Basudeb Dhara; Kalyan Singh

b-generalized skew derivations acting on Lie ideals in prime rings

Czechoslovak Mathematical Journal, Vol. 74 (2024), No. 2, 575–597

Persistent URL: `http://dml.cz/dmlcz/152459`

Czechoslovak Mathematical Journal, 74 (149) (2024), 575–597

b-GENERALIZED SKEW DERIVATIONS ACTING ON LIE IDEALS IN PRIME RINGS

Basudeb Dhara, Belda, Kalyan Singh, Kolkata

Received November 15, 2023. Published online June 6, 2024.

Abstract. Let R be any noncommutative prime ring of $\operatorname{char}(R) \neq 2, 3$, L a noncentral Lie ideal of R and F, G two nonzero b-generalized skew derivations of R. Suppose that

$$[F(u), u]G(u) = 0$$

for all $u \in L$. Then at least one of the following conclusions holds:
(1) $F(x) = \lambda x$ for all $x \in R$ and for some $\lambda \in C$, where C is the extended centroid of R;
(2) $R \subseteq M_2(K)$, the algebra of 2×2 matrices over a field K.

Keywords: derivation; b-generalized derivation; b-generalized skew derivation; Lie ideal; prime ring

MSC 2020: 16W25, 16N60

1. Introduction

At the very beginning of this article let us take a quick look to some notations and definitions. Throughout our present article, the prime ring, Lie ideals, derivations and b-generalized skew derivations play a crucial role. Therefore, we would like to denote R as a prime ring, $Z(R)$ the center of R, Q_r its right Martindale quotient ring. The center of Q_r denoted by $Z(Q_r) = C$ is called extended centroid of R. Here L is a Lie ideal of R satisfying the properties that L must be an additive subgroup of R along with $[L, R] \subseteq L$. A linear mapping $d\colon R \to R$ is a derivation on R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. A derivation of the form $d(x) = [p, x]$ for all $x \in R$ and for some $p \in R$ is called inner derivation induced by an element p in R. We re-call a linear mapping F from R to R a generalized derivation on R if there

This work is supported by a grant from Science and Engineering Research Board (SERB), New Delhi, India. Grant No. is MTR/2022/000568.

exists a derivation d on R such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. The map $F(x) = px + xq$ for all $x \in R$ and for some $p, q \in R$ is a very simple type of generalized derivation, which is called inner. Any derivation or generalized derivation which is not inner is known as outer derivation or outer generalized derivation, respectively.

A linear mapping $F\colon R \to Q_r$ is said to be a b-generalized derivation on R if there exists a derivation d on R such that

$$F(xy) = F(x)y + bxd(y)$$

for all $x, y \in R$ and $b \in Q_r$. By the definition, we see that generalized derivation is a 1-generalized derivation and the map $F(x) = ax + bxc$ for all $x \in R$, where $a, b, c \in Q_r$, is an example of b-generalized derivation, which is called inner b-generalized derivation.

Let α be an automorphism of R. A linear map $d\colon R \to Q_r$, associated with $\alpha \in \mathrm{Aut}(R)$, is called a skew derivation on R if

$$d(xy) = d(x)y + \alpha(x)d(y)$$

for all $x, y \in R$.

A linear map $F\colon R \to R$ is said to be a generalized skew derivation on R if there exists a skew derivation d of R associated with $\alpha \in \mathrm{Aut}(R)$ such that

$$F(xy) = F(x)y + \alpha(x)d(y)$$

for all $x, y \in R$. Here d is said to be an associated skew derivation of F and α is called an associated automorphism of G. The map $F(x) = ax - \alpha(x)b$ is an example of generalized skew derivation. This is called inner generalized skew derivation of R. There are many research articles which have investigated generalized skew derivations from various points of view.

De Filippis in [6] introduced a new map called b-generalized skew derivation, which is a common generalization of all above maps. Let $b \in Q_r$. A linear map $F\colon R \to Q_r$ is said to be a b-generalized skew derivation of R if there exists a skew derivation d of R such that

$$F(xy) = F(x)y + b\alpha(x)d(y)$$

holds for all $x, y \in R$. The map $F(x) = ax + b\alpha(x)c$ is an example of b-generalized skew derivation of R, which is called inner b-generalized skew derivation of R.

Let R be a prime ring with center $Z(R)$, L a noncentral Lie ideal of R and d be a derivation of R. A well known result of Posner (see [18]) states that if commutator $[d(x), x] \in Z(R)$ for all $x \in R$, then either $d = 0$ or R is commutative. There are many generalizations of Posner result in literature in many directions by a number of authors. Lee in [15] and then Lanski in [13] extended Posner's result to Lie ideal in prime rings.

Vukman in [19] proved that if R is a prime ring of $\mathrm{char}(R) \neq 2$ such that $[d(x), x]d(x) = 0$ for all $x \in R$, then $d = 0$.

Then De Filippis et al. in [8] studied the above result replacing d with a generalized derivation F of R. More precisely, they proved the following result:

Let R be a prime ring of $\mathrm{char}(R) \neq 2$, L a noncentral Lie ideal of R and F a nonzero generalized derivation of R. If $[F(u), u]F(u) = 0$, for all $u \in L$, then one of the following holds:

(1) there exists $\eta \in C$ such that $F(x) = \eta x$ for all $x \in R$;

(2) R satisfies the standard identity s_4 and there exist $a \in U$ and $\eta \in C$ such that $F(x) = ax + xa + \eta x$ for all $x \in R$, where U is the Utumi quotient ring of R.

Recently, in [1], Albas et al. studied a situation considering two different generalized derivations F and G. They obtained the following:

Let R be a prime ring of $\mathrm{char}(R) \neq 2$, L a noncentral Lie ideal of R and F, G two nonzero generalized derivations of R. If $[F(u), u]G(u) = 0$ for all $u \in L$. Then one of the following holds:

(1) there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;

(2) $R \subseteq M_2(K)$, the ring of 2×2-matrices over the field K, and there exist $a \in U$ and $\lambda \in C$ such that $F(x) = ax + xa + \lambda x$ for all $x \in R$, where U is the Utumi quotient ring of R.

Our motivation in the present article is to investigate the situation when F and G both are b-generalized skew derivations of R. More concisely, we will prove the following:

Theorem 1.1. *Let R be any noncommutative prime ring of* $\mathrm{char}(R) \neq 2, 3$, *$L$ a noncentral Lie ideal of R and F, G two nonzero b-generalized skew derivations of R. Suppose that*

$$[F(u), u]G(u) = 0$$

for all $u \in L$. Then at least one of the following conclusions holds:

(1) *$F(x) = \lambda x$ for all $x \in R$ and for some $\lambda \in C$;*

(2) *$R \subseteq M_2(K)$, the algebra of 2×2-matrices over a field K.*

2. F and G be inner b-generalized skew derivations

Lemma 2.1. *Let $R = M_k(K)$ be an algebra of all $k \times k$-matrices over the infinite field K and $k \geqslant 3$. Suppose for some $a, c, A, B, D, P, Q \in R$,*

(2.1)
$$au^2cu + au^2AuD + AuBucu + AuBuAuD - uaucu - uauAuD - uAuPu - uAuQuD = 0$$

for all $u \in [R,R]$. Then one of the following holds:
(1) $A \in Z(R)$;
(2) $B \in Z(R)$ *with* $a + AB \in Z(R)$;
(3) $D \in Z(R)$ *with* $c + AD = 0$.

Proof. Let e_{ij} be the square matrix having 1 in its (i,j)th entry and all other entries being zero. First, we assume that A, B and D are not central matrices. Then by [7], Lemma 1, there exists an inner automorphism φ of R such that $\varphi(A)$, $\varphi(B)$ and $\varphi(D)$ have all the entries nonzero. By hypothesis

$$\begin{aligned}(2.2)\qquad \varphi(a)u^2\varphi(c)u + \varphi(a)u^2\varphi(A)u\varphi(D) + \varphi(A)u\varphi(B)u\varphi(c)u \\ + \varphi(A)u\varphi(B)u\varphi(A)u\varphi(D) - u\varphi(a)u\varphi(c)u \\ - u\varphi(a)u\varphi(A)u\varphi(D) - u\varphi(A)u\varphi(P)u \\ - u\varphi(A)u\varphi(Q)u\varphi(D) = 0\end{aligned}$$

for all $u \in [R,R]$.

Choose $u = [x_1, x_2] = e_{ij}$ for any $i \neq j$ and then (2.2) reduces to

$$\begin{aligned}(2.3)\qquad \varphi(A)e_{ij}\varphi(B)e_{ij}\varphi(c)e_{ij} + \varphi(A)e_{ij}\varphi(B)e_{ij}\varphi(A)e_{ij}\varphi(D) \\ - e_{ij}(\varphi(a)e_{ij}\varphi(c) + \varphi(A)e_{ij}\varphi(P))e_{ij} \\ - e_{ij}(\varphi(a)e_{ij}\varphi(A) + \varphi(A)e_{ij}\varphi(Q))e_{ij}\varphi(D) = 0.\end{aligned}$$

Now multiplying e_{ij} from both sides of (2.3), we obtain

$$e_{ij}(\varphi(A)e_{ij}\varphi(B)e_{ij}\varphi(A)e_{ij}\varphi(D))e_{ij} = 0,$$

that is,

$$\varphi(A)_{ji}\varphi(B)_{ji}\varphi(D)_{ji} = 0.$$

This implies either $\varphi(A)_{ji} = 0$ or $\varphi(B)_{ji} = 0$ or $\varphi(D)_{ji} = 0$, a contradiction. Thus, we conclude that either A or B or D is central.

To prove our conclusion, we need to consider the following two cases:

Case (i): When $B \in Z(R)$. By hypothesis

$$(2.4)\qquad au^2cu + au^2AuD + ABu^2cu + ABu^2AuD - uaucu - uauAuD - uAuPu - uAuQuD = 0$$

for all $u \in [R,R]$. Now for $i \neq j \neq k$, we choose $u = [x_1, x_2] = [e_{ji} + e_{ik}, e_{ii} + e_{ik}] = e_{jk} + e_{ji} - e_{ik}$ in (2.4), i.e., $u^2 = [x_1, x_2]^2 = -e_{jk}$ and then multiplying e_{kk} from left and e_{jj} from right, we obtain

$$(2.5)\qquad -e_{kk}(a + AB)e_{jk}A(e_{jk} + e_{ji} - e_{ik})De_{jj} = 0.$$

Again, for $i \neq j \neq k$ we choose $u = [x_1, x_2] = [e_{ii} + e_{ji}, e_{jj} + e_{ik}] = e_{jk} - e_{ji} + e_{ik}$ in (2.4), i.e., $u^2 = [x_1, x_2]^2 = -e_{jk}$ and then multiplying (2.4) by e_{kk} from left and by e_{jj} from right and we obtain

$$-e_{kk}(a + AB)e_{jk}A(e_{jk} - e_{ji} + e_{ik})De_{jj} = 0. \tag{2.6}$$

Comparing (2.5) and (2.6), we get

$$2e_{kk}(a + AB)e_{jk}Ae_{jk}De_{jj} = 0,$$

that is,

$$2(a + AB)_{kj}A_{kj}D_{kj} = 0.$$

Since $\operatorname{char}(K) \neq 2$, by the same argument as above, we conclude that either A is central or D is central or $a + AB$ is central.

If D is central, we have from (2.4) that

$$(a + AB)u^2(c + AD)u - uau(c + AD)u - uAuPu - uADuQu = 0 \tag{2.7}$$

for all $u \in [R, R]$.

Now replacing $u = [x_1, x_2] = [e_{ji} + e_{ik}, -e_{ii} + e_{ik}] = e_{jk} - e_{ji} + e_{ik}$ in the above relation, and then left multiplying by e_{kk} and right by e_{ii} we have

$$e_{kk}(a + AB)e_{jk}(c + AD)e_{ji} = 0,$$

which implies $(a + AB)_{kj}(c + AD)_{kj} = 0$. By the same argument as before it yields either $a + AB \in Z(R)$ or $c + AD \in Z(R)$. If $c + AD = 0$, we have our conclusion (3). Assume that $0 \neq c + AD \in Z(R)$. Then by (2.7),

$$(c + AD)\{(a + AB)u^3 - uau^2\} - uAuPu - uADuQu = 0 \tag{2.8}$$

for all $u \in [R, R]$. Now replacing $u = [x_1, x_2] = [-e_{ij} + e_{ji}, e_{ii}] = e_{ij} + e_{ji}$ and then multiplying e_{kk} from the left and e_{ij} from the right we get $0 = e_{kk}(c + AD) \times (a + AB)e_{jj}$. Since $0 \neq c + AD \in Z(R)$, it gives $(a + AB)_{kj} = 0$. Thus, $a + AB$ is diagonal and hence central.

Case (ii): When $D \in Z(R)$. By hypothesis

$$au^2cu + au^2ADu + AuBucu + AuBuADu - uaucu - uauADu - uAuPu - uADuQu = 0 \tag{2.9}$$

for all $u \in [R, R]$. Now we may choose $[x_1, x_2] = e_{ij}$ for $i \neq j$ in (2.9) and we have

$$Ae_{ij}Be_{ij}(c + AD)e_{ij} - e_{ij}ae_{ij}(c + AD)e_{ij} - e_{ij}Ae_{ij}(P + DQ)e_{ij} = 0. \tag{2.10}$$

Multiplying (2.10) by e_{ij} from the left we get

$$e_{ij}Ae_{ij}Be_{ij}(c+AD)e_{ij}=0,$$

that is,

$$A_{ji}B_{ji}(c+AD)_{ji}=0.$$

Then one of A, B and $(c+AD)$ must be central as above. If $c+AD=0$, we have our conclusion (3).

In the case when $0\neq(c+AD)\in Z(R)$, (2.9) reduces to

$$a(c+AD)u^3+A(c+AD)uBu^2-uaucu-uauADu-uAuPu-uADuQu=0. \tag{2.11}$$

Now for $i\neq j\neq k$ we choose $[x_1,x_2]=[e_{ji}+e_{ik},e_{ii}+e_{ik}]=e_{jk}+e_{ji}-e_{ik}$ in (2.11) and then left multiplying (2.11) by e_{kk} and using $[x_1,x_2]^3=0$, we get

$$-e_{kk}A(c+AD)(e_{jk}+e_{ji}-e_{ik})Be_{jk}=0. \tag{2.12}$$

Again for $i\neq j\neq k$ we choose $[x_1,x_2]=[e_{ji}+e_{ik},-e_{ii}+e_{ik}]=e_{jk}-e_{ji}+e_{ik}$ in (2.11) and then multiplying (2.11) by e_{kk} from the left, we obtain

$$-e_{kk}A(c+AD)(e_{jk}-e_{ji}+e_{ik})Be_{jk}=0. \tag{2.13}$$

Comparing (2.12) and (2.13), we get

$$-2e_{kk}A(c+AD)e_{jk}Be_{jk}=0,$$

that is,

$$2A_{kj}B_{kj}=0.$$

Since $\mathrm{char}(K)\neq 2$, we conclude that either A is central or B is central as above. If B is central, then the conclusion follows by Case (i). $\square$

Lemma 2.2. *Suppose $R=M_k(K)$, the algebra of all $k\times k$-matrices over the field K having characteristic value different from 2 and 3 and $k\geqslant 3$. Suppose for some $a,c,A,B,D,P,Q\in R$,*

$$au^2cu+au^2AuD+AuBucu+AuBuAuD-uaucu-uauAuD-uAuPu-uAuQuD=0$$

for all $u\in[R,R]$. Then one of the following holds:
(1) $A\in Z(R)$;
(2) $B\in Z(R)$ with $a+AB\in Z(R)$;
(3) $D\in Z(R)$ with $c+AD=0$.

Proof. Assume

$$\begin{aligned}\phi(x_1,x_2) = a[x_1,x_2]^2c[x_1,x_2] &+ a[x_1,x_2]^2A[x_1,x_2]D\\ &+ A[x_1,x_2]B[x_1,x_2]c[x_1,x_2] + A[x_1,x_2]B[x_1,x_2]A[x_1,x_2]D\\ &- [x_1,x_2]a[x_1,x_2]c[x_1,x_2] - [x_1,x_2]a[x_1,x_2]A[x_1,x_2]D\\ &- [x_1,x_2]A[x_1,x_2]P[x_1,x_2] - [x_1,x_2]A[x_1,x_2]Q[x_1,x_2]D.\end{aligned}$$

If we assume K is an infinite field, then we have our conclusions from Lemma 2.1. Therefore, we may consider K to be a finite field and extend K to an infinite field E. Let $\overline{R} = M_k(E) = R \otimes_K E$. The generalized polynomial $\phi(x_1,x_2)$ is a multi-homogeneous of multi-degree $(3,3)$ in the indeterminates x_1, x_2 as well as a generalized polynomial identity for R. Hence, the complete linearization of $\phi(x_1,x_2)$ is a multilinear generalized polynomial $\chi(x_1,x_2,y_1,y_2,z_1,z_2)$. Moreover,

$$\chi(x_1,x_2,x_1,x_2,x_1,x_2) = 2^2\cdot 3^2\phi(x_1,x_2).$$

Clearly, the multilinear polynomial $\chi(x_1,x_2,y_1,y_2,z_1,z_2)$ is a generalized polynomial identity for R as well as for $\overline{R}$. Since $\operatorname{char}(K)\neq 2$ and $\operatorname{char}(K)\neq 3$, then $\phi(x_1,x_2)=0$ for all $x_1,x_2\in\overline{R}$. Hence, by Lemma 2.1, the conclusion follows. □

Similarly, we can prove the following lemmas.

Lemma 2.3. *Suppose $R = M_k(K)$, the algebra of all $k\times k$-matrices over the field K having characteristic value different from 2 and $k\geqslant 3$. Suppose for some $c, A, A', D, P, P' \in R$*

$$A'ucu + A'uAuD - uPu - uP'uD = 0$$

for all $u\in[R,R]$. Then one of the following holds:
(1) *$A\in Z(R)$;*
(2) *$A'\in Z(R)$;*
(3) *$D\in Z(R)$.*

Proof. Let K be an infinite field. On the contrary, we assume that A, A' and D are not central matrices. Then by [7], Lemma 1, there exists an inner automorphism φ of R such that $\varphi(A)$, $\varphi(A')$ and $\varphi(D)$ have all the entries nonzero. By hypothesis

$$\varphi(A')u\varphi(c)u + \varphi(A')u\varphi(A)u\varphi(D) - u\varphi(P)u - u\varphi(P')u\varphi(D) = 0 \tag{2.14}$$

for all $u\in[R,R]$. Choose $u=[x_1,x_2]=e_{ij}$ for any $i\neq j$ and then (2.14) reduces to

$$\varphi(A')e_{ij}\varphi(c)e_{ij} + \varphi(A')e_{ij}\varphi(A)e_{ij}\varphi(D) - e_{ij}\varphi(P)e_{ij} - e_{ij}\varphi(P')e_{ij}\varphi(D) = 0.$$

Multiplying e_{ij} from both sides in the above relation, we have $\varphi(A')_{ji}\varphi(A)_{ji} \times \varphi(D)_{ji} = 0$, a contradiction. Thus, we conclude that either A or A' or D is central. When K is a finite field, the same argument as in Lemma 2.2 can be adopted to get our conclusion. $\square$

Lemma 2.4. *Suppose $R = M_k(K)$, the algebra of all $k \times k$-matrices over the field K having characteristic value different from 2 and $k \geqslant 3$. Suppose for some $a, A, B, D', P \in R$,*

$$au^2D' + AuBuD' - uauD' - uAuP = 0$$

for all $u \in [R, R]$. Then one of the following holds:
(1) $A \in Z(R)$;
(2) $B \in Z(R)$;
(3) $D' \in Z(R)$.

Proof. Let K be an infinite field.

On the contrary, we assume that A, B and D' are not central matrices. Then by [7], Lemma 1, there exists an inner automorphism φ of R such that $\varphi(A)$, $\varphi(B)$ and $\varphi(D')$ have all the entries nonzero. By hypothesis

$$\varphi(a)u^2\varphi(D') + \varphi(A)u\varphi(B)u\varphi(D') - u\varphi(a)u\varphi(D') - u\varphi(A)u\varphi(P) = 0 \tag{2.15}$$

for all $u \in [R, R]$. Replacing $u = [x_1, x_2] = e_{ij}$ for any $i \neq j$ in (2.15) and then multiplying e_{ij} from both sides we have $\varphi(A)_{ji}\varphi(B)_{ji}\varphi(D')_{ji} = 0$, a contradiction. Thus, we conclude that either A or B or D' is central. When K is a finite field, the same argument as in Lemma 2.2 can be adopted to get our conclusion. $\square$

Lemma 2.5. *Let R be a prime ring. If for some $a, c, A, B, D \in R$,*

$$[a[x_1, x_2] + A[x_1, x_2]B, [x_1, x_2]](c[x_1, x_2] + A[x_1, x_2]D) = 0$$

is a trivial generalized polynomial identity (GPI) for R, then one of the following holds:
(i) $A \in C$;
(ii) $B \in C$ *with* $a + AB \in C$;
(iii) $D \in C$ *with* $c + AD = 0$.

Proof. On the contrary, assume that $A \notin C$. Let

$$\psi(x_1, x_2) = [a[x_1, x_2] + A[x_1, x_2]B, [x_1, x_2]](c[x_1, x_2] + A[x_1, x_2]D). \tag{2.16}$$

Since the above generalized polynomial identity (GPI) is satisfied by R, then it would be satisfied by Q_r, see [3]. Let $T = Q_r *_C C\{x_1, x_2\}$ be the free product of Q_r and $C\{x_1, x_2\}$ the free C-algebra of noncommutating indeterminates x_1, x_2. Then $\psi(x_1, x_2)$ is a zero element in the free product T. Replacing $[x_1, x_2] = X$ in (2.16), we get

$$[aX + AXB, X](cX + AXD) = 0 \in T. \tag{2.17}$$

Case 1: $\{1, a, A\}$ is linearly C-dependent. There exist some $\alpha_1, \alpha_2, \alpha_3 \in C$ such that $\alpha_1 + \alpha_2 a + \alpha_3 A = 0$. Since $A \notin C$, $\alpha_2 \neq 0$ and so we can write $a = \lambda A + \mu$ for some $\lambda, \mu \in C$.

In the case when $\lambda = 0$, $a = \mu \in C$ and (2.17) reduces to

$$[AXB, X](cX + AXD) = 0 \in T.$$

Since $\{1, A\}$ is linearly C-independent, then we get

$$AXBX(cX + AXD) = 0 \in T.$$

This implies $B = 0$ or $D \in C$ with $c + AD = 0$. In any case we have the conclusion.

Therefore, we assume $\lambda \neq 0$, then $a = \lambda A + \mu$ for some suitable $\lambda, \mu \in C$. Now we derive from (2.17) that

$$[(\lambda A + \mu)X + AXB, X](cX + AXD) = 0 \in T,$$

that is,

$$[\lambda AX + AXB, X](cX + AXD) = 0 \in T,$$

i.e.,

$$[AX(\lambda + B), X](cX + AXD) = 0 \in T.$$

Since $A \notin C$, the above equation reduces to

$$AX(\lambda + B)X(cX + AXD) = 0 \in T.$$

This implies $B + \lambda = 0$ or $D \in C$ with $c + AD = 0$. If $B + \lambda = 0$, then $a = \lambda A + \mu = -AB + \mu$, i.e., $a + AB \in C$. Thus, the conclusion follows.

Case 2: $\{1, a, A\}$ is linearly C-independent. From (2.17) it follows that

$$AXBX(cX + AXD) = 0 \in T.$$

This implies $B = 0$ or $D \in C$ with $c + AD = 0$. If $B = 0$, then (2.17) reduces to $[a, X]X(cX + AXD) = 0 \in T$. This implies $a \in C$ or $D \in C$ with $c + AD = 0$. In all the cases we have our conclusions. □

Proposition 2.6. *Let R be a noncommutative prime ring of characteristic not 2 and 3 with extended centroid C and Q_r its right Martindale quotient ring. Suppose $F(x) = ax + AxB$ and $G(x) = cx + AxD$ for all $x \in R$ and for some $a, c, A, B, D \in Q_r$. If*

$$[F(u), u]G(u) = 0$$

for all $u \in [R, R]$, then at least one of the following conclusions holds:
(1) *$A \in C$;*
(2) *$B \in C$ with $a + AB \in C$;*
(3) *$D \in C$ with $c + AD = 0$;*
(4) *$R \subseteq M_2(K)$, the ring of all 2×2 matrices over the field K.*

Proof. By hypothesis

$$\psi(x_1, x_2) = [a[x_1, x_2] + A[x_1, x_2]B, [x_1, x_2]](c[x_1, x_2] + A[x_1, x_2]D) = 0$$

for all $x_1, x_2 \in R$.

Then by Lemma 2.5, $\psi(x_1, x_2)$ is a nontrivial generalized polynomial identity for R and by [3], $\psi(x_1, x_2)$ is a nontrivial generalized polynomial identity for Q_r.

If C is infinite, then $Q_r \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C, satisfies $\psi(x_1, x_2)$. Since both Q_r and $Q_r \otimes_C \overline{C}$ are prime and centrally closed by [10], Theorems 2.5 and 3.5, then R can be replaced by Q_r or $Q_r \otimes_C \overline{C}$ according to whether C is finite or infinite, respectively. Therefore, R is centrally closed over C, which is either finite or algebraically closed. According to Martindale's theorem (see [17]), R is a primitive ring with a nonzero socle $H = \operatorname{soc}(R)$ with its associative division ring being C. Again R is isomorphic to the dense ring of linear transformations on a vector space V over C on the basis of Jacobson's theorem, see [12], P–75. First we assume that $\dim_C(V) = k$ is finite, then $R \cong M_k(C)$. If $k = 2$, then conclusion (4) follows. Thus, assume $k \geqslant 3$. We re-write the above identity in the form

$$\begin{aligned}\psi(x_1, x_2) = {} & a[x_1, x_2]^2 c[x_1, x_2] + a[x_1, x_2]^2 A[x_1, x_2]D + A[x_1, x_2]B[x_1, x_2]c[x_1, x_2] \\ & + A[x_1, x_2]B[x_1, x_2]A[x_1, x_2]D - [x_1, x_2]a[x_1, x_2]c[x_1, x_2] \\ & - [x_1, x_2]a[x_1, x_2]A[x_1, x_2]D - [x_1, x_2]A[x_1, x_2]P[x_1, x_2] \\ & - [x_1, x_2]A[x_1, x_2]Q[x_1, x_2]D = 0,\end{aligned}$$

where $P = Bc$ and $Q = BA$. By Lemma 2.1, we get our conclusions.

Next we consider that $\dim_C(V) = \infty$. Assume first that $A \notin C$, $B \notin C$ and $D \notin C$. Since $\dim_C(V)$ is infinite, then $[Q_r, Q_r]$ is dense on Q_r by [20], Lemma 2. Therefore, Q_r satisfies

(2.18)
$$ax^2cx + ax^2AxD + AxBxcx + AxBxAxD - xaxcx - xaxAxD - xAxPx - xAxQxD = 0.$$

Since $A \notin C$, $B \notin C$ and $D \notin C$, they cannot centralize $H = \mathrm{soc}(R)$ and moreover, H cannot satisfy $s_4(x_1, x_2, x_3, x_4)$ because of its infinite dimensionality. So there exists t_1, t_2, t_3, t_4, t_5, t_6, $t_7 \in H$ such that

▷ $[A, t_1] \neq 0$,
▷ $[B, t_2] \neq 0$,
▷ $[D, t_3] \neq 0$,
▷ $s_4(t_4, t_5, t_6, t_7) \neq 0$.

Now by Litoff's theorem [11], there exists $e^2 = e \in H$ such that $t_1, t_2, t_3, At_1, t_1A, Bt_2, t_2B, Dt_3, t_3D \in eQ_re$ and $eQ_re \cong M_m(C)$ with $m = \dim_C Ve$ for $m \geqslant 2$. By (2.18), since Q_r satisfies

$$\begin{aligned} e(ax^2cx + ax^2AxD + AxBxcx + AxBxAxD - xaxcx \\ - xaxAxD - xAxPx - xAxQxD)e = 0, \end{aligned}$$

eQ_re satisfies

$$\begin{aligned} (eae)x^2(ece)x + (eae)x^2(eAe)x(eDe) + (eAe)x(eBe)x(ece)x \\ + (eAe)x(eBe)x(eAe)x(eDe) - x(eae)x(ece)x \\ - x(eae)x(eAe)x(eDe) - x(eAe)x(ePe)x \\ - x(eAe)x(eQe)x(eDe) = 0. \end{aligned}$$

By Lemma 2.1, we obtain at least one of the following conclusions

(1) $eAe \in Ce$, which is a contradiction to the choice of t_1 in H;
(2) $eBe \in Ce$, which is a contradiction to the choice of t_2 in H;
(3) $eDe \in Ce$, which is a contradiction to the choice of t_3 in H;
(4) $eQ_re \cong M_2(C)$, i.e., eQ_re satisfies s_4 which contradicts with the choices of t_4, t_5, t_6, t_7 in H.

Thus, when $\dim_C(V) = \infty$, we have proved that either $A \in C$ or $B \in C$ or $D \in C$. If $A \in C$, then conclusion (1) is obtained. Thus, to obtain our remaining conclusions, we consider either $B \in C$ or $D \in C$.

On the other hand, by (2.18), Q_r satisfies

$$[ax + AxB, x](cx + AxD) = 0. \tag{2.19}$$

Replacing x with $x + 1$ in the above relation we have

$$[ax + AxB + a + AB, x](cx + AxD + c + AD) = 0. \tag{2.20}$$

By using (2.19), the above relation yields

$$[ax + AxB, x](c + AD) + [a + AB, x](cx + AxD + c + AD) = 0. \tag{2.21}$$

Replacing x with $-x$

$$[ax + AxB, x](c + AD) + [a + AB, x](cx + AxD - c - AD) = 0. \tag{2.22}$$

Adding and subtracting relations (2.21) and (2.22), we get by using $\mathrm{char}(R) \neq 2$ that

$$[ax + AxB, x](c + AD) + [a + AB, x](cx + AxD) = 0 \tag{2.23}$$

and

$$[a + AB, x](c + AD) = 0, \tag{2.24}$$

respectively. By [18], the last relation yields either $a + AB \in C$ or $c + AD = 0$.

Thus, up to now, we have proved that either $B \in C$ or $D \in C$ and either $a + AB \in C$ or $c + AD = 0$. Thus, we have four possible cases:

(1) $B \in C$ and $a + AB \in C$;
(2) $B \in C$ and $c + AD = 0$;
(3) $D \in C$ and $a + AB \in C$;
(4) $D \in C$ and $c + AD = 0$.

Since the first and last case gives our conclusions (2) and (3), we need to consider the remaining cases, which are given below.

Case (i): When $B \in C$ and $c + AD = 0$. In this case, by (2.23)

$$[a + AB, x](cx + AxD) = 0.$$

Assuming $A' = a + AB$, we have from above that

$$A'xcx + A'xAxD - xA'cx - xA'AxD = 0.$$

Assume that $A \notin C$, $A' \notin C$ and $D \notin C$. Then there exist t_1, t_2, t_3, t_4, t_5, t_6, $t_7 \in H$ such that

▷ $[A, t_1] \neq 0$,
▷ $[A', t_2] \neq 0$,
▷ $[D, t_3] \neq 0$,
▷ $s_4(t_4, t_5, t_6, t_7) \neq 0$.

Then by Litoff's theorem (see [11]), there exists $e^2 = e \in H$ such that t_1, t_2, t_3, At_1, $t_1A, A't_2, t_2A', Dt_3, t_3D \in eQ_re$ and $eQ_re \cong M_m(C)$ with $m = \dim_C Ve$ for $m \geqslant 2$. By (2.25), we can write as before that eQ_re satisfies

$$(eA'e)x(ece)x + (eA'e)x(eAe)x(eDe) - x(eA'ce)x - x(eA'Ae)x(eDe) = 0.$$

By Lemma 2.3, we obtain at least one of the following conclusions

(1) $eAe \in Ce$, which is a contradiction to the choice of t_1 in H;

(2) $eA'e \in Ce$, which is a contradiction to the choice of t_2 in H;

(3) $eDe \in Ce$, which is a contradiction to the choice of t_3 in H;

(4) $eQ_re \cong M_2(C)$, i.e., eQ_re satisfies s_4 which contradicts with the choices of t_4, t_5, t_6, t_7 in H.

Thus, we conclude that either $A \in C$ or $a + AB \in C$ or $D \in C$. Hence, conclusions (1), (2) and (3) follow.

Case (ii): When $D \in C$ and $a + AB \in C$. In this case, (2.23) reduces to

$$[ax + AxB, x](c + AD) = 0. \tag{2.25}$$

Assuming $D' = c + AD$, we have from above that

$$ax^2D' + AxBxD' - xaxD' - xAxBD' = 0.$$

Assume that $A \notin C$, $B \notin C$ and $D' \notin C$. Then there exist $t_1, t_2, t_3, t_4, t_5, t_6, t_7 \in H$, such that

▷ $[A, t_1] \neq 0$,

▷ $[B, t_2] \neq 0$,

▷ $[D', t_3] \neq 0$,

▷ $s_4(t_4, t_5, t_6, t_7) \neq 0$.

Then by Litoff's theorem (see [11]), there exists $e^2 = e \in H$ such that t_1, t_2, t_3, At_1, $t_1A, Bt_2, t_2B, D't_3, t_3D' \in eQ_re$ and $eQ_re \cong M_m(C)$ with $m = \dim_C Ve$ for $m \geqslant 2$. By (2.26), we can write as before that eQ_re satisfies

$$(eae)x^2(eD'e) + (eAe)x(eBe)x(eD'e) - x(eae)x(eD'e) - x(eAe)x(eBD'e) = 0.$$

By Lemma 2.4, we obtain at least one of the following conclusions

(1) $eAe \in Ce$, which is a contradiction to the choice of t_1 in H;

(2) $eBe \in Ce$, which is a contradiction to the choice of t_2 in H;

(3) $eD'e \in Ce$, which is a contradiction to the choice of t_3 in H;

(4) $eQ_re \cong M_2(C)$, i.e., eQ_re satisfies s_4, which contradicts with the choices of t_4, t_5, t_6, t_7 in H.

Thus, we conclude that either $A \in C$ or $B \in C$ or $c + AD \in C$. Let $c + AD = 0$. Then conclusions (1), (2) and (3) follow. Hence, assume that $0 \neq c + AD \in C$. Then by (2.25), Q_r satisfies

$$[ax + AxB, x] = 0. \tag{2.26}$$

Then by Theorem 1.1 of [16] we have $B \in C$ with $a + AB \in C$. This is our conclusion (2). Hence, the proposition is proved. □

Proposition 2.7. *Let R be a noncommutative prime ring of characteristic not 2 and 3 with extended centroid C and Q_r its right Martindale quotient ring. Suppose F and G are two nonzero b-generalized skew derivations of R, defined as $F(x) = ax + bpxp^{-1}u$ and $G(x) = cx + bpxp^{-1}v$, respectively, for all $x \in R$ and for some $a, b, c, p, u, v \in Q_r$ with some invertible p. If*

$$[F(u), u]G(u) = 0$$

for all $u \in [R, R]$, then one of the following conclusions holds:
(1) *there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;*
(2) *$R \subseteq M_2(K)$, the ring of all 2×2 matrices over the field K.*

Proof. By Proposition 2.6, one of the following holds:
(1) $bp \in C$; in this case F and G become generalized derivations and hence the conclusion follows by [1], Proposition 2.12;
(2) $p^{-1}u$, $a + bu \in C$; in this case $F(x) = \lambda x$ for all $x \in R$, where $\lambda = a + bu \in C$;
(3) $p^{-1}v \in C$ and $c + bv = 0$; in this case $G = 0$, a contradiction;
(4) $R \subseteq M_2(K)$, the ring of all 2×2 matrices over the field K. □

Proposition 2.8. *Let R be a noncommutative prime ring of characteristic not 2 and 3 with extended centroid C and Q_r its right Martindale quotient ring. Suppose $\alpha \in \mathrm{Aut}(R)$ and F, G are two nonzero b-generalized skew derivations of R, defined as $F(x) = ax + b\alpha(x)u$ and $G(x) = cx + b\alpha(x)v$, respectively, for all $x \in R$ and for some $a, b, c, u, v \in Q_r$. If*

$$[F(u), u]G(u) = 0$$

for all $u \in [R, R]$, then at least one the following conclusions holds:
(1) *$F(x) = \lambda x$ for all $x \in R$ and for some $\lambda \in C$;*
(2) *$R \subseteq M_2(K)$, the ring of all 2×2 matrices over a field K.*

Proof. If α is an inner automorphism or $b = 0$, then the conclusion follows by Proposition 2.7. Therefore, we assume that α is an outer automorphism on R and $b \neq 0$. By hypothesis, R as well as Q_r satisfy

$$[ax + b\alpha(x)u, x](cx + b\alpha(x)v) = 0$$

for all $x \in [Q_r, Q_r]$. Since α is outer, Q_r satisfies

$$[a[x_1, x_2] + b[y_1, y_2]u, [x_1, x_2]](c[x_1, x_2] + b[y_1, y_2]v) = 0. \tag{2.27}$$

Assuming $y_1 = y_2 = 0$, we have from above that

$$[a[x_1, x_2], [x_1, x_2]]c[x_1, x_2] = 0. \tag{2.28}$$

Thus, by using (2.28), equation (2.27) reduces to

$$\begin{aligned}(2.29)\qquad [a[x_1,x_2],[x_1,x_2]]b[y_1,y_2]v + [b[y_1,y_2]u,[x_1,x_2]]c[x_1,x_2]\\ + [b[y_1,y_2]u,[x_1,x_2]]b[y_1,y_2]v = 0\end{aligned}$$

for all $x_1, x_2, y_1, y_2 \in Q_r$. Now putting y_1 with $-y_1$ in (2.29) we get

$$\begin{aligned}(2.30)\qquad -[a[x_1,x_2],[x_1,x_2]]b[y_1,y_2]v - [b[y_1,y_2]u,[x_1,x_2]]c[x_1,x_2]\\ + [b[y_1,y_2]u,[x_1,x_2]]b[y_1,y_2]v = 0.\end{aligned}$$

By adding (2.29) and (2.30) and then using $\mathrm{char}(R) \neq 2$, we have that Q_r satisfies

$$[b[y_1,y_2]u,[x_1,x_2]]b[y_1,y_2]v = 0.$$

Above equation can be written as

$$(b[y_1,y_2]u)X(b[y_1,y_2]v) - X(b[y_1,y_2]u)(b[y_1,y_2]v) = 0$$

for all $X \in [Q_r, Q_r]$. By Lemma 2.2 in [5] either $b[y_1,y_2]u \in C$ for all $y_1, y_2 \in Q_r$ or $b[y_1,y_2]v = 0$ for all $y_1, y_2 \in Q_r$. Since $b \neq 0$, this implies either $u = 0$ or $v = 0$.

If $u = 0$, by (2.29), Q_r satisfies $[a[x_1,x_2],[x_1,x_2]](b[y_1,y_2]v) = 0$. Right multiplying by $[x_1,x_2]$ and applying [1], we must have either $a \in C$ or $b[y_1,y_2]v = 0$, i.e., $v = 0$. The first case gives $F(x) = ax$ for all $x \in R$ which is conclusion (1) and the second case gives $G = 0$, a contradiction.

If $v = 0$, by (2.29), Q_r satisfies $[b[y_1,y_2]u,[x_1,x_2]]c[x_1,x_2] = 0$. Left multiplying by $[x_1,x_2]$ we have $[[x_1,x_2](b[y_1,y_2]u),[x_1,x_2]]c[x_1,x_2] = 0$. Then by [1], either $b[y_1,y_2]u \in C$, i.e., $u = 0$ or $c = 0$. If $c = 0$, then $G = 0$, a contradiction. Thus, $u = v = 0$. Then F, G becomes generalized derivations and so conclusion follows by [1]. □

3. Proof of the main theorem

Lemma 3.1. *Let R be a prime ring of $\mathrm{char}(R) \neq 2, 3$, $0 \neq b \in R$, α an automorphism of R. If R does not embed $M_2(K)$ for a field K and R satisfies*

(3.1)

$$[b(t_1x_2 + \alpha(x_1)t_2 - t_2x_1 - \alpha(x_2)t_1),[x_1,x_2]]b(t_1x_2 + \alpha(x_1)t_2 - t_2x_1 - \alpha(x_2)t_1) = 0,$$

then R is commutative.

P r o o f. Assume that R is noncommutative. If $\alpha = \mathrm{Id}$, then R satisfies

$$[b([t_1, x_2] + [x_1, t_2]), [x_1, x_2]]b([t_1, x_2] + [x_1, t_2]) = 0. \tag{3.2}$$

In particular, replacing t_1 with $[q, x_1]$ and t_2 with $[q, x_2]$ for some $q \notin C$, we have

$$[bq[x_1, x_2] - b[x_1, x_2]q, [x_1, x_2]](bq[x_1, x_2] - b[x_1, x_2]q) = 0.$$

By Proposition 2.6, $b \in C$. Then (3.2) reduces to

$$[[t_1, x_2] + [x_1, t_2], [x_1, x_2]]([t_1, x_2] + [x_1, t_2]) = 0, \tag{3.3}$$

which is a polynomial identity for R. Thus, by Lemma 1 of [14], there exists a field C such that $R \subseteq M_k(C)$, $k \geqslant 2$, such that $M_k(C)$ satisfies (3.3). But for $t_1 = 0$, $t_2 = e_{21}$, $x_1 = e_{11}$, $x_2 = e_{12}$, we have

$$0 = [[t_1, x_2] + [x_1, t_2], [x_1, x_2]]([t_1, x_2] + [x_1, t_2]) = [[e_{11}, e_{21}], [e_{11}, e_{12}]][e_{11}, e_{21}] = e_{21},$$

a contradiction. Thus, we may assume that $\alpha \neq \mathrm{Id}$ map.

Next assume that α is inner, i.e., there exists invertible $p \in Q_r$ such that $\alpha(x) = pxp^{-1}$, where $p \notin C$. Thus, from (3.1), we have

$$[b(t_1x_2+px_1p^{-1}t_2-t_2x_1-px_2p^{-1}t_1), [x_1, x_2]]b(t_1x_2+px_1p^{-1}t_2-t_2x_1-px_2p^{-1}t_1) = 0.$$

Replacing t_1 with pt_1 and t_2 with pt_2, we get

$$[bp([t_1, x_2] + [x_1, t_2]), [x_1, x_2]]bp([t_1, x_2] + [x_1, t_2]) = 0.$$

This is the same as (3.2). Thus, by the same argument as above, we have $bp \in C$. Since $bp \neq 0$ (as $bp = 0$ implies $b = 0$, a contradiction), we have

$$[[t_1, x_2] + [x_1, t_2], [x_1, x_2]]([t_1, x_2] + [x_1, t_2]) = 0.$$

This is the same as (3.3) and so it leads to a contradiction.

Next assume that α is not inner. By (3.1), R satisfies

$$[b(t_1x_2 + y_1t_2 - t_2x_1 - y_2t_1), [x_1, x_2]]b(t_1x_2 + y_1t_2 - t_2x_1 - y_2t_1) = 0.$$

In particular, assuming $y_1 = x_1$ and $y_2 = x_2$, we have that R satisfies

$$[b([t_1, x_2] + [x_1, t_2]), [x_1, x_2]]b([t_1, x_2] + [x_1, t_2]) = 0.$$

This is the same as (3.2). Thus, by the same argument the conclusion follows. $\square$

Lemma 3.2. *Let R be a prime ring of* $\operatorname{char}(R) \neq 2, 3$, $0 \neq b \in R$, *α an automorphism of R. If R does not embed $M_2(K)$ for a field K and R satisfies*

$$(3.4)\quad [b(t_1x_2+\alpha(x_1)t_2-t_2x_1-\alpha(x_2)t_1),[x_1,x_2]]\{(c+bq)[x_1,x_2]-b\alpha([x_1,x_2])q\}=0,$$

then one of the following holds:
(i) *R is commutative;*
(ii) *$\alpha = Id$, $q \in C$ with $c = 0$;*
(iii) *$\alpha(x) = pxp^{-1}$ and $p^{-1}q \in C$ with $c = 0$;*
(iv) *$q = c = 0$.*

Proof. Assume that R is noncommutative. If $\alpha = \mathrm{Id}$, then R satisfies

$$(3.5)\qquad [b([t_1,x_2]+[x_1,t_2]),[x_1,x_2]]\{(c+bq)[x_1,x_2]-b[x_1,x_2]q\}=0.$$

In particular, replacing t_1 with $[q',x_1]$ and t_2 with $[q',x_2]$ for some $q' \notin C$ we have

$$(3.6)\qquad [bq'[x_1,x_2]-b[x_1,x_2]q',[x_1,x_2]]\{(c+bq)[x_1,x_2]-b[x_1,x_2]q\}=0.$$

By Proposition 2.6, $b \in C$ or $q \in C$ with $c = 0$. If $b \in C$, (3.6) reduces to

$$[[q',[x_1,x_2]],[x_1,x_2]]\{(c+bq)[x_1,x_2]-[x_1,x_2]bq\}=0.$$

By [1], $bq \in C$ with $c = 0$. As $b \neq 0$, we have $q \in C$ with $c = 0$.

Next assume that α is inner, i.e., there exists invertible $p \in Q_r$ such that $\alpha(x) = pxp^{-1}$, where $p \notin C$. Thus, from (3.4) we have

$$[b(t_1x_2+px_1p^{-1}t_2-t_2x_1-px_2p^{-1}t_1),[x_1,x_2]]\{(c+bq)[x_1,x_2]-bp[x_1,x_2]p^{-1}q\}=0.$$

Replacing t_1 with pt_1 and t_2 with pt_2, we get

$$(3.7)\qquad [bp([t_1,x_2]+[x_1,t_2]),[x_1,x_2]]\{(c+bq)[x_1,x_2]-bp[x_1,x_2]p^{-1}q\}=0.$$

In particular, for $t_2 = 0$ and $t_1 = x_1$, R satisfies

$$[bp[x_1,x_2],[x_1,x_2]]\{(c+bq)[x_1,x_2]-bp[x_1,x_2]p^{-1}q\}=0.$$

By Proposition 2.6, $bp \in C$ or $p^{-1}q \in C$ with $c = 0$. If $bp \in C$, then by (3.7) we have

$$[([t_1,x_2]+[x_1,t_2]),[x_1,x_2]]\{(c+bq)[x_1,x_2]-bp[x_1,x_2]p^{-1}q\}=0.$$

Again assuming $t_1 = [p', x_1]$ and $t_2 = [p', x_2]$ for some $p' \notin C$ and then using Proposition 2.6, we may conclude that $p^{-1}q \in C$ with $c = 0$. Thus, in all the cases we have $p^{-1}q \in C$ with $c = 0$.

Next assume that α is not inner. By (3.4), R satisfies

$$[b(t_1x_2 + y_1t_2 - t_2x_1 - y_2t_1), [x_1, x_2]]\{(c + bq)[x_1, x_2] - b[y_1, y_2]q\} = 0. \tag{3.8}$$

In particular, assuming $y_1 = x_1$ and $y_2 = x_2$, R satisfies

$$[b([t_1, x_2] + [x_1, t_2]), [x_1, x_2]]\{(c + bq)[x_1, x_2] - b[x_1, x_2]q\} = 0.$$

This is the same as (3.5). Thus, by the same argument we have $q \in C$ with $c = 0$. If $q = 0$, we have conclusion (iv).

Assume $q \neq 0$. Then (3.8) reduces to

$$[b(t_1x_2 + y_1t_2 - t_2x_1 - y_2t_1), [x_1, x_2]]b([x_1, x_2] - [y_1, y_2]) = 0. \tag{3.9}$$

In particular, for $y_1 = y_2 = 0$ and $t_1 = x_1$ and $t_2 = x_2$, we have

$$[b[x_1, x_2], [x_1, x_2]]b[x_1, x_2] = 0.$$

By Proposition 2.6, $b \in C$. Then (3.9) reduces to

$$[(t_1x_2 + y_1t_2 - t_2x_1 - y_2t_1), [x_1, x_2]]([x_1, x_2] - [y_1, y_2]) = 0. \tag{3.10}$$

This is a polynomial identity for R. Thus, by Lemma 1 of [14], there exists a field C such that $R \subseteq M_k(C)$, $k \geqslant 2$, such that $M_k(C)$ satisfies (3.10). But for $t_1 = 0$, $t_2 = e_{21}$, $x_1 = e_{11}$, $x_2 = e_{12}$, $y_1 = y_2 = 0$, we have

$$0 = [(t_1x_2 + y_1t_2 - t_2x_1 - y_2t_1), [x_1, x_2]]([x_1, x_2] - [y_1, y_2]) = e_{12},$$

a contradiction. □

Lemma 3.3. *Let R be a prime ring of $\mathrm{char}(R) \neq 2, 3$, $0 \neq b \in R$, α an automorphism of R. If R does not embed $M_2(K)$ for a field K and R satisfies*

$$[a[x_1, x_2] + b(c[x_1, x_2] - \alpha([x_1, x_2])c), [x_1, x_2]]b(t_1x_2 + \alpha(x_1)t_2 - t_2x_1 - \alpha(x_2)t_1) = 0, \tag{3.11}$$

then one of the following holds:

(i) *R is commutative;*
(ii) *$\alpha = \mathrm{Id}$, $a, c \in C$;*
(iii) *$\alpha(x) = pxp^{-1}$ and $a, p^{-1}c \in C$;*
(iv) *$\alpha(x) = pxp^{-1}$ and $a, bc, bp \in C$;*
(v) *$a, c \in C$ with $bc = 0$.*

P r o o f. Assume that R is noncommutative. If $\alpha = \mathrm{Id}$, then R satisfies

$$[(a+bc)[x_1,x_2]-b[x_1,x_2]c,[x_1,x_2]]b([t_1,x_2]+[x_1,t_2])=0.$$

In particular, replacing t_1 with $[q',x_1]$ and t_2 with $[q',x_2]$ for some $q'\notin C$, we have

$$[(a+bc)[x_1,x_2]-b[x_1,x_2]c,[x_1,x_2]]b(q'[x_1,x_2]-[x_1,x_2]q')=0. \tag{3.12}$$

By Proposition 2.6, $b\in C$ or $a,c\in C$. If $b\in C$, (3.12) reduces to

$$[(a+bc)[x_1,x_2]-[x_1,x_2]bc,[x_1,x_2]](q'[x_1,x_2]-[x_1,x_2]q')=0.$$

By [1], $a,bc\in C$. Since $0\neq b\in C$, we have $c\in C$. Thus, in all the cases we have $a,c\in C$. This gives conclusion (ii).

Next assume that α is inner, i.e., there exists invertible $p\in Q_r$ such that $\alpha(x)=pxp^{-1}$, where $p\notin C$. Thus, from (3.11), we have

$$[(a+bc)[x_1,x_2]-bp[x_1,x_2]p^{-1}c,[x_1,x_2]]b(t_1x_2+px_1p^{-1}t_2-t_2x_1-px_2p^{-1}t_1)=0.$$

Replacing t_1 with pt_1 and t_2 with pt_2, we get

$$[(a+bc)[x_1,x_2]-bp[x_1,x_2]p^{-1}c,[x_1,x_2]]bp([t_1,x_2]+[x_1,t_2])=0.$$

In particular, for $t_2=0$ and $t_1=x_1$,

$$[(a+bc)[x_1,x_2]-bp[x_1,x_2]p^{-1}c,[x_1,x_2]]bp[x_1,x_2]=0. \tag{3.13}$$

By Proposition 2.6, $bp\in C$ or $p^{-1}c,a\in C$. As $bp=0$ implies $b=0$, a contradiction, thus by (3.13), $0\neq bp\in C$ implies

$$[(a+bc)[x_1,x_2]-[x_1,x_2]bc,[x_1,x_2]][x_1,x_2]=0.$$

By [1], $a,bc\in C$. Thus, conclusions (iii) and (iv) are obtained.

Next assume that α is not inner. By (3.11), R satisfies

$$[(a+bc)[x_1,x_2]-b[y_1,y_2]c,[x_1,x_2]]b(t_1x_2+y_1t_2-t_2x_1-y_2t_1)=0.$$

In particular, assuming $y_1=x_1$ and $y_2=x_2$, R satisfies

$$[(a+bc)[x_1,x_2]-b[y_1,y_2]c,[x_1,x_2]]b([t_1,x_2]+[x_1,t_2])=0.$$

This is the same as (3.12). Thus, by the same argument we have $a, c \in C$. Hence, the above equation reduces to

$$[bc([x_1, x_2] - [y_1, y_2]), [x_1, x_2]]b([t_1, x_2] + [x_1, t_2]) = 0.$$

Right multiplying by c, we have

$$[bc([x_1, x_2] - [y_1, y_2]), [x_1, x_2]]bc([t_1, x_2] + [x_1, t_2]) = 0.$$

Assuming $y_1 = y_2 = 0$, $t_1 = x_1$ and $t_2 = 0$ and then using Proposition 2.6, we have $bc \in C$. Then from above

$$(bc)^2[[y_1, y_2], [x_1, x_2]]([t_1, x_2] + [x_1, t_2]) = 0.$$

This implies that either $bc = 0$ or R satisfies $[[y_1, y_2], [x_1, x_2]]([t_1, x_2] + [x_1, t_2]) = 0$.

The last one is a polynomial identity for R. Thus, by Lemma 1 of [14], there exists a field C such that $R \subseteq M_k(C)$, $k \geqslant 2$, such that $M_k(C)$ satisfies

$$[[y_1, y_2], [x_1, x_2]]([t_1, x_2] + [x_1, t_2]) = 0.$$

But for $t_1 = 0$, $t_2 = e_{12}$, $x_1 = e_{11}$, $x_2 = e_{12}$, $y_1 = e_{22}$, $y_2 = e_{21}$, we have

$$0 = [[y_1, y_2], [x_1, x_2]]([t_1, x_2] + [x_1, t_2]) = -e_{12},$$

a contradiction. $\square$

Now we are ready to prove our Theorem 1.1.

P r o o f of Theorem 1.1. Since char $R \neq 2$ and L is a Lie ideal which is not central valued on R, then by Lemma 1 of [2], there is a nonzero ideal I of R such that $[I, I] \subseteq L$. Moreover, as L is noncentral, R must be noncommutative.

Since F and G are b-generalized skew derivations, by Lemma 3.3 of [9], we have $F(x) = ax + bd(x)$, $G(x) = cx + b\delta(x)$ for all $x \in R$ and $a, b, c \in Q_r$, where d, δ are two skew derivations on R.

Any skew derivations on R can be extended uniquely to Q_r and R and Q_r satisfy the same GPIs with a single skew derivation, see [4], Theorem 2. Hence, by our hypothesis, Q_r satisfies

$$[a[x_1, x_2] + bd([x_1, x_2]), [x_1, x_2]](c[x_1, x_2] + b\delta([x_1, x_2])) = 0$$

for all $x_1, x_2 \in Q_r$. Using the definition of d as well as δ, we get

$$\begin{aligned}(3.14)\quad & [a[x_1, x_2] + b(d(x_1)x_2 + \alpha(x_1)d(x_2) - d(x_2)x_1 - \alpha(x_2)d(x_1)), [x_1, x_2]] \\ & \times (c[x_1, x_2] + b(\delta(x_1)x_2 + \alpha(x_1)\delta(x_2) - \delta(x_2)x_1 - \alpha(x_2)\delta(x_1))) = 0\end{aligned}$$

for all $x_1, x_2 \in Q_r$. If $b = 0$, then F and G would be generalized derivations on R and so the conclusion easily follows from [1], Theorem 2 and also if d and δ both are simultaneously inner skew derivations on R, then the result follows from Proposition 2.8. Therefore, we are going to analyze the situation considering d and δ both are not inner simultaneously on R along with $b \neq 0$.

Case (i): d and δ are linearly C-independent modulo SD_{int}. Since none of d and δ are Q_r-inner, so by Theorem 1 of [4], (3.14) yields

$$[a[x_1, x_2] + b(t_1x_2 + \alpha(x_1)t_2 - t_2x_1 - \alpha(x_2)t_1), [x_1, x_2]] \\ \times (c[x_1, x_2] + b(w_1x_2 + \alpha(x_1)w_2 - w_2x_1 - \alpha(x_2)w_1)) = 0$$

for all $x_1, x_2, t_1, t_2, w_1, w_2 \in Q_r$.

In particular, Q_r satisfies blended component

$$[b(t_1x_2 + \alpha(x_1)t_2 - t_2x_1 - \alpha(x_2)t_1), [x_1, x_2]]b(w_1x_2 + \alpha(x_1)w_2 - w_2x_1 - \alpha(x_2)w_1) = 0$$

for all $x_1, x_2, t_1, t_2, w_1, w_2 \in Q_r$. Assuming $w_1 = t_1$ and $w_2 = t_2$ and then applying Lemma 3.1, we conclude that R is commutative, a contradiction.

Case (ii): d and δ are linearly C-dependent modulo SD_{int}. In this situation, there exist $\gamma, \eta \in C$, $h \in Q_r$ and $\psi \in \text{Aut}(R)$ such that $\gamma d(y) + \eta\delta(y) = hy - \psi(y)h$ for all $y \in R$. If $\gamma = 0$, then $\eta \neq 0$. So we have $\delta(y) = a'y - \psi(y)a'$, where $a' = \eta^{-1}h$ and for all $y \in R$. As each inner skew derivation is associated with an automorphism uniquely, so $\psi = \alpha$ in this scenario. Since δ is inner, so d must be an outer skew derivation on R. In this case, from (3.14) we get

$$[a[x_1, x_2] + b(d(x_1)x_2 + \alpha(x_1)d(x_2) - d(x_2)x_1 - \alpha(x_2)d(x_1)), [x_1, x_2]] \\ \times (c[x_1, x_2] + b(a'[x_1, x_2] - \alpha([x_1, x_2])a')) = 0$$

for all $x_1, x_2 \in Q_r$. Since d is an outer skew derivation on R, we have

$$[a[x_1, x_2] + b(t_1x_2 + \alpha(x_1)t_2 - t_2x_1 - \alpha(x_2)t_1), [x_1, x_2]] \\ \times (c[x_1, x_2] + b(a'[x_1, x_2] - \alpha([x_1, x_2])a')) = 0$$

for all $x_1, x_2 \in Q_r$. In particular, Q_r satisfies blended component

$$[b(t_1x_2 + \alpha(x_1)t_2 - t_2x_1 - \alpha(x_2)t_1), [x_1, x_2]](c[x_1, x_2] + b(a'[x_1, x_2] - \alpha([x_1, x_2])a')) = 0.$$

By applying Lemma 3.2, either R is commutative or $G = 0$. In all the cases, we have contradiction.

Next assume that $\gamma \neq 0$. Then $d(x) = \varrho\delta(x) + a_1 x - \sigma(x)a_1$ for all $x \in R$, where $\varrho = -\gamma^{-1}\eta$ and $a_1 = \gamma^{-1}h$. In this case, from (3.14) we get

$$\begin{aligned}[a[x_1, x_2] + b(\varrho\delta(x_1)x_2 + \alpha(x_1)\varrho\delta(x_2) - \varrho\delta(x_2)x_1 - \alpha(x_2)\varrho\delta(x_1)) \\ + b(a_1[x_1, x_2] - \sigma([x_1, x_2])a_1), [x_1, x_2]] \\ \times (c[x_1, x_2] + b(\delta(x_1)x_2 + \alpha(x_1)\delta(x_2) - \delta(x_2)x_1 - \alpha(x_2)\delta(x_1))) = 0\end{aligned}$$

for all $x_1, x_2 \in Q_r$. Since δ is outer, we have

$$\begin{aligned}[a[x_1, x_2] + b(\varrho t_1 x_2 + \alpha(x_1)\varrho t_2 - \varrho t_2 x_1 - \alpha(x_2)\varrho t_1) \\ + b(a_1[x_1, x_2] - \sigma([x_1, x_2])a_1), [x_1, x_2]] \\ \times (c[x_1, x_2] + b(t_1 x_2 + \alpha(x_1)t_2 - t_2 x_1 - \alpha(x_2)t_1)) = 0\end{aligned} \tag{3.15}$$

for all $x_1, x_2 \in Q_r$. In particular, Q_r satisfies blended component

$$[b(\varrho t_1 x_2 + \alpha(x_1)\varrho t_2 - \varrho t_2 x_1 - \alpha(x_2)\varrho t_1), [x_1, x_2]](b(t_1 x_2 + \alpha(x_1)t_2 - t_2 x_1 - \alpha(x_2)t_1)) = 0.$$

If $\varrho \neq 0$, then by Lemma 3.1, R is commutative, a contradiction. Thus, assume that $\varrho = 0$ and then d must be inner skew derivation. In this case, by the same argument $\sigma = \alpha$. Thus, by (3.15)

$$\begin{aligned}[a[x_1, x_2] + b(a_1[x_1, x_2] - \alpha([x_1, x_2])a_1), [x_1, x_2]] \\ \times (c[x_1, x_2] + b(t_1 x_2 + \alpha(x_1)t_2 - t_2 x_1 - \alpha(x_2)t_1)) = 0\end{aligned}$$

for all $x_1, x_2 \in Q_r$. In particular, R satisfies blended component

$$[a[x_1, x_2] + b(a_1[x_1, x_2] - \alpha([x_1, x_2])a_1), [x_1, x_2]](b(t_1 x_2 + \alpha(x_1)t_2 - t_2 x_1 - \alpha(x_2)t_1)) = 0$$

for all $x_1, x_2 \in Q_r$. Then by Lemma 3.3, $F(x) = ax$ for all $x \in R$, with $a \in C$. Thus, the proof is completed. $\square$

Acknowledgements. The authors are thankful to the anonymous referee for their very carefully reading and giving constructive comments that led to significant improvement in the readability of the paper.

References

[1] *E. Albaş, N. Argaç, V. De Filippis*: Posner's second theorem and some related annihilating conditions on Lie ideals. Filomat *32* (2018), 1285–1301. zbl MR doi

[2] *J. Bergen, I. N. Herstein, J. W. Kerr*: Lie ideals and derivations of prime rings. J. Algebra *71* (1981), 259–267. zbl MR doi

[3] *C.-L. Chuang*: GPIs having coefficients in Utumi quotient rings. Proc. Am. Math. Soc. *103* (1988), 723–728. zbl MR doi

[4] *C.-L. Chuang, T.-K. Lee*: Identities with a single skew derivation. J. Algebra *288* (2005), 59–77. zbl MR doi

[5] *B. Dhara, N. Argac, E. Albas*: Vanishing derivations and co-centralizing generalized derivations on multilinear polynomials in prime rings. Commun. Algebra *44* (2016), 1905–1923. zbl MR doi

[6] *V. De Filippis*: Annihilators and power values of generalized skew derivations on Lie ideals. Can. Math. Bull. *59* (2016), 258–270. zbl MR doi

[7] *V. De Filippis, O. M. Di Vincenzo*: Vanishing derivations and centralizers of generalized derivations on multilinear polynomials. Commun. Algebra *40* (2012), 1918–1932. zbl MR doi

[8] *V. De Filippis, G. Scudo, M. S. Tammam El-sayiad*: An identity with generalized derivations on Lie ideals, right ideals and Banach algebras. Czech. Math. J. *62* (2012), 453–468. zbl MR doi

[9] *V. De Filippis, F. Wei*: An Engel condition with X-generalized skew derivations on Lie ideals. Commun. Algebra *46* (2018), 5433–5446. zbl MR doi

[10] *T. S. Erickson, W. S. Martindale III, J. M. Osborn*: Prime nonassociative algebras. Pac. J. Math. *60* (1975), 49–63. zbl MR doi

[11] *C. Faith, Y. Utumi*: On a new proof of Litoff's theorem. Acta Math. Acad. Sci. Hung. *14* (1963), 369–371. zbl MR doi

[12] *N. Jacobson*: Structure of Rings. Colloquium Publications 37. AMS, Providence, 1964. zbl MR doi

[13] *C. Lanski*: Differential identities, Lie ideals, and Posner's theorems. Pac. J. Math. *134* (1988), 275–297. zbl MR doi

[14] *C. Lanski*: An Engel condition with derivation. Proc. Am. Math. Soc. *118* (1993), 731–734. zbl MR doi

[15] *P. H. Lee, T. K. Lee*: Lie ideals of prime rings with derivations. Bull. Inst. Math., Acad. Sin. *11* (1983), 75–80. zbl MR

[16] *C.-K. Liu*: An Engel condition with b-generalized derivations. Linear Multilinear Algebra *65* (2017), 300–312. zbl MR doi

[17] *W. S. Martindale III*: Prime rings satisfying generalized polynomial identity. J. Algebra *12* (1969), 576–584. zbl MR doi

[18] *E. C. Posner*: Derivations in prime rings. Proc. Am. Math. Soc. *8* (1957), 1093–1100. zbl MR doi

[19] *J. Vukman*: On derivations in prime rings and Banach algebras. Proc. Am. Math. Soc. *116* (1992), 877–884. zbl MR doi

[20] *T.-L. Wong*: Derivations with power-central values on multilinear polynomials. Algebra Colloq. *3* (1996), 369–378. zbl MR

Authors' addresses: Basudeb Dhara (corresponding author), Department of Mathematics, Belda College, Belda, Deuli, Paschim Medinipur, 721424, West Bengal, India, e-mail: `basu_dhara@yahoo.com`; Kalyan Singh, Department of Mathematics, Jadavpur University, Jadavpur, Kolkata, 700032, West Bengal, India, e-mail: `kalyank4singh@gmail.com`.