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BILINEAR FRACTIONAL HARDY-TYPE OPERATORS WITH ROUGH KERNELS ON CENTRAL MORREY SPACES WITH VARIABLE EXPONENTS

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Abstract. We introduce a type of n-dimensional bilinear fractional Hardy-type operators with rough kernels and prove the boundedness of these operators and their commutators on central Morrey spaces with variable exponents. Furthermore, the similar definitions and results of multilinear fractional Hardy-type operators with rough kernels are obtained.

Keywords: bilinear fractional Hardy operator; rough kernel; central Morrey space; variable exponent

MSC 2020: 42B20, 42B35

1. Introduction

With the development of partial differential equations, nonlinear analysis and other fields, modern harmonic analysis uses its unique views and methods to classify functions, which leads to various special theories of function spaces, such as Hardy space, Herz space and Morrey space. As early as 1931, Orlicz in [22] began to express and study the theory of variable exponent L^p space. However, it was not until 1991, after the publication of the work of Kováčik and Rákosník (see [18]), the variable function spaces attracted wide attention. In [18], the authors extended classical Lebesgue and Sobolev spaces to variable Lebesgue and Sobolev spaces, and established the properties of these variable function spaces. In addition, variable function spaces are widely used in fluid mechanics and differential equations with non-standard growth conditions (see [14], [23]), which prompted many scholars to do a lot of work around

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variable exponent spaces and get more results, and further explored other variable exponent spaces, such as variable Besov space, variable Triebel-Lizorkin space, variable Hardy space, variable Herz space and so on, see [5], [17], [21], [39], [40], [41].

In recent years, the boundedness of many important operators in harmonic analysis on variable exponent function spaces has been proved. For example, in [2], [15], [26], [27] and [29]–[34], the authors studied the boundedness of some integral operators on variable function spaces through the properties of variable L^p spaces, respectively. On the other hand, since the work of Torchinsky (see [28]), the development of Morrey space has gradually become the mainstream of modern harmonic analysis. Many scholars have studied some properties of central Morrey spaces or central BMO spaces (see [1], [8], [10], [20], [25], [38]), which made the boundedness of operators on central Morrey spaces become more and more perfect. Influenced by the development of variable function space, Fu et al. in [11] extended the classical central Morrey spaces from constant exponent to variable exponent in 2019. They defined central Morrey spaces and λ-central BMO spaces with variable exponent and gave the estimates of singular integral operators and their commutators, which made the further development of the theory of central Morrey spaces. With the concept and properties of variable function spaces, naturally some scholars will study the boundedness of operators on them. From 2019 to 2022, some scholars have studied the boundedness of several classes of operators and commutators on variable λ-central Morrey spaces. For example, Wang et al. discussed the boundedness of multilinear singular integral operators and multilinear fractional integral operators in [35]–[37], expanding the study of [11]. In 2022, Hussain et al. mentioned the boundedness of Hardy operator on variable λ-central Morrey space in [16].

In 1920, Hardy first proposed the concept of one-dimensional Hardy operator, see [13]. Later, more and more people studied and developed the definition and various forms of Hardy-type operators. In 1995, Christ and Grafakos in [3] extended the definition of Hardy operator from one dimension to n dimension, and established the boundedness of n-dimensional Hardy operator on L^p space. Fu et al. in [9] further generalized Hardy operators, introduced n-dimensional fractional Hardy operators, and established the boundedness of their commutators on Lebesgue spaces and homogeneous Herz spaces. Subsequently, the boundedness of Hardy operators attracted many scholars to research, see [7], [8], [12], [15], [16], [19], [24], [42]. Among them, Fu et al. introduced the boundedness of n-dimensional rough Hardy operators and commutators in [12]. Hussain et al. in [16] obtained the estimates of fractional Hardy operators and commutators on variable λ-central Morrey spaces. These two articles give us a lot of inspiration. With the increasing influence of variable exponent function spaces on the development of mathematics, physics, information science and other fields, the boundedness of multilinear operators in variable exponent function space has become a hot topic in recent years. Therefore, this paper attempts to make a breakthrough in analyzing the boundedness of n-dimensional bilinear fractional Hardy operators with rough kernels on central Morrey spaces with variable exponent. On the basis of analyzing the boundedness of bilinear fractional Hardy operators, this paper analyzes the boundedness of adjoint bilinear fractional Hardy operators and their commutators, and then compares the results and methods of bilinear fractional operator boundedness, so as to obtain relevant conclusions of multilinear operators.

Let f_1, f_2 be locally integrable functions in $\mathbb{R}^n$ and $0 \leqslant \beta < 2n$. Suppose that $\mathbb{S}^{n-1}$ denote the unit sphere in $\mathbb{R}^n (n \geqslant 2)$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$. The n-dimensional bilinear fractional Hardy operator with rough kernels and its adjoint operator can be defined by

$$\mathcal{H}_{\vec{\Omega},\beta}(f_1, f_2)(x) = \frac{1}{|x|^{2n-\beta}} \int_{|(t_1,t_2)|\leqslant|x|} \Omega_1(x - t_1)\Omega_2(x - t_2) f_1(t_1) f_2(t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2$$

and

$$\mathcal{H}^*_{\vec{\Omega},\beta}(f_1, f_2)(x) = \frac{1}{|x|^{2n-\beta}} \int_{|(t_1,t_2)|>|x|} \Omega_1(x - t_1)\Omega_2(x - t_2) f_1(t_1) f_2(t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2,$$

where $x \in \mathbb{R}^n \backslash \{0\}$ and $\Omega_1, \Omega_2 \in L^s(\mathbb{S}^{n-1})$ are homogenous of degree zero for $1 \leqslant s < \infty$.

The commutators of the n-dimensional bilinear fractional Hardy operator with rough kernels and its adjoint operator are defined by

$$\mathcal{H}^{\vec{b}}_{\vec{\Omega},\beta}(f_1, f_2)(x) = \sum_{i=1}^{2} \mathcal{H}^{b_i}_{\vec{\Omega},\beta}(f_1, f_2)(x)$$

and

$$\mathcal{H}^{*,\vec{b}}_{\vec{\Omega},\beta}(f_1, f_2)(x) = \sum_{i=1}^{2} \mathcal{H}^{*,b_i}_{\vec{\Omega},\beta}(f_1, f_2)(x),$$

where

$$\begin{aligned}
&\mathcal{H}^{b_i}_{\vec{\Omega},\beta}(f_1, f_2)(x)\\
&\quad= \frac{1}{|x|^{2n-\beta}} \int_{|(t_1,t_2)|\leqslant|x|} \Omega_1(x - t_1)\Omega_2(x - t_2) f_1(t_1) f_2(t_2)(b_i(x) - b_i(t_i))\,\mathrm{d}t_1\,\mathrm{d}t_2,\\
&\mathcal{H}^{*,b_i}_{\vec{\Omega},\beta}(f_1, f_2)(x)\\
&\quad= \int_{|(t_1,t_2)|>|x|} \frac{1}{|t|^{2n-\beta}} \Omega_1(x - t_1)\Omega_2(x - t_2) f_1(t_1) f_2(t_2)(b_i(x) - b_i(t_i))\,\mathrm{d}t_1\,\mathrm{d}t_2,
\end{aligned}$$

$x \in \mathbb{R}^n \setminus \{0\}$, $|t| = |(t_1, t_2)| = \sqrt{t_1^2 + t_2^2}$ and Ω_1, $\Omega_2 \in L^s(\mathbb{S}^{n-1})$ are homogenous of degree zero for $1 \leqslant s < \infty$.

Next, let us explain the outline of this article. In Section 2, we first briefly review some standard notations and lemmas in variable Lebesgue spaces and give the definitions of λ-central BMO spaces and central Morrey spaces with variable exponents. In Section 3, we will establish the boundedness for n-dimensional bilinear fractional Hardy operator with rough kernels and its adjoint operator on central Morrey spaces with variable exponents. In Section 4, we will demonstrate the boundedness of the commutators of n-dimensional bilinear fractional Hardy operator with rough kernels and its adjoint operator on central Morrey spaces with variable exponents. Subsequently the boundedness of multilinear fractional Hardy-type operators and their commutators on central Morrey spaces with variable exponents will also be obtained in Section 5.

2. Function spaces with variable exponents

In this section, we are going to introduce some basic properties of variable Lebesgue spaces and definitions related to the variable exponent function spaces. Throughout this article, we denote by $|B|$ and χ_B the Lebesgue measure and characteristic functions as a measurable set $B \subset \mathbb{R}^n$, respectively.

Given an open set $E \subset \mathbb{R}^n$, and a measurable function $p(\cdot)\colon E \to [1,\infty)$, $L^{p(\cdot)}(E)$ denotes the set of measurable functions f on E such that for some $\lambda > 0$,

$$\int_E \Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)}\,\mathrm{d}x < \infty.$$

This set becomes a Banach function space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\Big\{\lambda > 0\colon \int_E \Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)}\,\mathrm{d}x \leqslant 1\Big\}.$$

These spaces are referred to as variable L^p spaces, since they generalized the standard L^p spaces.

The space $L^{p(\cdot)}_{\rm loc}(E)$ is defined by

$$L^{p(\cdot)}_{\rm loc}(E) := \{f\colon f \in L^{p(\cdot)}(F) \text{ for all compact subsets } F \subset E\}.$$

Define $\mathcal{P}(E)$ to be a set of $p(\cdot)\colon E \to [1,\infty)$ such that

$$p^- = \operatorname{ess\,inf}\{p(x)\colon x \in E\} > 1, \quad p^+ = \operatorname{ess\,sup}\{p(x)\colon x \in E\} < \infty$$

and $p'(\cdot)$ denotes the conjugate exponent of $p(\cdot)$ which satisfies

$$\frac{1}{p'(\cdot)} + \frac{1}{p(\cdot)} = 1.$$

Let $\mathcal{B}(E)$ be the set of $p(\cdot) \in \mathcal{P}(E)$ such that the Hardy-Littlewood maximal operator $\mathcal{M}$ defining

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{B_r \cap E} |f(y)| \, \mathrm{d}y$$

is bounded on $L^{p(\cdot)}(E)$, where $B_r = \{y \in \mathbb{R}^n \colon \, |x-y| < r\}$.

In variable L^p spaces there are some important lemmas as follows.

Lemma 2.1 ([4]). *Given an open set $E \subset \mathbb{R}^n$, if $p(\cdot) \in \mathcal{P}(E)$ and satisfies*

$$|p(x) - p(y)| \leqslant \frac{C}{-\log(|x-y|)}, \quad |x-y| \leqslant \frac{1}{2} \tag{2.1}$$

and

$$|p(x) - p(y)| \leqslant \frac{C}{\log(|x| + \mathrm{e})}, \quad |y| \geqslant |x|, \tag{2.2}$$

then $p(\cdot) \in \mathcal{B}(E)$, that is, the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded on $L^{p(\cdot)}(E)$.

Lemma 2.2 ([18], Generalized Hölder's inequality). *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, then fg is integrable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} |f(x)g(x)| \, \mathrm{d}x \leqslant r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where

$$r_p = 1 + \frac{1}{p^-} - \frac{1}{p^+}.$$

Lemma 2.3 ([17]). *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a constant $0 < \delta < 1$ and a positive constant C such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leqslant C \frac{|B|}{|S|}, \quad \text{and} \quad \frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leqslant C \Big(\frac{|S|}{|B|}\Big)^{\delta}.$$

Remark 2.1. Throughout this paper δ is the same as in Lemma 2.3.

Lemma 2.4 ([17]). *Suppose $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that for all balls B in $\mathbb{R}^n$,*

$$\frac{1}{|B|} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leqslant C.$$

Lemma 2.5 ([6]). *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy conditions* (2.1) *and* (2.2) *in Lemma* 2.1. *Then*

$$\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx \begin{cases} |Q|^{1/p(x)} & \text{if } |Q| \leqslant 2^n \text{ and } x \in Q, \\ |Q|^{1/p(\infty)} & \text{if } |Q| \geqslant 1 \end{cases}$$

for every cube (or ball) $Q \in \mathbb{R}^n$, where $p(\infty) = \lim\limits_{x\to\infty} p(x)$.

Lemma 2.6 ([6]). *Let $p(\cdot), q(\cdot), s(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ be such that*

$$\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}$$

for almost every $x \in \mathbb{R}^n$. Then

$$\|fg\|_{L^{s(\cdot)}(\mathbb{R}^n)} \leqslant C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{q(\cdot)}(\mathbb{R}^n)}$$

for $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{q(\cdot)}(\mathbb{R}^n)$.

Lemma 2.7 ([21]). *If $f \in L^s(\mathbb{R}^n)$ and $g \in L^{q(\cdot)}(\mathbb{R}^n)$, and $1/s + 1/q(\cdot) = 1/p(\cdot)$, then we have*

$$\|fg\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|f\|_{L^s(\mathbb{R}^n)}\|g\|_{L^{q(\cdot)}(\mathbb{R}^n)},$$

where C is a positive constant independent of f and g.

Now we recall that the λ-central BMO space with variable exponent and the central Morrey space with variable exponent in [11] are defined as follows.

Definition 2.1 ([11]). Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\lambda < n^{-1}$. The λ-central BMO space with variable exponent $\mathrm{CBMO}^{p(\cdot),\lambda}(\mathbb{R}^n)$ is defined by

$$\mathrm{CBMO}^{p(\cdot),\lambda}(\mathbb{R}^n) = \{f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)\colon \|f\|_{\mathrm{CBMO}^{p(\cdot),\lambda}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\mathrm{CBMO}^{p(\cdot),\lambda}(\mathbb{R}^n)} = \sup_{R>0} \frac{\|(f - f_{B(0,R)})\chi_{B(0,R)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}.$$

Definition 2.2 ([11]). Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$. The central Morrey space with variable exponent $\dot{\mathcal{B}}^{p(\cdot),\lambda}(\mathbb{R}^n)$ is defined by

$$\dot{\mathcal{B}}^{p(\cdot),\lambda}(\mathbb{R}^n) = \{f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)\colon \|f\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda}(\mathbb{R}^n)} = \sup_{R>0} \frac{\|f\chi_{B(0,R)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}.$$

Remark 2.2. Denote by $\mathrm{CMO}^{q(\cdot),\lambda}(\mathbb{R}^n)$ and $\mathcal{B}^{q(\cdot),\lambda}(\mathbb{R}^n)$ the inhomogeneous versions of the λ-central BMO spaces and central Morrey spaces with variable exponents, which are defined, respectively, by taking the supremum over $R \geqslant 1$ in Definitions 2.1 and 2.2 instead of $R > 0$ there.

Remark 2.3. Our results in this paper remain true for the inhomogeneous versions of λ-central BMO spaces and central Morrey spaces with variable exponents.

3. Bilinear fractional Hardy operator with rough kernels and its adjoint operator

We begin this section by illustrating and proving Theorem 3.1, which present the boundedness of bilinear fractional Hardy operator with rough kernels and its adjoint operator on central Morrey space with variable exponents. Here and subsequently, for simplicity, we write 2^kB with the same center as $B = B(0,R)$ and 2^k multiple of its radius, $C_k = 2^kB \setminus 2^{k-1}B$ for $k \in \mathbb{Z}$, where $\mathbb{Z}$ is the collection of all integers.

Theorem 3.1. *Let $p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and satisfy conditions* (2.1) *and* (2.2) *in Lemma* 2.1 *with $1/p(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot)$. Suppose that $\lambda = \lambda_1 + \lambda_2 + \beta/n$ and $\Omega_i \in L^s(\mathbb{S}^{n-1})$ with $s > p_i'(\cdot)$, $i = 1,2$.*
(1) *If $\lambda + \delta > 0$, then*

$$\|\mathcal{H}_{\vec{\Omega},\beta}(f_1,f_2)\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda}(\mathbb{R}^n)} \leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}.$$

(2) *If $\lambda < 0$ and $\lambda + \delta > 0$, then*

$$\|\mathcal{H}^*_{\vec{\Omega},\beta}(f_1,f_2)\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda}(\mathbb{R}^n)} \leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}.$$

To finish the proof of Theorem 3.1, we need the following lemma.

Lemma 3.1 ([12]). *For $x \in C_k$, $t \in C_i$ and $i \leqslant k$ we have $0 \leqslant |x-t| \leqslant |x| + |2^iB|^{1/n} \leqslant 2\cdot|2^kB|^{1/n}$, and*

$$\int_{C_i} |\Omega(x-t)|^s \,\mathrm{d}t \leqslant \int_0^{2\cdot|2^kB|^{1/n}} \int_{\mathcal{S}^{n-1}} |\Omega(x')|^s \,\mathrm{d}\sigma(x') r^{n-1} \,\mathrm{d}r \leqslant C|2^kB|.$$

Proof of Theorem 3.1. We can appropriately decompose the bilinear fractional Hardy operator in the following form:

(3.1)

$$\begin{aligned}
&|\mathcal{H}_{\vec{\Omega},\beta}(f_1,f_2)(x)\chi_B(x)| \\
&\quad \leqslant \sum_{k=-\infty}^{0} \frac{1}{|x|^{2n-\beta}} \bigg| \int_{|(t_1,t_2)|\leqslant |x|} \Omega_1(x-t_1)\Omega_2(x-t_2) f_1(t_1) f_2(t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2 \bigg| \cdot \chi_{C_k}(x) \\
&\quad \leqslant \sum_{k=-\infty}^{0} |2^k B|^{(\beta-2n)/n} \\
&\qquad \times \bigg| \int_{|t_1|\leqslant |x|} \Omega_1(x-t_1) f_1(t_1)\,\mathrm{d}t_1 \bigg| \cdot \bigg| \int_{|t_2|\leqslant |x|} \Omega_2(x-t_2) f_2(t_2)\,\mathrm{d}t_2 \bigg| \cdot \chi_{C_k}(x) \\
&\quad \leqslant \sum_{k=-\infty}^{0} |2^k B|^{(\beta-2n)/n} \\
&\qquad \times \int_{2^k B} |\Omega_1(x-t_1)||f_1(t_1)|\,\mathrm{d}t_1 \cdot \int_{2^k B} |\Omega_2(x-t_2)||f_2(t_2)|\,\mathrm{d}t_2 \cdot \chi_{C_k}(x).
\end{aligned}$$

The establishment of the above inequality cannot be separated from the following conclusions. First in view of the condition $s > p_1'(\cdot)$ and $\Omega_1 \in L^s(\mathbb{S}^{n-1})$, set $1/p_1'(\cdot) = 1/s + 1/q_1(\cdot)$, then by Lemmas 2.5, 2.7 and 3.1, we have

$$\begin{aligned}
(3.2)\qquad \|\Omega_1(x-\cdot)\chi_{2^k B}(\cdot)\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} &\leqslant C\|\Omega_1(x-\cdot)\chi_{2^k B}(\cdot)\|_{L^s(\mathbb{R}^n)} \|\chi_{2^k B}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\
&= C\bigg(\int_{2^k B} |\Omega_1(x-t)|^s\,\mathrm{d}t\bigg)^{1/s} \|\chi_{2^k B}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\
&\leqslant C|2^k B|^{1/s} \|\chi_{2^k B}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

When $|2^k B| \leqslant 2^n$ and $x \in 2^k B$, by Lemma 2.5 and $1/p_1'(\cdot) = 1/s + 1/q_1(\cdot)$ we have

$$\|\chi_{2^k B}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \approx |2^k B|^{1/q_1(x)} \approx \|\chi_{2^k B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} |2^k B|^{-1/s}.$$

When $|2^k B| \geqslant 1$, we have

$$\|\chi_{2^k B}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \approx |2^k B|^{1/q_1(\infty)} \approx \|\chi_{2^k B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} |2^k B|^{-1/s}.$$

So we obtain

$$\|\chi_{2^k B}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \approx \|\chi_{2^k B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} |2^k B|^{-1/s}.$$

Then, from Hölder's inequality to $p_1(\cdot)$ and $p_1'(\cdot)$ and Lemma 2.4, we can get the following result:

$$
\begin{aligned}
(3.3)\quad \int_{2^k B} |\Omega_1(x-t_1)||f_1(t_1)|\,\mathrm{d}t_1
&\leqslant C\|\Omega_1(x-\cdot)\chi_{2^kB}(\cdot)\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|f_1\chi_{2^kB}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C\|\chi_{2^kB}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|f_1\chi_{2^kB}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}|2^kB|^{1+\lambda_1}\frac{1}{|2^kB|}\|\chi_{2^kB}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{2^kB}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}|2^kB|^{1+\lambda_1}.
\end{aligned}
$$

Repeat the above two steps to obtain

$$
\int_{2^k B} |\Omega_2(x-t_2)||f_2(t_2)|\,\mathrm{d}t_2 \leqslant C\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}|2^kB|^{1+\lambda_2}.
$$

Taking the $L^{p(\cdot)}(\mathbb{R}^n)$ norm on both (3.1) sides, then through Lemma 2.3 and conditions $\lambda=\lambda_1+\lambda_2+\beta/n$ and $\lambda+\delta>0$, we have

$$
\begin{aligned}
&\|\mathcal{H}_{\overrightarrow{\Omega},\beta}(f_1,f_2)\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\quad\leqslant C\sum_{k=-\infty}^{0}|2^kB|^{\beta/n+\lambda_1+\lambda_2}\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\|\chi_{C_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\quad\leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\\
&\qquad\times\sum_{k=-\infty}^{0}\frac{|2^kB|^\lambda}{|B|^\lambda}\frac{\|\chi_{2^kB}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}|B|^\lambda\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\quad\leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\sum_{k=-\infty}^{0}2^{nk(\lambda+\delta)}|B|^\lambda\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\quad\leqslant \|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}|B|^\lambda\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Thus, we have

$$
\|\mathcal{H}_{\overrightarrow{\Omega},\beta}(f_1,f_2)\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda}(\mathbb{R}^n)}\leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}.
$$

In this way, we have proved the boundedness of bilinear fractional Hardy operator on the central Morrey spaces. Next, let us take a look at its adjoint operator. Through the definition of $\mathcal{H}^*_{\overrightarrow{\Omega},\beta}(f_1,f_2)(x)$, and repeating the steps (3.2) and (3.3) in $\chi_{2^jB}(x)$, we have

(3.4)

$$
\begin{aligned}
&|\mathcal{H}^*_{\overrightarrow{\Omega},\beta}(f_1,f_2)(x)\chi_B(x)| \\
&\quad \leqslant \sum_{k=-\infty}^{0}\left|\int_{|(t_1,t_2)|>|x|}\frac{1}{|t|^{2n-\beta}}\Omega_1(x-t_1)\Omega_2(x-t_2)f_1(t_1)f_2(t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2\right|\cdot\chi_{C_k}(x) \\
&\quad \leqslant \sum_{k=-\infty}^{0}\sum_{j=k+1}^{\infty}|2^jB|^{(\beta-2n)/n} \\
&\qquad \times\left|\int_{C_j}\Omega_1(x-t_1)f_1(t_1)\,\mathrm{d}t_1\right|\cdot\left|\int_{C_j}\Omega_2(x-t_2)f_2(t_2)\,\mathrm{d}t_2\right|\cdot\chi_{C_k}(x) \\
&\quad \leqslant \sum_{k=-\infty}^{0}\sum_{j=k+1}^{\infty}|2^jB|^{(\beta-2n)/n} \\
&\qquad \times\int_{2^jB}|\Omega_1(x-t_1)||f_1(t_1)|\,\mathrm{d}t_1\cdot\int_{2^jB}|\Omega_2(x-t_2)||f_2(t_2)|\,\mathrm{d}t_2\cdot\chi_{C_k}(x) \\
&\quad \leqslant C\sum_{k=-\infty}^{0}\sum_{j=k+1}^{\infty}|2^jB|^{(\beta-2n)/n} \\
&\qquad \times\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}|2^jB|^{1+\lambda_1}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}|2^jB|^{1+\lambda_2}\cdot\chi_{C_k}(x) \\
&\quad = C\sum_{k=-\infty}^{0}\sum_{j=k+1}^{\infty}|2^jB|^{\beta/n+\lambda_1+\lambda_2}\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\cdot\chi_{C_k}(x).
\end{aligned}
$$

Taking the $L^{p(\cdot)}(\mathbb{R}^n)$ norm on both (3.4) sides, then through Lemma 2.3 and conditions $\lambda=\lambda_1+\lambda_2+\beta/n$, $\lambda<0$ and $\lambda+\delta>0$, we have

$$
\begin{aligned}
&\|\mathcal{H}^*_{\overrightarrow{\Omega},\beta}(f_1,f_2)\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad \leqslant C\sum_{k=-\infty}^{0}\sum_{j=k+1}^{\infty}|2^jB|^{\beta/n+\lambda_1+\lambda_2}\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\|\chi_{C_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad \leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} \\
&\qquad \times\sum_{k=-\infty}^{0}\sum_{j=k+1}^{\infty}\frac{|2^jB|^{\lambda}}{|2^kB|^{\lambda}}\frac{|2^kB|^{\lambda}}{|B|^{\lambda}}\frac{\|\chi_{2^kB}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}|B|^{\lambda}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad \leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} \\
&\qquad \times\sum_{k=-\infty}^{0}\sum_{j=k+1}^{\infty}2^{n(j-k)\lambda}2^{nk(\lambda+\delta)}|B|^{\lambda}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad \leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}|B|^{\lambda}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Finally, we have

$$\|\mathcal{H}^{*}_{\overrightarrow{\Omega},\beta}(f_1,f_2)\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda}(\mathbb{R}^n)} \leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}.$$

The proof of Theorem 3.1 is completed. □

4. Commutators of bilinear fractional Hardy-type operators with rough kernels

In this section, we will prove the boundedness for commutators of bilinear fractional Hardy operators with rough kernels and their adjoint operators on central Morrey space with variable exponents.

Theorem 4.1. *Let $p(\cdot),p_1(\cdot),p_2(\cdot),p_3(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ and satisfy conditions* (2.1) *and* (2.2) *in Lemma* 2.1 *with $1/p(\cdot)=1/p_1(\cdot)+1/p_2(\cdot)+1/p_3(\cdot)$. Suppose that $\Omega_i\in L^s(\mathbb{S}^{n-1})$ with $s>p'(\cdot)$, $\lambda=\lambda_1+\lambda_2+\lambda_3+\beta/n$, $\lambda+\delta>0$ and $b_i\in\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)$, where $i=1,2$. Then*

$$\|\mathcal{H}^{\overrightarrow{b}}_{\overrightarrow{\Omega},\beta}(f_1,f_2)\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda}(\mathbb{R}^n)} \leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}.$$

Proof. By the definition of the bilinear fractional Hardy commutator with rough kernels, it is easy to see that

$$|\mathcal{H}^{\overrightarrow{b}}_{\overrightarrow{\Omega},\beta}(f_1,f_2)(x)| \leqslant |\mathcal{H}^{b_1}_{\overrightarrow{\Omega},\beta}(f_1,f_2)(x)|+|\mathcal{H}^{b_2}_{\overrightarrow{\Omega},\beta}(f_1,f_2)(x)|.$$

So we just need to prove

$$\|\mathcal{H}^{b_1}_{\overrightarrow{\Omega},\beta}(f_1,f_2)(x)\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda}(\mathbb{R}^n)} \leqslant C\|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)}\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}.$$

We can appropriately decompose the commutators of bilinear fractional Hardy operators in the following form:

$$\begin{aligned}(4.1)\quad &|\mathcal{H}^{b_1}_{\overrightarrow{\Omega},\beta}(f_1,f_2)(x)\chi_B(x)|\\ &\quad=\bigg|\frac{1}{|x|^{2n-\beta}}\int_{|(t_1,t_2)|<|x|}\Omega_1(x-t_1)f_1(t_1)(b_1(x)-b_1(t_1))\\ &\qquad\times\Omega_2(x-t_2)f_2(t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2\chi_B(x)\bigg|\\ &\quad\leqslant\frac{1}{|x|^{2n-\beta}}\bigg|\int_{|t_1|<|x|}\Omega_1(x-t_1)f_1(t_1)(b_1(x)-b_1(t_1))\,\mathrm{d}t_1\bigg|\\ &\qquad\times\bigg|\int_{|t_2|<|x|}\Omega_2(x-t_2)f_2(t_2)\,\mathrm{d}t_2\bigg|\cdot\chi_B(x)\end{aligned}$$

$$\begin{aligned}&\leqslant \sum_{k=-\infty}^{0} |2^kB|^{(\beta-2n)/n} \int_{2^kB} |\Omega_1(x-t_1)||f_1(t_1)||b_1(x)-b_1(t_1)|\,\mathrm{d}t_1\\&\quad\times \int_{2^kB} |\Omega_2(x-t_2)||f_2(t_2)|\,\mathrm{d}t_2\cdot\chi_{C_k}(x).\end{aligned}$$

Before proceeding to the next step, we have noticed that

$$\begin{aligned}(4.2)\qquad &\int_{2^kB} |\Omega_1(x-t_1)||f_1(t_1)||b_1(x)-b_1(t_1)|\,\mathrm{d}t_1\\&\quad\leqslant \int_{2^kB} |\Omega_1(x-t_1)||f_1(t_1)|\,\mathrm{d}t_1\cdot|(b_1(x)-(b_1)_{2^kB})\chi_{2^kB}(x)|\\&\quad\quad+\int_{2^kB} |\Omega_1(x-t_1)||f_1(t_1)||b_1(t_1)-(b_1)_{2^kB}|\,\mathrm{d}t_1\\&\quad=: D+E,\end{aligned}$$

where

$$\begin{aligned}D&=\int_{2^kB} |\Omega_1(x-t_1)||f_1(t_1)|\,\mathrm{d}t_1\cdot|(b_1(x)-(b_1)_{2^kB})\chi_{2^kB}(x)|,\\E&=\int_{2^kB} |\Omega_1(x-t_1)||f_1(t_1)||b_1(t_1)-(b_1)_{2^kB}|\,\mathrm{d}t_1.\end{aligned}$$

First, for the estimation of D, we need to use (3.2) and (3.3).

$$\begin{aligned}D&\leqslant C\|\Omega_1(x-\cdot)\chi_{2^kB}(\cdot)\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|f_1\chi_{2^kB}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\cdot|(b_1(x)-(b_1)_{2^kB})\chi_{2^kB}(x)|\\&\leqslant C\|\chi_{2^kB}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|f_1\chi_{2^kB}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\cdot|(b_1(x)-(b_1)_{2^kB})\chi_{2^kB}(x)|\\&\leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}|2^kB|^{1+\lambda_1}\cdot|(b_1(x)-(b_1)_{2^kB})\chi_{2^kB}(x)|.\end{aligned}$$

Next, we estimate E. In view of the condition $s>p'(\cdot)$, by $1/p(\cdot)=1/p_1(\cdot)+1/p_2(\cdot)+1/p_3(\cdot)$ we can get $1/p_1'(\cdot)=1/p'(\cdot)+1/p_2(\cdot)+1/p_3(\cdot)>1/s+1/p_2(\cdot)+1/p_3(\cdot)$. So we can take $1/p_1'(\cdot)=1/s+1/p_2(\cdot)+1/p_3(\cdot)+1/q(\cdot)$. By (3.2), (3.3) and Lemmas 2.4–2.7 we have

$$\begin{aligned}E&\leqslant C\|\Omega_1(x-\cdot)(b_1(\cdot)-(b_1)_{2^kB})\chi_{2^kB}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|f_1\chi_{2^kB}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\&\leqslant C\|\Omega_1(x-\cdot)\chi_{2^kB}\|_{L^{s}(\mathbb{R}^n)}\|(b_1(\cdot)-(b_1)_{2^kB})\chi_{2^kB}\|_{L^{p_3(\cdot)}(\mathbb{R}^n)}\\&\quad\times\|\chi_{2^kB}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{2^kB}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|f_1\chi_{2^kB}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\&\leqslant C|2^kB|^{1/s}\|(b_1(\cdot)-(b_1)_{2^kB})\chi_{2^kB}\|_{L^{p_3(\cdot)}(\mathbb{R}^n)}\\&\quad\times\|\chi_{2^kB}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{2^kB}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|f_1\chi_{2^kB}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\end{aligned}$$

$$
\begin{aligned}
&\leqslant C|2^kB|^{1/s}\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|\chi_{2^kB}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}|2^kB|^{\lambda_1}\|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)}\\
&\quad\times|2^kB|^{\lambda_3}\|\chi_{2^kB}\|_{L^{p_3(\cdot)}(\mathbb{R}^n)}\|\chi_{2^kB}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{2^kB}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)}\\
&\quad\times|2^kB|^{\lambda_1+\lambda_3}\|\chi_{2^kB}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{2^kB}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)}|2^kB|^{1+\lambda_1+\lambda_3}.
\end{aligned}
$$

So we can get the following $L^{p(\cdot)}(\mathbb{R}^n)$ norm:

$$
\begin{aligned}
&\left\|\int_{2^kB}|\Omega_1(\cdot-t_1)||f_1(t_1)||b_1(\cdot)-b_1(t_1)|\,\mathrm{d}t_1\cdot\chi_{C_k}(\cdot)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\qquad\leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}|2^kB|^{1+\lambda_1}\cdot\|(b_1(\cdot)-(b_1)_{2^kB})\cdot\chi_{C_k}(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\qquad\quad+C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)}|2^kB|^{1+\lambda_1+\lambda_3}\|\chi_{C_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\qquad\leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)}|2^kB|^{1+\lambda_1+\lambda_3}\|\chi_{2^kB}\|_{L^{p(\cdot)}(\mathbb{R}^n)},
\end{aligned}
$$

where

$$
\begin{aligned}
&\|(b_1(\cdot)-(b_1)_{2^kB})\cdot\chi_{C_k}(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\quad\leqslant C\|(b_1-(b_1)_{2^kB})\chi_{2^kB}\|_{L^{p_3(\cdot)}(\mathbb{R}^n)}\|\chi_{2^kB}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{2^kB}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\\
&\quad\leqslant C\|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)}|2^kB|^{\lambda_3}\|\chi_{2^kB}\|_{L^{p_3(\cdot)}(\mathbb{R}^n)}\|\chi_{2^kB}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{2^kB}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\\
&\quad\leqslant C\|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)}|2^kB|^{\lambda_3}\|\chi_{2^kB}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

From (3.2) and (3.3) we have

$$
\begin{aligned}
\int_{2^kB}|\Omega_2(x-t_2)||f_2(t_2)|dt_2&\leqslant C\|\Omega_2(x-\cdot)\chi_{2^kB}(\cdot)\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\|f_2\chi_{2^kB}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C\|\chi_{2^kB}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\|f_2\chi_{2^kB}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}|2^kB|^{1+\lambda_2}.
\end{aligned}
$$

Then, taking the $L^{p(\cdot)}(\mathbb{R}^n)$ norm on both (4.1) sides, we can get

$$
\begin{aligned}
&\|\mathcal{H}^{b_1}_{\vec{\Omega},\beta}(f_1,f_2)\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\quad\leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)}\\
&\qquad\times\sum_{k=-\infty}^{0}|2^kB|^{(\beta-2n)/n}|2^kB|^{1+\lambda_1+\lambda_3}|2^kB|^{1+\lambda_2}\|\chi_{2^kB}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\quad\leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)}\\
&\qquad\times\sum_{k=-\infty}^{0}|2^kB|^{\beta/n+\lambda_1+\lambda_2+\lambda_3}\|\chi_{2^kB}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Available after finishing and by the conditions $\lambda=\lambda_1+\lambda_2+\lambda_3+\beta/n$ and $\lambda+\delta>0$ we have

$$\begin{aligned}
&\|\mathcal{H}^{b_1}_{\overrightarrow{\Omega},\beta}(f_1,f_2)\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\quad\leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)}|B|^{\lambda}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\qquad\times\sum_{k=-\infty}^{0}\frac{|2^kB|^{\lambda}}{|B|^{\lambda}}\frac{\|\chi_{2^kB}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\\
&\quad\leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)}|B|^{\lambda}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\qquad\times\sum_{k=-\infty}^{0}2^{nk(\lambda+\delta)}\\
&\quad\leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)}|B|^{\lambda}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Finally, we can obtain

$$\|\mathcal{H}^{b_1}_{\overrightarrow{\Omega},\beta}(f_1,f_2)\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda}(\mathbb{R}^n)}\leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)}.$$

From the above proof and by the condition $b_i\in\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)$, we have

$$\|\mathcal{H}^{\overrightarrow{b}}_{\overrightarrow{\Omega},\beta}(f_1,f_2)\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda}(\mathbb{R}^n)}\leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}.$$

The proof of Theorem 4.1 is completed. □

Theorem 4.2. *Let $p(\cdot),p_1(\cdot),p_2(\cdot),p_3(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ and satisfy conditions* (2.1) *and* (2.2) *in Lemma* 2.1 *with $1/p(\cdot)=1/p_1(\cdot)+1/p_2(\cdot)+1/p_3(\cdot)$. Suppose that $\Omega_i\in L^s(\mathbb{S}^{n-1})$ with $s>p'(\cdot)$, $\lambda=\lambda_1+\lambda_2+\lambda_3+\beta/n<0$, $\lambda+\delta>0$, $0\leqslant\lambda_3<1/n$ and $b_i\in\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)$, where $i=1,2$. Then*

$$\|\mathcal{H}^{*,\overrightarrow{b}}_{\overrightarrow{\Omega},\beta}(f_1,f_2)\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda}(\mathbb{R}^n)}\leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}.$$

P r o o f. It is easy to see that

$$|\mathcal{H}^{*,\overrightarrow{b}}_{\overrightarrow{\Omega},\beta}(f_1,f_2)(x)|\leqslant|\mathcal{H}^{*,b_1}_{\overrightarrow{\Omega},\beta}(f_1,f_2)(x)|+|\mathcal{H}^{*,b_2}_{\overrightarrow{\Omega},\beta}(f_1,f_2)(x)|.$$

In the same way, we just need to prove

$$\|\mathcal{H}^{*,b_1}_{\overrightarrow{\Omega},\beta}(f_1,f_2)(x)\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda}(\mathbb{R}^n)}\leqslant C\|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)}\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}.$$

In order to express it more clearly, we can decompose $|\mathcal{H}^{*,b_1}_{\vec{\Omega},\beta}(f_1,f_2)\chi_B|$ in the following form:

(4.3)

$$\begin{aligned}
&|\mathcal{H}^{*,b_1}_{\vec{\Omega},\beta}(f_1,f_2)(x)\chi_B(x)|\\
&=\left|\int_{|(t_1,t_2)|>|x|}\frac{\Omega_1(x-t_1)\Omega_2(x-t_2)f_1(t_1)f_2(t_2)(b_1(x)-b_1(t_1))}{|t|^{2n-\beta}}\,\mathrm{d}t_1\,\mathrm{d}t_2\cdot\chi_B(x)\right|\\
&\leqslant\sum_{k=-\infty}^{0}\left|\int_{|(t_1,t_2)|>|x|}\frac{\Omega_1(x-t_1)\Omega_2(x-t_2)f_1(t_1)f_2(t_2)(b_1(x)-b_1(t_1))}{|t|^{2n-\beta}}\,\mathrm{d}t_1\,\mathrm{d}t_2\right|\chi_{C_k}(x)\\
&\leqslant\sum_{k=-\infty}^{0}\sum_{j=k+1}^{\infty}|2^jB|^{(\beta-2n)/n}\int_{2^jB}|\Omega_1(x-t_1)||f_1(t_1)||b_1(x)-b_1(t_1)|\,\mathrm{d}t_1\\
&\quad\times\int_{2^jB}|\Omega_2(x-t_2)||f_2(t_2)|\,\mathrm{d}t_2\cdot\chi_{C_k}(x)\\
&\leqslant C\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\sum_{k=-\infty}^{0}\sum_{j=k+1}^{\infty}|2^jB|^{(\beta-2n)/n}|2^jB|^{1+\lambda_2}\\
&\quad\times\int_{2^jB}|\Omega_1(x-t_1)||f_1(t_1)||b_1(x)-b_1(t_1)|\,\mathrm{d}t_1\cdot\chi_{C_k}(x),
\end{aligned}$$

where

$$\begin{aligned}
\int_{2^jB}|\Omega_2(x-t_2)||f_2(t_2)|\,\mathrm{d}t_2&\leqslant C\|\Omega_2(x-\cdot)\chi_{2^jB}(\cdot)\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\|f_2\chi_{2^jB}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C\|\chi_{2^jB}\|_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\|f_2\chi_{2^jB}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}|2^jB|^{1+\lambda_2}.
\end{aligned}$$

Now we need to estimate the remaining part and we rewrite

$$\begin{aligned}
F=&\sum_{k=-\infty}^{0}\sum_{j=k+1}^{\infty}|2^jB|^{(\beta-2n)/n}|2^jB|^{1+\lambda_2}\\
&\times\int_{2^jB}|\Omega_1(x-t_1)||f_1(t_1)||b_1(x)-b_1(t_1)|\,\mathrm{d}t_1\cdot\chi_{C_k}(x).
\end{aligned}$$

In order to facilitate the operation, we first need to perform a simple decomposition of F:

$$\begin{aligned}
F\leqslant&\sum_{k=-\infty}^{0}\sum_{j=k+1}^{\infty}|2^jB|^{\beta/n-1+\lambda_2}\int_{2^jB}|\Omega_1(x-t_1)||f_1(t_1)||b_1(x)-(b_1)_{2^kB}|\,\mathrm{d}t_1\cdot\chi_{C_k}(x)\\
&+\sum_{k=-\infty}^{0}\sum_{j=k+1}^{\infty}|2^jB|^{\beta/n-1+\lambda_2}\int_{2^jB}|\Omega_1(x-t_1)||f_1(t_1)||b_1(t_1)-(b_1)_{2^kB}|\,\mathrm{d}t_1\cdot\chi_{C_k}(x)\\
&=:F_1+F_2,
\end{aligned}$$

where

$$F_1=\sum_{k=-\infty}^{0}\sum_{j=k+1}^{\infty}|2^jB|^{\beta/n-1+\lambda_2}\int_{2^jB}|\Omega_1(x-t_1)||f_1(t_1)||b_1(x)-(b_1)_{2^kB}|\,\mathrm{d}t_1\cdot\chi_{C_k}(x)$$

and

$$F_2=\sum_{k=-\infty}^{0}\sum_{j=k+1}^{\infty}|2^jB|^{\beta/n-1+\lambda_2}\int_{2^jB}|\Omega_1(x-t_1)||f_1(t_1)||b_1(t_1)-(b_1)_{2^kB}|\,\mathrm{d}t_1\cdot\chi_{C_k}(x).$$

The estimate of F_1 is relatively simple. We can refer to the estimate of D in Theorem 4.1. By using Lemma 2.3, $\lambda<0$ and $\lambda+\delta>0$ we have

$$\begin{aligned}
\|F_1\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)}\\
&\quad\times\sum_{k=-\infty}^{0}\sum_{j=k+1}^{\infty}|2^jB|^{\lambda}\|\chi_{C_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)}\\
&\quad\times\sum_{k=-\infty}^{0}\sum_{j=k+1}^{\infty}\frac{|2^jB|^{\lambda}}{|2^kB|^{\lambda}}\frac{|2^kB|^{\lambda}}{|B|^{\lambda}}\frac{\|\chi_{2^kB}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}|B|^{\lambda}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)}\\
&\quad\times\sum_{k=-\infty}^{0}\sum_{j=k+1}^{\infty}2^{n(j-k)\lambda}2^{kn(\lambda+\delta)}|B|^{\lambda}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)}|B|^{\lambda}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Next we will focus on the estimate of F_2, and we find that

$$\begin{aligned}
F_2 &= \sum_{k=-\infty}^{0}\sum_{j=k+1}^{\infty}|2^jB|^{\beta/n-1+\lambda_2}\\
&\quad\times\int_{2^jB}|\Omega_1(x-t_1)||f_1(t_1)||b_1(t_1)-(b_1)_{2^kB}|\,\mathrm{d}t_1\cdot\chi_{C_k}(x)\\
&\leqslant C\sum_{k=-\infty}^{0}\sum_{j=k+1}^{\infty}|2^jB|^{\beta/n-1+\lambda_2}\\
&\quad\times\int_{2^jB}|\Omega_1(x-t_1)||f_1(t_1)||(b_1)_{2^jB}-(b_1)_{2^kB}|\,\mathrm{d}t_1\cdot\chi_{C_k}(x)\\
&\quad+C\sum_{k=-\infty}^{0}\sum_{j=k+1}^{\infty}|2^jB|^{\beta/n-1+\lambda_2}\\
&\quad\times\int_{2^jB}|\Omega_1(x-t_1)||f_1(t_1)||b_1(t_1)-(b_1)_{2^jB}|\,\mathrm{d}t_1\cdot\chi_{C_k}(x)\\
&=: F_{21}+F_{22},
\end{aligned}$$

where

$$F_{21} = C \sum_{k=-\infty}^{0} \sum_{j=k+1}^{\infty} |2^j B|^{\beta/n-1+\lambda_2} \times \int_{2^j B} |\Omega_1(x-t_1)||f_1(t_1)|\,\mathrm{d}t_1 |(b_1)_{2^j B} - (b_1)_{2^k B}| \cdot \chi_{C_k}(x)$$

and

$$F_{22} = C \sum_{k=-\infty}^{0} \sum_{j=k+1}^{\infty} |2^j B|^{\beta/n-1+\lambda_2} \times \int_{2^j B} |\Omega_1(x-t_1)||f_1(t_1)||b_1(t_1) - (b_1)_{2^j B}|\,\mathrm{d}t_1 \cdot \chi_{C_k}(x).$$

For F_{22}, we can refer to the estimate of E in Theorem 4.1. Noting that $1/p_1'(\cdot) = 1/s + 1/p_2(\cdot) + 1/p_3(\cdot) + 1/q(\cdot)$ and by using Lemmas 2.4–2.7 we can get

$$\begin{aligned}
F_{22} &\leqslant C \sum_{k=-\infty}^{0} \sum_{j=k+1}^{\infty} |2^j B|^{\beta/n-1+\lambda_2} \|\Omega_1(x-\cdot)(b_1(\cdot) - (b_1)_{2^j B})\chi_{2^j B}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \\
&\quad \times \|f_1 \chi_{2^j B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \cdot \chi_{C_k}(x) \\
&\leqslant C \sum_{k=-\infty}^{0} \sum_{j=k+1}^{\infty} |2^j B|^{\beta/n-1+\lambda_2} |2^j B|^{1/s} \|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \|\chi_{2^j B}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} |2^j B|^{\lambda_1} \\
&\quad \times \|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)} |2^j B|^{\lambda_3} \|\chi_{2^j B}\|_{L^{p_3(\cdot)}(\mathbb{R}^n)} \|\chi_{2^j B}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\
&\quad \times \|\chi_{2^j B}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \cdot \chi_{C_k}(x) \\
&\leqslant C \|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)} \sum_{k=-\infty}^{0} \sum_{j=k+1}^{\infty} |2^j B|^{\beta/n+\lambda_1+\lambda_2+\lambda_3} \cdot \chi_{C_k}(x).
\end{aligned}$$

Taking the $L^{p(\cdot)}(\mathbb{R}^n)$ norm on both sides of F_{22} and using Lemma 2.3, $\beta/n + \lambda_1 + \lambda_2 + \lambda_3 = \lambda < 0$ and $\lambda + \delta > 0$:

$$\begin{aligned}
\|F_{22}\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leqslant C \|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)} \\
&\quad \times \sum_{k=-\infty}^{0} \sum_{j=k+1}^{\infty} |2^j B|^{\beta/n+\lambda_1+\lambda_2+\lambda_3} \|\chi_{C_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\leqslant C \|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)} \\
&\quad \times \sum_{k=-\infty}^{0} \sum_{j=k+1}^{\infty} 2^{n(j-k)\lambda} 2^{kn(\lambda+\delta)} |B|^{\lambda} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\leqslant C \|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)} |B|^{\lambda} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Let us focus on F_{21}. We can find

$$\begin{aligned}F_{21} &\leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\sum_{k=-\infty}^{0}\sum_{j=k+1}^{\infty}|2^jB|^{\beta/n+\lambda_2+\lambda_1}\cdot|(b_1)_{2^jB}-(b_1)_{2^kB}|\cdot\chi_{C_k}(x)\\ &\leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)}\\ &\quad\times\sum_{k=-\infty}^{0}\sum_{j=k+1}^{\infty}(j-k)|2^jB|^{\beta/n+\lambda_1+\lambda_2+\lambda_3}\cdot\chi_{C_k}(x),\end{aligned}$$

where we need the following boundedness for $|(b_1)_{2^jB}-(b_1)_{2^kB}|$:

$$\begin{aligned}&|(b_1)_{2^jB}-(b_1)_{2^kB}|\\ &\quad\leqslant\sum_{m=k}^{j-1}|(b_1)_{2^{m+1}B}-(b_1)_{2^mB}|=\sum_{m=k}^{j-1}\frac{1}{|2^mB|}\int_{2^mB}|(b_1)_{2^{m+1}B}-b_1(x)|\,\mathrm{d}x\\ &\quad=C\sum_{m=k}^{j-1}\frac{1}{|2^{m+1}B|}\int_{2^{m+1}B}|(b_1)_{2^{m+1}B}-b_1(x)|\,\mathrm{d}x\\ &\quad\leqslant C\sum_{m=k}^{j-1}\frac{1}{|2^{m+1}B|}\|(b_1-(b_1)_{2^{m+1}B})\chi_{2^{m+1}B}\|_{L^{p_3(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{m+1}B}\|_{L^{p_3'(\cdot)}(\mathbb{R}^n)}\\ &\quad\leqslant C\sum_{m=k}^{j-1}\|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)}|2^{m+1}B|^{\lambda_3}\frac{1}{|2^{m+1}B|}\\ &\qquad\times\|\chi_{2^{m+1}B}\|_{L^{p_3(\cdot)}(\mathbb{R}^n)}\|\chi_{2^{m+1}B}\|_{L^{p_3'(\cdot)}(\mathbb{R}^n)}\\ &\quad\leqslant C\sum_{m=k}^{j-1}\|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)}|2^{m+1}B|^{\lambda_3}\\ &\quad\leqslant C(j-k)\|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)}|2^jB|^{\lambda_3}.\end{aligned}$$

In the same way and under the same conditions, we can get

$$\|F_{21}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)}|B|^{\lambda}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Then, taking the $L^{p(\cdot)}(\mathbb{R}^n)$ norm on both sides of (4.3) with the help of the above results and using Minkowski's inequality, we can get

$$\begin{aligned}&\|\mathcal{H}^{*,b_1}_{\vec{\Omega},\beta}(f_1,f_2)\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\quad\leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)}|B|^{\lambda}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Thus,

$$\|\mathcal{H}^{*,b_1}_{\vec{\Omega},\beta}(f_1,f_2)\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda}(\mathbb{R}^n)}\leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}\|b_1\|_{\mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)}.$$

Therefore, by the condition $b_i \in \mathrm{CBMO}^{p_3(\cdot),\lambda_3}(\mathbb{R}^n)$, we have

$$\|\mathcal{H}^{*,\overrightarrow{b}}_{\overrightarrow{\Omega},\beta}(f_1, f_2)\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda}(\mathbb{R}^n)} \leqslant C\|f_1\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{\dot{\mathcal{B}}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)}.$$

The proof of Theorem 4.2 is completed. □

5. Multilinear fractional Hardy-type operators with rough kernels

At the end of this paper, we want to give the boundedness of multilinear n-dimensional fractional Hardy operator with rough kernels and its related operators.

Let f_i be locally integrable functions in $\mathbb{R}^n$ and $0 \leqslant \beta < mn$, $i = 1, \ldots, m$. The multilinear n-dimensional fractional Hardy operator with rough kernels and its adjoint operator can be defined by

$$\begin{aligned}\mathcal{H}_{\overrightarrow{\Omega},\beta}&(f_1, \ldots, f_m)(x)\\ &= \frac{1}{|x|^{mn-\beta}} \int_{|(t_1,\ldots,t_m)|\leqslant|x|} \Omega_1(x-t_1)\ldots\Omega_m(x-t_m)f_1(t_1)\ldots f_m(t_m)\,\mathrm{d}t_1\ldots\,\mathrm{d}t_m\end{aligned}$$

and

$$\begin{aligned}\mathcal{H}^{*}_{\overrightarrow{\Omega},\beta}&(f_1, \ldots, f_m)(x)\\ &= \frac{1}{|x|^{mn-\beta}} \int_{|(t_1,\ldots,t_m)|>|x|} \Omega_1(x-t_1)\ldots\Omega_m(x-t_m)f_1(t_1)\ldots f_m(t_m)\,\mathrm{d}t_1\ldots\,\mathrm{d}t_m,\end{aligned}$$

where $x \in \mathbb{R}^n \setminus \{0\}$ and $\Omega_i \in L^s(\mathbb{S}^{n-1})(i = 1, \ldots, m)$ are homogenous of degree zero for $1 \leqslant s < \infty$.

The commutators of the multilinear n-dimensional fractional Hardy operator with rough kernels and its adjoint operator can be defined by

$$\mathcal{H}^{\overrightarrow{b}}_{\overrightarrow{\Omega},\beta}(f_1, \ldots, f_m)(x) = \sum_{i=1}^{m} \mathcal{H}^{b_i}_{\overrightarrow{\Omega},\beta}(f_1, \ldots, f_m)(x)$$

and

$$\mathcal{H}^{*,\overrightarrow{b}}_{\overrightarrow{\Omega},\beta}(f_1, \ldots, f_m)(x) = \sum_{i=1}^{m} \mathcal{H}^{*,b_i}_{\overrightarrow{\Omega},\beta}(f_1, \ldots, f_m)(x),$$

where

$$\begin{aligned}\mathcal{H}^{b_i}_{\overrightarrow{\Omega},\beta}(f_1, \ldots, f_m)(x) &= \frac{1}{|x|^{mn-\beta}} \int_{|(t_1,\ldots,t_m)|\leqslant|x|} \Omega_1(x-t_1)\ldots\Omega_m(x-t_m)\\ &\quad\times f_1(t_1)\ldots f_m(t_m)(b_i(x)-b_i(t_i))\,\mathrm{d}t_1\ldots\,\mathrm{d}t_m,\\ \mathcal{H}^{*,b_i}_{\overrightarrow{\Omega},\beta}(f_1, \ldots, f_m)(x) &= \int_{|(t_1,\ldots,t_m)|>|x|} \frac{1}{|t|^{mn-\beta}}\Omega_1(x-t_1)\ldots\Omega_m(x-t_m)\\ &\quad\times f_1(t_1)\ldots f_m(t_m)(b_i(x)-b_i(t_i))\,\mathrm{d}t_1\ldots\,\mathrm{d}t_m,\end{aligned}$$

$x \in \mathbb{R}^n \setminus \{0\}$, $|t| = |(t_1, t_2, \ldots, t_m)| = \sqrt{t_1^2 + t_2^2 + \ldots + t_m^2}$ and $\Omega_i \in L^s(\mathbb{S}^{n-1})$ $(i = 1, \ldots, m)$ are homogenous of degree zero for $1 \leqslant s < \infty$.

Next, we refer to the previous bilinear content and give two theorems about the boundedness of the multilinear Hardy-type operators and their commutators.

Theorem 5.1. *For $i = 1, 2, \ldots, m$, let $p(\cdot), p_i(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and satisfy conditions (2.1) and (2.2) in Lemma 2.1 with $1/p(\cdot) = \sum\limits_{i=1}^{m} 1/p_i(\cdot)$. Suppose that $\Omega_i \in L^s(\mathbb{S}^{n-1})$ with $s > p'(\cdot)$ and $\lambda = \sum\limits_{i=1}^{m} \lambda_i + \beta/n$.*

(1) *If $\lambda + \delta > 0$, then*

$$\|\mathcal{H}_{\vec{\Omega},\beta}(f_1, \ldots, f_m)\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda}(\mathbb{R}^n)} \leqslant C \prod_{i=1}^{m} \|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}.$$

(2) *If $\lambda < 0$ and $\lambda + \delta > 0$, then*

$$\|\mathcal{H}^{*}_{\vec{\Omega},\beta}(f_1, \ldots, f_m)\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda}(\mathbb{R}^n)} \leqslant C \prod_{i=1}^{m} \|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}.$$

Theorem 5.2. *Let $1 < p(\cdot) < \infty$, $p(\cdot), p_i(\cdot), p_{m+1}(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and satisfy conditions (2.1) and (2.2) in Lemma 2.1 with $1/p(\cdot) = \sum\limits_{j=1}^{m+1} 1/p_j(\cdot)$. Suppose that $\Omega_i \in L^s(\mathbb{S}^{n-1})$ with $s > p'(\cdot)$, $\lambda = \sum\limits_{j=1}^{m+1} \lambda_j + \beta/n$ and $b_i \in \mathrm{CBMO}^{p_{m+1}(\cdot),\lambda_{m+1}}(\mathbb{R}^n)$, where $i = 1, 2, \ldots, m$.*

(1) *If $\lambda + \delta > 0$, then*

$$\|\mathcal{H}^{\vec{b}}_{\vec{\Omega},\beta}(f_1, \ldots, f_m)\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda}(\mathbb{R}^n)} \leqslant C \prod_{i=1}^{m} \|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}.$$

(2) *If $\lambda + \delta > 0$, $\lambda < 0$ and $0 \leqslant \lambda_{m+1} < n^{-1}$, then*

$$\|\mathcal{H}^{*,\vec{b}}_{\vec{\Omega},\beta}(f_1, \ldots, f_m)\|_{\dot{\mathcal{B}}^{p(\cdot),\lambda}(\mathbb{R}^n)} \leqslant C \prod_{i=1}^{m} \|f_i\|_{\dot{\mathcal{B}}^{p_i(\cdot),\lambda_i}(\mathbb{R}^n)}.$$

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References

[1] *J. Alvarez, J. Lakey, M. Guzmán-Partida*: Spaces of bounded λ-central mean oscillation, Morrey spaces, and λ-central Carleson measures. Collect. Math. *51* (2000), 1–47. zbl MR

[2] *C. Capone, D. Cruz-Uribe, A. Fiorenza*: The fractional maximal operator and fractional integrals on variable L^p spaces. Rev. Mat. Iberoam. *23* (2007), 743–770. zbl MR doi

[3] *M. Christ, L. Grafakos*: Best constants for two nonconvolution inequalities. Proc. Am. Math. Soc. *123* (1995), 1687–1693. zbl MR doi

[4] *D. Cruz-Uribe, A. Fiorenza, C. J. Neugebauer*: The maximal function on variable L^p spaces. Ann. Acad. Sci. Fenn., Math. *28* (2003), 223–238. zbl MR

[5] *D. Cruz-Uribe, D. Wang*: Variable Hardy spaces. Indiana Univ. Math. J. *63* (2014), 447–493. zbl MR doi

[6] *L. Diening, P. Harjulehto, P. Hästö, M. Růžička*: Lebesgue and Sobolev Spaces with Variable Exponents. Lecture Notes in Mathematics 2017. Springer, Berlin, 2011. zbl MR doi

[7] *Z. W. Fu, S. L. Gong, S. Z. Lu, W. Yuan*: Weighted multilinear Hardy operators and commutators. Forum Math. *27* (2015), 2825–2851. zbl MR doi

[8] *Z. Fu, Y. Lin*: λ-central BMO estimates for commutators of higher dimensional fractional Hardy operators. Acta Math. Sin., Chin. Ser. *53* (2010), 925–932. (In Chinese.) zbl MR doi

[9] *Z. Fu, Z. Liu, S. Lu, H. Wang*: Characterization for commutators of n-dimensional fractional Hardy operators. Sci. China, Ser. A *50* (2007), 1418–1426. zbl MR doi

[10] *Z. Fu, S. Lu, S. Shi*: Two characterizations of central BMO space via the commutators of Hardy operators. Forum Math. *33* (2021), 505–529. zbl MR doi

[11] *Z. Fu, S. Lu, H. Wang, L. Wang*: Singular integral operators with rough kernels on central Morrey spaces with variable exponent. Ann. Acad. Sci. Fenn., Math. *44* (2019), 505–522. zbl MR doi

[12] *Z. Fu, S. Lu, F. Zhao*: Commutators of n-dimensional rough Hardy operators. Sci. China, Math. *54* (2011), 95–104. zbl MR doi

[13] *G. H. Hardy*: Note on a theorem of Hilbert. Math. Z. *6* (1920), 314–317. zbl MR doi

[14] *P. Harjulehto, P. Hästö, Ú. V. Lê, M. Nuortio*: Overview of differential equations with non-standard growth. Nonlinear Anal., Theory Methods Appl., Ser. A *72* (2010), 4551–4574. zbl MR doi

[15] *K.-P. Ho*: Hardy's inequality on Hardy-Morrey spaces with variable exponents. Mediterr. J. Math. *14* (2017), Article ID 79, 19 pages. zbl MR doi

[16] *A. Hussain, M. Asim, F. Jarad*: Variable λ-central Morrey space estimates for the fractional Hardy operators and commutators. J. Math. *2022* (2022), Article ID 5855068, 12 pages. MR doi

[17] *M. Izuki*: Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization. Anal. Math. *36* (2010), 33–50. zbl MR doi

[18] *O. Kováčik, J. Rákosník*: On spaces $L^{p(x)}$ and $W^{k,p(x)}$. Czech. Math. J. *41* (1991), 592–618. zbl MR doi

[19] *Y. Mizuta, A. Nekvinda, T. Shimomura*: Optimal estimates for the fractional Hardy operator. Stud. Math. *227* (2015), 1–19. zbl MR doi

[20] *Y. Mizuta, T. Ohno, T. Shimomura*: Boundedness of maximal operators and Sobolev's theorem for non-homogeneous central Morrey spaces of variable exponent. Hokkaido Math. J. *44* (2015), 185–201. zbl MR doi

[21] *E. Nakai, Y. Sawano*: Hardy spaces with variable exponents and generalized Campanato spaces. J. Funct. Anal. *262* (2012), 3665–3748. zbl MR doi

[22] *W. Orlicz*: Über konjugierte Exponentenfolgen. Stud. Math. *3* (1931), 200–212. (In German.) zbl doi

[23] *M. Růžička*: Electrorheological Fluids: Modeling and Mathematical Theory. Lecture Notes in Mathematics 1748. Springer, Berlin, 2000. zbl MR doi

[24] *S. Shi, Z. Fu, S. Lu*: On the compactness of commutators of Hardy operators. Pac. J. Math. *307* (2020), 239–256. zbl MR doi

[25] *S. Shi, S. Lu*: Characterization of the central Campanato space via the commutator operator of Hardy type. J. Math. Anal. Appl. *429* (2015), 713–732. zbl MR doi

[26] *J. Tan, Z. Liu*: Some boundedness of homogeneous fractional integrals on variable exponent function spaces. Acta Math. Sin., Chin. Ser. *58* (2015), 309–320. (In Chinese.) zbl MR doi

[27] *J. Tan, Z. Liu, J. Zhao*: On some multilinear commutators in variable Lebesgue spaces. J. Math. Inequal. *11* (2017), 715–734. zbl MR doi

[28] *A. Torchinsky*: Real-Variable Methods in Harmonic Analysis. Pure and Applied Mathematics 123. Academic Press, New York, 1986. zbl MR

[29] *H. Wang*: Commutators of Marcinkiewicz integrals on Herz spaces with variable exponent. Czech. Math. J. *66* (2016), 251–269. zbl MR doi

[30] *H. Wang*: The continuity of commutators on Herz-type Hardy spaces with variable exponent. Kyoto J. Math. *56* (2016), 559–573. zbl MR doi

[31] *H. Wang*: Commutators of singular integral operator on Herz-type Hardy spaces with variable exponent. J. Korean Math. Soc. *54* (2017), 713–732. zbl MR doi

[32] *H. Wang, Z. Fu*: Estimates of commutators on Herz-type spaces with variable exponent and applications. Banach J. Math. Anal. *15* (2021), Article ID 36, 27 pages. zbl MR doi

[33] *H. Wang, F. Liao*: Boundedness of singular integral operators on Herz-Morrey spaces with variable exponent. Chin. Ann. Math., Ser. B *41* (2020), 99–116. zbl MR doi

[34] *H. Wang, Z. Liu*: Boundedness of singular integral operators on weak Herz type spaces with variable exponent. Ann. Funct. Anal. *11* (2020), 1108–1125. zbl MR doi

[35] *H. Wang, J. Xu*: Multilinear fractional integral operators on central Morrey spaces with variable exponent. J. Inequal. Appl. *2019* (2019), Article ID 311, 23 pages. zbl MR doi

[36] *H. Wang, J. Xu, J. Tan*: Boundedness of multilinear singular integrals on central Morrey spaces with variable exponents. Front. Math. China *15* (2020), 1011–1034. zbl MR doi

[37] *L. Wang*: Multilinear Calderón-Zygmund operators and their commutators on central Morrey spaces with variable exponent. Bull. Korean Math. Soc. *57* (2020), 1427–1449. zbl MR doi

[38] *Q. Y. Wu, Z. W. Fu*: Weighted p-adic Hardy operators and their commutators on p-adic central Morrey spaces. Bull. Malays. Math. Sci. Soc. (2) *40* (2017), 635–654. zbl MR doi

[39] *J. Xu*: Variable Besov and Triebel-Lizorkin spaces. Ann. Acad. Sci. Fenn., Math. *33* (2008), 511–522. zbl MR

[40] *D. Yang, W. Yuan, C. Zhuo*: Triebel-Lizorkin type spaces with variable exponents. Banach J. Math. Anal. *9* (2015), 146–202. zbl MR doi

[41] *D. Yang, C. Zhuo, W. Yuan*: Besov-type spaces with variable smoothness and integrability. J. Funct. Anal. *269* (2015), 1840–1898. zbl MR doi

[42] *F. Zhao, Z. Fu, S. Lu*: M_p weights for bilinear Hardy operators on $\mathbb{R}^n$. Collect. Math. *65* (2014), 87–102. zbl MR doi

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