to the exact one, whose maximum error (ME) is 7.4×10^{-4} as shown in Figure 2. This accuracy is much better than that given in [12]. For Example 2 we employ ($n = 10$) eigenvalues to recover $q(x)$ as shown in Figure 3, where ME $= 8.12 \times 10^{-3}$ as shown in Figure 4.

Then we recover a nonsmooth potential function in Figure 5, where $n = 10$ and ME $= 3.16 \times 10^{-2}$ as shown in Figure 6. The last potential to be recovered is a discontinuous one as shown in Figure 7, where the Gibbs phenomenon appears.

7. Conclusions

The problem of solving the symmetric Sturm-Liouville eigenvalue problem is transformed to the matrix eigenvalue problem with a special $\mathbf{A} := [a_{ij}]$, $a_{ij} = 0$ if $i + j$ is odd. Consequently, all the closed-form eigenvalues can be sequentially obtained due to the special structure of matrix $\mathbf{A}$. We have developed a fictitious time integration method to obtain the higher-index eigenvalues. A quite difficult inverse potential problem of a symmetric inverse Sturm-Liouville problem has been solved by specifying a few eigenvalues. The novel method is highly effective and very accurate to reconstruct the symmetric potential functions with some supplementary data from lower-index eigenvalues. Although for nonsmooth and discontinuous potential functions the recovered results are still acceptable.

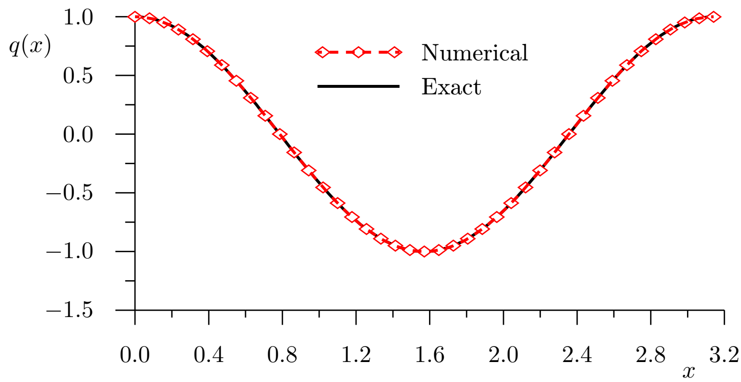

Figure 1. For Example 1 of the inverse potential Sturm-Liouville problems solved by the novel method comparing the numerical solution with the exact one.

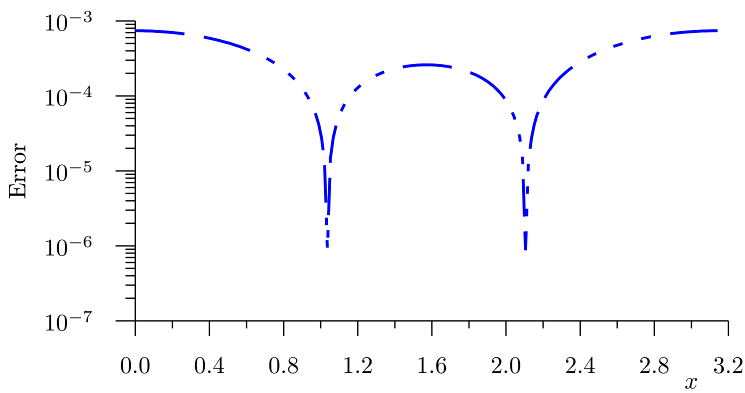

Figure 2. For Example 1 of the inverse potential Sturm-Liouville problems solved by the novel method and showing the absolute errors.

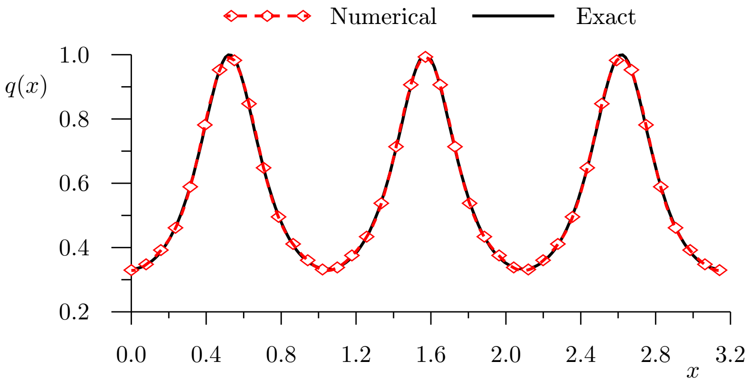

Figure 3. For Example 2 of the inverse potential Sturm-Liouville problems solved by the novel method comparing the numerical solution with the exact one.

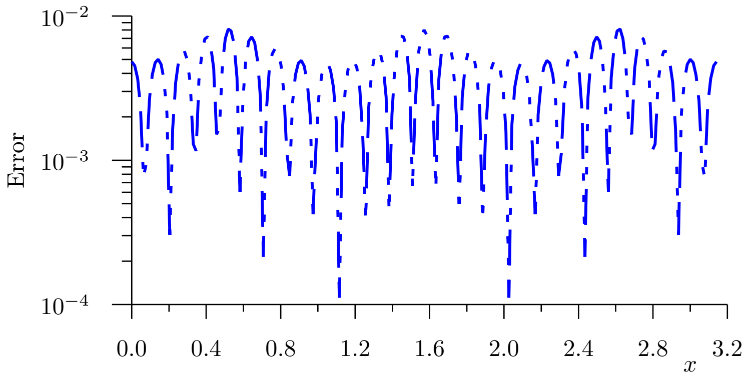

Figure 4. For Example 2 of the inverse potential Sturm-Liouville problems solved by the novel method and showing the absolute errors.

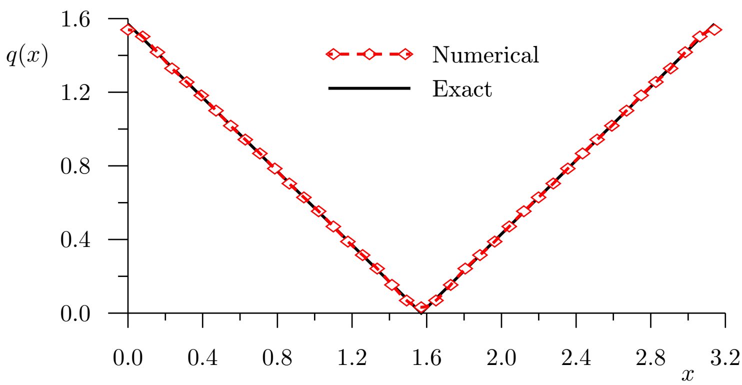

Figure 5. For Example 3 of the inverse potential Sturm-Liouville problems solved by the novel method comparing the numerical solution with the exact one.

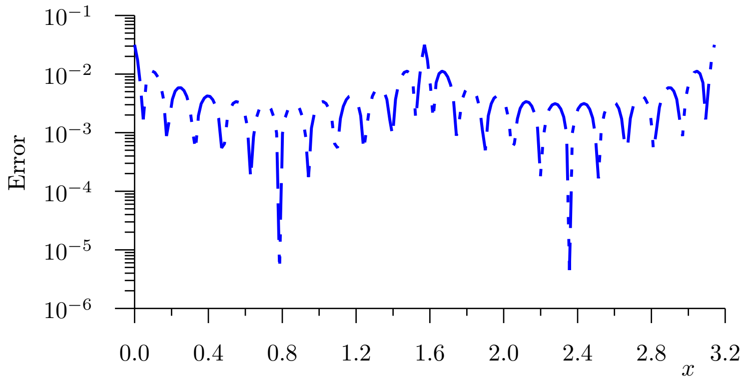

Figure 6. For Example 3 of the inverse potential Sturm-Liouville problems solved by the novel method and showing the absolute errors.

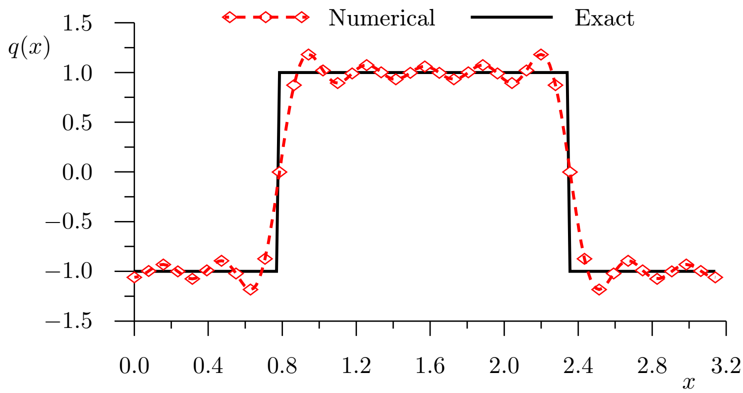

Figure 7. For Example 4 of the inverse potential Sturm-Liouville problems solved by the novel method, comparing the numerical solution with the exact one.

Appendix

In this appendix we derive the explicit form of the Jacobian matrix. However, we only do it for $n = 6$ and other n-dimensional Jacobian matrix can be done similarly. In (50) we take $n = 6$ as a demonstrative example:

(A1)

$$\mathbf{A} = \begin{bmatrix} 1+q_1-q_2 & 0 & q_2-q_3 & 0 & q_3-q_4 & 0 \\ 0 & 4+q_1-q_3 & 0 & q_2-q_4 & 0 & q_3-q_5 \\ q_2-q_3 & 0 & 9+q_1-q_4 & 0 & q_2-q_5 & 0 \\ 0 & q_2-q_4 & 0 & 16+q_1-q_5 & 0 & q_2-q_6 \\ q_3-q_4 & 0 & q_2-q_5 & 0 & 25+q_1-q_6 & 0 \\ 0 & q_3-q_5 & 0 & q_2-q_6 & 0 & 36+q_1 \end{bmatrix}$$

whose characteristic equation reads as

$$\text{(A2)}\quad \begin{vmatrix} a_{11}-\lambda & 0 & a_{13} & 0 & a_{15} & 0 \\ 0 & a_{22}-\lambda & 0 & a_{24} & 0 & a_{26} \\ a_{13} & 0 & a_{33}-\lambda & 0 & a_{35} & 0 \\ 0 & a_{24} & 0 & a_{44}-\lambda & 0 & a_{46} \\ a_{15} & 0 & a_{35} & 0 & a_{55}-\lambda & 0 \\ 0 & a_{26} & 0 & a_{46} & 0 & a_{66}-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} a_{11}-\lambda & a_{13} & a_{15} \\ a_{13} & a_{33}-\lambda & a_{35} \\ a_{15} & a_{35} & a_{55}-\lambda \end{vmatrix} \begin{vmatrix} a_{22}-\lambda & a_{24} & a_{26} \\ a_{24} & a_{44}-\lambda & a_{46} \\ a_{26} & a_{46} & a_{66}-\lambda \end{vmatrix} = 0.$$

From (52) and (A2) we have

$$\text{(A3)}\qquad J_{ik} = \frac{\partial F_i(\lambda_i)}{\partial q_k} = \frac{\partial P_i(\lambda_i)}{\partial q_k} Q_i(\lambda_i) + P_i(\lambda_i) \frac{\partial Q_i(\lambda_i)}{\partial q_k},$$

where

$$\text{(A4)}\quad \frac{\partial P_i(\lambda_i)}{\partial q_k} = \begin{vmatrix} a_{11}^k & a_{13}^k & a_{15}^k \\ a_{13} & a_{33}-\lambda_i & a_{35} \\ a_{15} & a_{35} & a_{55}-\lambda_i \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11}-\lambda_i & a_{13} & a_{15} \\ a_{13}^k & a_{33}^k & a_{35}^k \\ a_{15} & a_{35} & a_{55}-\lambda_i \end{vmatrix} + \begin{vmatrix} a_{11}-\lambda_i & a_{13} & a_{15} \\ a_{13} & a_{33}-\lambda_i & a_{35} \\ a_{15}^k & a_{35}^k & a_{55}^k \end{vmatrix},$$

$$\text{(A5)}\quad \frac{\partial Q_i(\lambda_i)}{\partial q_k} = \begin{vmatrix} a_{22}^k & a_{24}^k & a_{26}^k \\ a_{24} & a_{44}-\lambda_i & a_{46} \\ a_{26} & a_{46} & a_{66}-\lambda_i \end{vmatrix}$$

$$+ \begin{vmatrix} a_{22}-\lambda_i & a_{24} & a_{26} \\ a_{24}^k & a_{44}^k & a_{46}^k \\ a_{26} & a_{46} & a_{66}-\lambda_i \end{vmatrix} + \begin{vmatrix} a_{22}-\lambda_i & a_{24} & a_{26} \\ a_{24} & a_{44}-\lambda_i & a_{46} \\ a_{26}^k & a_{46}^k & a_{66}^k \end{vmatrix}.$$

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Authors' addresses: *Chein-Shan Liu*, Center of Excellence for Ocean Engineering, National Taiwan Ocean University, No. 2, Beining Rd., Zhongzheng Dist., Keelung City 202301, Taiwan, e-mail: `csliu@ntou.edu.tw`; *Botong Li* (corresponding author), School of Mathematics and Physics, University of Science and Technology Beijing, Haidian District, Beijing 100083, P. R. China, e-mail: `libotong@ustb.edu.cn`.