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C^*-BASIC CONSTRUCTION BETWEEN NON-BALANCED QUANTUM DOUBLES

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Abstract. For finite groups X, G and the right G-action on X by group automorphisms, the non-balanced quantum double $D(X;G)$ is defined as the crossed product $(\mathbb{C}X^{\rm op})^* \rtimes \mathbb{C}G$. We firstly prove that $D(X;G)$ is a finite-dimensional Hopf C^*-algebra. For any subgroup H of G, $D(X;H)$ can be defined as a Hopf C^*-subalgebra of $D(X;G)$ in the natural way. Then there is a conditonal expectation from $D(X;G)$ onto $D(X;H)$ and the index is $[G;H]$. Moreover, we prove that an associated natural inclusion of non-balanced quantum doubles is the crossed product by the group algebra.

Keywords: non-balanced quantum double; C^*-basic construction; crossed product; action

MSC 2020: 46L05, 16S35

1. Introduction

Jones index theory has found important applications in algebraic quantum field theory, conformal field theory and so on, see [10], [13]. Especially, the basic construction plays an important role in constructing subfactors with the index $4\cos^2(\pi/n)$, $n = 3, 4, \ldots$ Later, Jones index theory has been generalized to general inclusions of factors by Kosaki, see [11], Loi, see [12], Hiai, see [6], Pimsner and Popa, see [16]. On the basis of other works, Watatani in [20] considered a conditional expectation $\Gamma\colon B \to A$ for a C^*-algebra B and its C^*-subalgebra A, and then introduced the C^*-basic construction $\langle B, \gamma_A\rangle_{C^*}$.

As we all know, the quantum double (see [5]) of Drinfeld plays an important role in the field of mathematical physics. The quasitriangular structure not only provides a way to find solutions for the Yang-Baxter equation of statistical mechanics, but also

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leads to a braiding in the category of representations. Usually, quantum double $D(G)$ is defined as the crossed product $(\mathbb{C}G^{\mathrm{op}})^* \rtimes \mathbb{C}G$. In [4], the authors had shown that the basic construction for a certain natural inclusion of infinite crossed product algebras is the crossed product by the quantum double. In [2], we considered the field algebra on Hopf spin models, and had shown that the C^*-basic construction for the field and the observable algebra is the crossed product of the field algebra by the quantum double of a Hopf algebra.

For a finite group G and a subgroup H of G, we can define $D(G;H)$ as the crossed product $C(G) \rtimes \mathbb{C}H$ with respect to the adjoint action. If $H = G$, then $D(G;H)$ reduces to the double algebra $D(G)$. The main difference is that $D(G)$ is quasi-triangular while $D(G;H)$ is not, see [17]. Jiang and Zhu in [8] constructed the map $E\colon D(G) \to D(G;H)$ and then proved that E is a conditional expectation of index-finite type. In [22], the authors considered the basic construction for $D(G)$ and $D(G;H)$ and gave the concrete form. In the quantum double, two groups are the same. In order to generalize the quantum double, Chen and Li in [3] defined non-balanced quantum double $D(X;G)$ as $(\mathbb{C}X^{\mathrm{op}})^* \rtimes \mathbb{C}G$, where G and X are two finite groups and the right G-action on X. Inspired by [3] and [22], we firstly define *-operation on $D(X;G)$ such that $D(X;G)$ becomes a Hopf C^*-algebra. Moreover, for any subgroup H of G, one can define a Hopf C^*-subalgebra $D(X;H)$ of $D(X;G)$, and consider C^*-basic construction for $D(X;G)$ and $D(X;H)$, which generalizes the results in [22].

The paper is organized as follows. In Section 2, we collect the main definitions and results on Hopf C^*-algebras and the conditional expectation, which are used in the sequel. In Section 3, for finite groups X, G and the right G-action on X by group automorphisms, the non-balanced quantum double $D(X;G)$ is defined as the crossed product $(\mathbb{C}X^{\mathrm{op}})^* \rtimes \mathbb{C}G$. There is a *-structure on $(\mathbb{C}X^{\mathrm{op}})^* \rtimes \mathbb{C}G$ such that it becomes a Hopf C^*-algebra. The ordinary quanum double of a finite group is a special case of this construction. Moreover, for any subgroup H of G, the non-balanced quantum double $D(X;H)$ can be naturally defined as a Hopf C^*-subalgebra of $D(X;G)$. Consider the conditional expectation Φ from $D(X;G)$ onto $D(X;H)$, then it is of index-finite type with Index $\Phi = [G;H]1_{D(X;G)}$. Based on the results of Section 3 we can construct in Section 4 the crossed product and prove that the C^*-basic construction $\langle D(X;G), \varphi\rangle_{C^*}$ is the crossed product up to isomorphism.

2. Preliminaries

Definition 2.1 ([9])**.** Let $(K, m, \iota, \Delta, \varepsilon, S)$ be a Hopf algebra. Then K is said to be a Hopf *-algebra if there exists an antilinear involution * on K which satisfies the following conditions:

(1) the map * is an antimorphism of real algebras, as well as a morphism of real coalgebras. In other words, for any $a \in K$,

$$\Delta(a^*) = \Delta(a)^*, \quad \varepsilon(a^*) = \overline{\varepsilon(a)};$$

(2) the map * is compatible with the antipode S of K, which means

$$(S \circ *)^2 = \mathrm{id}.$$

If K is a Hopf *-algebra of finite dimension as well as a C^*-algebra, then K is said to be a Hopf C^*-algebra.

Remark 2.1.

(1) For any finite group G, the group algebra $\mathbb{C}G$ is a finite dimensional Hopf C^*-algebra. Denote by G^{op} the opposite group of G (i.e., with opposite muliplication), then $\mathbb{C}G^{\mathrm{op}}$ is also a Hopf C^*-algebra.
(2) If K is a Hopf C^*-algebra of finite dimension, so is K^*, which is the dual of K.

One can refer to some definitions and properties on Hopf algebras (see [14], [18]), and on Hopf C^*-algebras, see [9], [19]. Also we shall use the summation convention, which is standard in Hopf algebra theory: $\Delta(a) = \sum\limits_{(a)} a_{(1)} \otimes a_{(2)}$.

In this section, suppose that B is a C^*-algebra over $\mathbb{C}$ and A a C^*-subalgebra with a common unit 1_B.

Definition 2.2 ([20])**.** A linear map $\Gamma\colon\ B \to A$ is called a *conditional expectation* if it satisfies the following conditions:

(1) $\Gamma(1_B) = 1_B$;
(2) (bimodular property) for $a_1, a_2 \in A$, $b \in B$,

$$\Gamma(a_1 b a_2) = a_1 \Gamma(b) a_2;$$

(3) Γ is positive, i.e., for any $b \in B$, $\Gamma(b^* b)$ is a positive element in A.

Actually, if $\Gamma\colon\ B \to A$ is a conditional expectation, Γ is a projection of norm one, see [1].

Definition 2.3 ([20])**.** Let $\Gamma\colon\ B \to A$ be a conditional expectation. A finite family $\{(u_i, u_i^*) \in B \times B,\ i = 1, 2, \ldots, n\}$ is called a *quasi-basis* for Γ if for all $x \in B$,

$$\sum_{i=1}^{n} u_i \Gamma(u_i^* x) = x = \sum_{i=1}^{n} \Gamma(x u_i) u_i^*.$$

Moreover, if there exists a quasi-basis for Γ, we call Γ of index-finite type. In this case the index of Γ can be defined by

$$\text{Index } \Gamma = \sum_{i=1}^{n} u_i u_i^*.$$

Remark 2.2.
(1) If Γ is of index-finite type, then Index Γ is in the center of B and does not depend on the choice of quasi-basis. And the existence of a quasi-basis guarantees that conditional expectation Γ is non-degenerate, see [20].
(2) Let G be a finite group, H a subgroup of G and $\Gamma\colon \mathbb{C}G \to \mathbb{C}H$ the canonical conditional expectation. Then Index Γ is exactly $[G;H]$, the index of H in G.

3. C^*-index between non-balanced quantum doubles

Suppose that G and X are finite groups, e is the unit of G, we have two group algebras $\mathbb{C}G$ and $\mathbb{C}X^{\mathrm{op}}$.

Definition 3.1. The non-balanced quantum double $D(X;G)$ is defined as the crossed product $(\mathbb{C}X^{\mathrm{op}})^* \rtimes \mathbb{C}G$ with respect to the right action by group automorphisms.

Chen and Li in [3] have proved that $D(X;G)$ is a Hopf algebra with the structure maps in the following defined by

$$\begin{aligned}
(x,g)(y,h) &= \delta_{x^g,y}(x,gh), && \text{(multiplication)}\\
1_{D(X;G)} &= \sum_{x\in X}(x,e), && \text{(unit)}\\
\Delta(x,g) &= \sum_{a\in X}(a^{-1}x,g)\otimes(a,g), && \text{(comultiplication)}\\
\varepsilon(x,g) &= \delta_{1_X,x}, && \text{(counit)}\\
S(x,g) &= ((x^g)^{-1},g^{-1}), && \text{(antipode)}
\end{aligned}$$

for (x,g), $(y,h) \in D(X;G)$, where 1_X is the unit of X, x^g denotes the G-action and

$$\delta_{x,y} = \begin{cases} 1 & \text{if } x = y,\\ 0 & \text{if } x \neq y.\end{cases}$$

Moreover, we can define $(x,g)^* = (x^g, g^{-1})$ on the basis elements and extend it antilinearly to the Hopf algebra $D(X;G)$.

Theorem 3.1. *The algebra $D(X;G)$ becomes a Hopf C^*-algebra.*

P r o o f. Firstly, $D(X;G)$ is a *-algebra, i.e., $(x,g)^{**}=(x,g)$ and $((x,g)(y,h))^*=(y,h)^*(x,g)^*$ for any $(x,g),(y,h)\in D(X;G)$. In fact,

$$(x,g)^{**}=(x^g,g^{-1})^*=((x^g)^{g^{-1}},g^{-1})=(x,g),$$

and

$$\begin{aligned}((x,g)(y,h))^*&=(xy^{g^{-1}},gh)^*=((xy^{g^{-1}})^{gh},h^{-1}g^{-1})=(x^{gh}y^h,h^{-1}g^{-1})\\&=(y^hx^{gh},h^{-1}g^{-1})=(y^h,h^{-1})(x^g,g^{-1})=(y,h)^*(x,g)^*.\end{aligned}$$

In the sequel we will show that Δ and ε are *-homomorphisms and $S(S(x,g)^*)^*=(x,g)$ for any $(x,g)\in D(X;G)$, which yields that $D(X;G)$ is a Hopf *-algebra. Note that

$$\begin{aligned}(\Delta(x,g))^*&=\Big(\sum_{a\in X}(a^{-1}x,g)\otimes(a,g)\Big)^*=\sum_{a\in X}((a^{-1}x)^g,g^{-1})\otimes(a^g,g^{-1})\\&=\sum_{a\in X}((a^{-1})^gx^g,g^{-1})\otimes(a^g,g^{-1})=\sum_{b\in X}(b^{-1}x^g,g^{-1})\otimes(b,g^{-1})\\&=\Delta(x^g,g^{-1})=\Delta((x,g)^*).\end{aligned}$$

Similarly, we can show that ε is also a *-homomorphism, and

$$S(S(x,g)^*)^*=S(S(x^g,g^{-1}))^*=S(x^{-1},g)^*=S((x^{-1})^g,g^{-1})=(x,g).$$

Moreover, there is a unique element $z=|G|^{-1}\sum\limits_{g\in G}(1_X,g)$, called an integral, satisfying $(x,g)z=z(x,g)=\varepsilon(x,g)z$, $(x,g)\in D(X;G)$, and $\varepsilon(z)=1$. Then one can define the inner product $\langle(x,g)(y,h)\rangle=z((y,h)^*(x,g))$ for any (x,g), $(y,h)\in D(X,G)$ such that $D(X,G)$ can be turned into a Hilbert space. For any $(x,g)\in D(X,G)$, the map λ defined as $(\lambda(x,g))(y,h)=(x,g)(y,h)$ is a faithful *-representation on $D(X,G)$. Hence, $D(X,G)$ is a Hopf C^*-algebra of finite dimension. $\square$

Remark 3.1.

(1) If $X=G$ and the G-action is given by conjugation (i.e., $x^g=g^{-1}xg$), then $D(X;G)$ reduces to the quantum double $D(G)$. Note that $(\mathbb{C}X^{\mathrm{op}})^*$ and $\mathbb{C}G$ can be viewed as Hopf C^*-subalgebras of $D(X;G)$ via the canonical embeddings $i_{(X^{\mathrm{op}})^*}x=(x,e)$ and $i_Gg=(1_X,g)$.

(2) Moreover, if X is a normal subgroup of G, we can get the quasi-triangular Hopf C^*-algebra $D(X;G)$, which is not a Hopf subalgebra of the quantum double $D(G)$, see [21].

(3) If G is a subgroup of X, the Hopf C^*-algebra $D(X;G)$ is considered in [7].

For any subgroup H of G, we can define the space $D(X;H)$ as the crossed product $(\mathbb{C}X^{\mathrm{op}})^*\rtimes\mathbb{C}H$. There exists a natural Hopf structure such that $D(X;H)$ is a Hopf C^*-subalgebra of $D(X;G)$ with the common unit.

Consider the natural mapping $\Phi\colon\ D(X;G)\to D(X;H)$ defined by

$$\Phi\Bigg(\sum_{x\in X,\, g\in G} k_{x,g}(x,g)\Bigg)=\sum_{x\in X,\, g\in H} k_{x,g}(x,g)$$

for any $k_{x,g}\in\mathbb{C}$ and $(x,g)\in D(X;G)$.

Theorem 3.2. *The map Φ is a conditional expectation from $D(X;G)$ onto $D(X;H)$ of index-finite type and* Index $\Phi=[G;H]1_{D(X;G)}$.

Proof. We only show that Φ satisfies bimodular property and is positive.

(2) For $(y,g)\in D(X;G)$, $(x,h)\in D(X;H)$,

$$\begin{aligned}\Phi((y,g)(x,h))&=\Phi(\delta_{y^g,x}(y,gh)=\begin{cases}\delta_{y^g,x}(y,gh) & \text{if } g\in H,\\ 0 & \text{if } g\notin H,\end{cases}=\begin{cases}(y,g)(x,h) & \text{if } g\in H,\\ 0 & \text{if } g\notin H,\end{cases}\\ &=\Phi(y,g)(x,h).\end{aligned}$$

Similarly, $\Phi((x,h)(y,g))=(x,h)\Phi(y,g)$.

(3) For $(y,g)\in D(X;G)$ we have

$$\Phi((y,g)^*(y,g))=\Phi((y^g,g^{-1})(y,g))=\Phi(y^g,e)=(y^g,e)=(y,g)^*(y,g).$$

Hence, Φ is a conditional expectation.

Let $u_i=\sum\limits_{x\in X}(x,t_i)$ and $u_i{}^*=\sum\limits_{y\in X}(y,t_i^{-1})$, $i=1,2,\dots,[G;H]$, we can compute

$$\begin{aligned}&\sum_i u_i\Phi\Bigg(u_i{}^*\sum_{x\in X,\, g\in G}k_{x,g}(x,g)\Bigg)\\ &=\sum_i u_i\Phi\Bigg(\sum_{y,x\in X,\, g\in G}(y,t_i^{-1})k_{x,g}(x,g)\Bigg)=\sum_i u_i\Phi\Bigg(\sum_{y,x\in X,\, g\in G}\delta_{y,x^{t_i}}k_{x,g}(y,t_i^{-1}g)\Bigg)\\ &=\sum_i u_i\Phi\Bigg(\sum_{x\in X,\, h\in G}k_{x,t_ih}(x^{t_i},h)\Bigg)=\sum_i\sum_{y\in X}(y,t_i)\sum_{x\in X,\, h\in H}k_{x,t_ih}(x^{t_i},h)\\ &=\sum_i\sum_{x\in X,h\in H}k_{x,t_ih}((x^{t_i})^{t_i^{-1}},t_ih)=\sum_i\sum_{x\in X,\, h\in H}k_{x,t_ih}(x,t_ih)=\sum_{x\in X,\, g\in G}k_{x,g}(x,g).\end{aligned}$$

Similarly, $\sum\limits_i\Phi\Big(\sum\limits_{x\in X,\, g\in G}k_{x,g}(x,g)u_i\Big)u_i{}^*=\sum\limits_{x\in X,\, g\in G}k_{x,g}(x,g)$. Thus,

$$\text{Index }\Phi=\sum_i u_iu_i{}^*=\sum_i\sum_{x,y\in X}(x,t_i)(y,t_i^{-1})=\sum_i\sum_{x\in X}(x,e)=[G;H]1_{D(X;G)}.$$

□

In the following we consider the C^*-basic construction for the conditional expectation Φ.

Let $\mathfrak{L}_{D(X;H)}(D(X;G))$ be the C^*-algebra of all adjointable right $D(X;H)$-module homomorphism. In the case, the element in $D(X;G)$ can be viewed as the left multiplication operator in $\mathfrak{L}_{D(X;H)}(D(X;G))$, and Φ as a projection in $\mathfrak{L}_{D(X;H)}(D(X;G))$, denoted by φ. Then the C^*-basic construction $\langle D(X;G),\varphi\rangle_{C^*}$ is the C^*-subalgebra of $\mathfrak{L}_{D(X;H)}(D(X;G))$ generated by $\{(x,g)\colon (x,g)\in D(X;G)\}$ and φ.

Remark 3.2. The C^*-algebra $\langle D(X;G),\varphi\rangle_{C^*}$ can be linearly generated by the set $\{(x,g)\varphi(y,h)\colon (x,g),(y,h)\in D(X;G)\}$. Indeed, $\varphi=\sum\limits_{x,y\in X}(x,e)\varphi(y,e)$. And for $(x,g)\in D(X;G)$, we have that

$$\begin{aligned}\sum_{i=1}^{k}(x,t_i)\varphi(x^{t_i},t_i^{-1}g)(y,h)&=\sum_{i=1}\delta_{x^{t_i},y^{g^{-1}t_i}}(x,t_i)\Phi(x^{t_i},t_i^{-1}gh)\\&=\delta_{x^{t_j},y^{g^{-1}t_j}}(x,t_j)(x^{t_j},t_j^{-1}gh)=\delta_{x^{t_j},y^{g^{-1}t_j}}(x,gh)\\&=\delta_{x,y^{g^{-1}}}(x,gh)=(x,g)(y,h),\end{aligned}$$

where we use the property that for $gh\in G$, there exists t_j such that $gh\in t_jH$. Hence, $(x,g)=\sum\limits_{i=1}^{k}(x,t_i)\varphi(x^{t_i},t_i^{-1}g)$.

4. C^*-basic construction for the conditional expectation Φ

In this section, we will prove that there is a left action α of $\mathbb{C}G$ on $C(G/H\times X)$, and the resulting crossed product C^*-algebra $C(G/H\times X)\rtimes_\alpha\mathbb{C}G$ is C^*-isomorphic to the C^*-basic construction $\langle D(X;G),\varphi\rangle_{C^*}$.

Suppose that H is a subgroup of the group G with the index $[G;H]=k$. Let G/H be the set of all left cosets of H, which means $G=\bigcup\limits_i t_iH$, and $i\neq j$ induces that $t_iH\cap t_jH=\emptyset$, denoted by $G/H=\{[t_1],[t_2],\dots,[t_k]\}$, and $C(G/H\times X)$ the C^*-algebra of complex functions on $G/H\times X$. Then we can choose the set $\{[t_i]\colon i=1,2,\dots,k\}$ as a linear basis for the C^*-algebra $C(G/H)$. Since $C(G/H\times X)\cong C(G/H)\otimes C(X)$, then $C(G/H\times X)$ has a linear basis $\{(t_i,x)\colon i=1,2,\dots,k;\ x\in X\}$, where we use (t_i,x) for $([t_i],\delta_x)$, and $\{\delta_x\colon x\in X\}$ is a linear basis of $C(X)$.

For any $g\in G$, $(t_i,x)\in C(G/H\times X)$, the map $\alpha\colon\mathbb{C}G\times C(G/H\times X)\to C(G/H\times X)$ is defined by

$$\alpha(g\times(t_i,x))=(gt_i,x^{g^{-1}}),$$

and linearly extended both on $\mathbb{C}G$ and $C(G/H\times X)$. Then α is an automorphic action. Under this action of $\mathbb{C}G$ on $C(G/H\times X)$, we can obtain the crossed product, denoted by $C(G/H\times X)\rtimes\mathbb{C}G$, which can be generated by $\{(t_i,x),g\colon (t_i,x)\in D(X;G),\ g\in G;\ i=1,2,\dots,k\}$.

Using the linear basis elements $(t_i, x; g)$ of $C(G/H \times X) \rtimes \mathbb{C}G$, the multiplication and *-operation are defined as

$$(t_i, x; g)(t_j, y; h) = \delta_{[t_i],[gt_j]}\delta_{x^g,y}(t_i, x; gh), \quad (t_i, x; g)^* = (g^{-1}t_i, x^g; g^{-1}).$$

Clearly, $C(G/H \times X) \rtimes \mathbb{C}G$ is a unital C^*-algebra with the unit $\sum\limits_{x\in X}\sum\limits_{i=1}^{k}(t_i, x; e)$.

Proposition 4.1. *The elements* $\sum\limits_{i}(t_i, x; g)$ *and* $\sum\limits_{y\in X}(t_1, y; e)$ *satisfy*

(1) $\sum\limits_{y\in X}(t_1, y; e)\sum\limits_{i}(t_i, x; g)\sum\limits_{y\in X}(t_1, y; e) = \delta_{[t_1],[g]}\sum\limits_{i}(t_i, x; g)\sum\limits_{y\in X}(t_1, y; e)$;

(2) $\sum\limits_{x\in X}(t_1, x; e)$ *is a projection in* $C(G/H \times X) \rtimes \mathbb{C}G$, *i.e.*,

$$\Big(\sum_{x\in X}(t_1, x; e)\Big)^2 = \Big(\sum_{x\in X}(t_1, x; e)\Big)^* = \sum_{x\in X}(t_1, x; e).$$

P r o o f. (1) Notice that

$$\begin{aligned}\sum_{y\in X}(t_1, y; e)\sum_{i}(t_i, x; g)\sum_{y\in X}(t_1, y; e) &= \sum_{y\in X}(t_1, y; e)\sum_{i}\sum_{y\in X}\delta_{[t_i],[g]}\delta_{x^g,y}(t_i, x; g)\\ &= \sum_{y\in X}(t_1, y; e)(g, x; g) = \delta_{[t_1],[g]}(t_1, x; g),\end{aligned}$$

and

$$\begin{aligned}&\delta_{[t_1],[g]}\sum_{i}(t_i, x; g)\sum_{y\in X}(t_1, y; e)\\ &\quad= \delta_{[t_1],[g]}(g, x; g) = \delta_{[t_1],[g]}\sum_{i}\sum_{y\in X}\delta_{[t_i],[g]}\delta_{x^g,y}(t_i, x; g) = \delta_{[t_1],[g]}(g, x; g).\end{aligned}$$

Hence, the proof is completed. □

(2) We can compute that

$$\begin{aligned}\Big(\sum_{x\in X}(t_1, x; e)\Big)^2 &= \sum_{x,y\in X}(t_1, x; e)(t_1, y; e) = \sum_{x,y\in X}\delta_{x,y}(t_1, x; e)\\ &= \sum_{x\in X}(t_1, x; e) = \sum_{x\in X}(t_1, x; e)^*.\end{aligned}$$

□

Theorem 4.1. *The* C^**-basic construction* $\langle D(X; G), \varphi\rangle_{C^*}$ *is* C^**-isomorphic to the crossed product* C^**-algebras* $C(G/H \times X) \rtimes \mathbb{C}G$.

P r o o f. By Remark 3.2, we have

$$\langle D(X;G),\varphi\rangle_{C^*}=\overline{\{(x,g)\varphi(y,h)\colon\ (x,g),(y,h)\in D(X;G)\}}^{\|\cdot\|},$$

and then define the map $\Pi\colon\ \langle D(X;G),\varphi\rangle_{C^*}\to C(G/H\times X)\rtimes\mathbb{C}G$ as

$$\Pi((x,g)\varphi(y,h))=\delta_{x^g,y}(g,x;gh).$$

Then Π is well-defined. Moreover, we have

$$\begin{aligned}
&\Pi((x_1,g_1)\varphi(y_1,h_1)(x_2,g_2)\varphi(y_2,h_2))\\
&\quad=\Pi(\delta_{y_1^{h_1},x_2}(x_1,g_1)\varphi(y_1,h_1g_2)\varphi(y_2,h_2))\\
&\quad=\Pi(\delta_{[h_1g_2],[t_1]}\delta_{y_1^{h_1},x_2}(x_1,g_1)(y_1,h_1g_2)\varphi(y_2,h_2))\\
&\quad=\Pi(\delta_{[h_1g_2],[t_1]}\delta_{y_1^{h_1},x_2}\delta_{x_1^{g_1},y_1}(x_1,g_1h_1g_2)\varphi(y_2,h_2))\\
&\quad=\Pi(\delta_{[h_1g_2],[t_1]}\delta_{y_1^{h_1},x_2}\delta_{x_1^{g_1},y_1}(x_1,g_1h_1g_2)\varphi(y_2,h_2))\\
&\quad=\delta_{[h_1g_2],[t_1]}\delta_{y_1^{h_1},x_2}\delta_{x_1^{g_1},y_1}\delta_{x_1^{g_1h_1g_2},y_2}(g_1h_1g_2,x_1;g_1h_1g_2h_2)\\
&\quad=\delta_{[g_1h_1g_2],[g_1]}\delta_{x_1^{g_1},y_1}\delta_{x_2^{g_2},y_2}\delta_{x_1^{g_1h_1},x_2}(g_1,x_1;g_1h_1g_2h_2)\\
&\quad=(g_1,x_1;g_1h_1)(g_2,x_2;g_2h_2)\\
&\quad=\Pi((x_1,g_1)\varphi(y_1,h_1))\Pi((x_2,g_2)\varphi(y_2,h_2)),
\end{aligned}$$

and

$$\begin{aligned}
\Pi((x,g)\varphi(y,h))^*&=\Pi((y,h)^*\varphi(x,g)^*)=\Pi((y^h,h^{-1})\varphi(x^g,g^{-1}))\\
&=\delta_{x^{gh},y^h}(h^{-1},y^h;h^{-1}g^{-1})=\delta_{x^g,y}(h^{-1},x^{gh};h^{-1}g^{-1})\\
&=(\delta_{x^g,y}(g,x;gh))^*=(\Pi((x,g)\varphi(y,h)))^*.
\end{aligned}$$

Hence, Π preserves the algebra structure and $*$-operation, and it follows from Theorem 2.1.7 of [15] that Π is norm-decreasing.

Again, for any $(t_i,x;g)\in C(G/H\times X)\rtimes\mathbb{C}G$, we can take $(x,t_i)\varphi(x^{t_i},t_i^{-1}g)$ in $\langle D(X;G),\varphi\rangle_{C^*}$ such that

$$\Pi((x,t_i)\varphi(x^{t_i},t_i^{-1}g))=(t_i,x;g),$$

which means that Π is surjective. Combining this with the open mapping theorem, we know that Π is C^*-isomorphic. □

Remark 4.1. It follows from Proposition 2.10.11 in [20] that the C^*-basic construction does not depend on the choice of conditional expectations. Combined with Theorem 4.1, we know that the crossed product C^*-algebra $C(G/H\times X)\rtimes\mathbb{C}G$ does not depend on the choice of Φ.

Example 4.1.

(1) Let $X = G$ and consider the map $\Phi\colon D(G) \to D(G;H)$ defined by

$$\Phi\colon D(G) \to D(G;H), \qquad \sum_{g,h\in G} k_{g,h}(g,h) \mapsto \sum_{g\in G,h\in H} k_{g,h}(g,h),$$

where $k_{g,h} \in \mathbb{C}$ and $(g,h) \in D(G)$. Then Φ is a conditional expectation from $D(G)$ onto $D(G;H)$. Using Theorem 4.1, the C^*-basic construction $\langle D(G), \varphi\rangle_{C^*}$ is the crossed product $C(G/H \times G) \rtimes \mathbb{C}G$, which is the main result in [22].

(2) Let $X = G$ and consider the map $\Phi\colon \mathbb{C}G \to \mathbb{C}H$ defined by

$$\Phi\Big(\sum_{g\in G} \lambda_g g\Big) = \sum_{h\in H} \lambda_h h,$$

where $\lambda_g \in \mathbb{C}$ for $g \in G$. Then Φ is a conditional expectation from $\mathbb{C}G$ onto $\mathbb{C}H$. And there is an inclusion $i\colon \mathbb{C}G \to D(G)$ given by $i(g) = \sum\limits_{f\in G}(f,g)$, which together with Theorem 4.1 yields that the C^*-basic construction $\langle \mathbb{C}G, \varphi\rangle_{C^*}$ is the crossed product $C(G/H) \rtimes \mathbb{C}G$ in [20].

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