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A NOTE ON AVERAGE BEHAVIOUR OF THE FOURIER COEFFICIENTS OF jth SYMMETRIC POWER L-FUNCTION OVER CERTAIN SPARSE SEQUENCE OF POSITIVE INTEGERS

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Abstract. Let $j \geqslant 2$ be a given integer. Let H_k^* be the set of all normalized primitive holomorphic cusp forms of even integral weight $k \geqslant 2$ for the full modulo group $\mathrm{SL}(2,\mathbb{Z})$. For $f \in H_k^*$, denote by $\lambda_{\mathrm{sym}^j f}(n)$ the nth normalized Fourier coefficient of jth symmetric power L-function ($L(s,\mathrm{sym}^j f)$) attached to f. We are interested in the average behaviour of the sum

$$\sum_{\substack{n=a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2\leqslant x\\ (a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}} \lambda_{\mathrm{sym}^j f}^2(n),$$

where x is sufficiently large, which improves the recent work of A. Sharma and A. Sankaranarayanan (2023).

Keywords: cusp form; Fourier coefficient; symmetric power L-function

MSC 2020: 11F11, 11F30, 11F66

1. Introduction

Let H_k^* be the set of all normalized primitive holomorphic cusp forms of even integral weight $k \geqslant 2$ for the full modulo group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Then for each $f \in H_k^*$, let $\lambda_f(n)$ be the normalized nth Fourier coefficient of the Fourier expansion of $f(z)$ at the cusp ∞, i.e.,

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} \mathrm{e}^{2\pi i n z},$$

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where $\Im(z) > 0$. Then the L-function attached to $\lambda_f(n)$ is defined as

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}$$

for $\Re(s) > 1$, where $\lambda_f(n)$ are the eigenvalues of all the Hecke operators. It is well known that $\lambda_f(1) = 1$ and $\lambda_f(n)$ is real and satisfies the multiplicative property

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\Big(\frac{mn}{d^2}\Big),$$

where $m \geqslant 1$ and $n \geqslant 1$ are positive integers. Moreover, in 1974, Deligne in [2] proved the Ramanujan-Petersson conjecture

$$|\lambda_f(n)| \leqslant d(n),$$

where $d(n)$ is the divisor function. In other words, Deligne proved that for any prime p there exist complex numbers $\alpha(p)$ and $\beta(p)$ such that

$$\alpha(p) + \beta(p) = \lambda_f(p), \quad \alpha(p)\beta(p) = |\alpha(p)| = |\beta(p)| = 1.$$

Therefore, $L(s, f)$ can be written as

$$L(s, f) = \prod_p \Big(1 - \frac{\alpha(p)}{p^s}\Big)^{-1} \Big(1 - \frac{\beta(p)}{p^s}\Big)^{-1} \quad (\Re(s) > 1).$$

The jth symmetric power L-function attached to $f \in H^*$ is defined as

$$L(s, \mathrm{sym}^j f) := \prod_p \prod_{m=0}^{j} \Big(1 - \frac{\alpha_p^{j-m}\beta_p^m}{p^s}\Big)^{-1} \quad (\Re(s) > 1),$$

which can also be expressed as the Dirichlet series

$$L(s, \mathrm{sym}^j f) = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s} = \prod_p \Bigg(1 + \sum_{m \geqslant 1} \frac{\lambda_{\mathrm{sym}^j f}(p^m)}{p^{ms}}\Bigg).$$

Note that $\lambda_{\mathrm{sym}^j f}(n)$ is a real multiplicative function, and

$$L(s, \mathrm{sym}^0 f) = \zeta(s), \quad L(s, \mathrm{sym}^1 f) = L(s, f).$$

The average behaviour of Fourier coefficients attached to cusp forms is an interesting and important topic in modern number theory, and has been considered by a number of authors in various aspects. In 2006, Fomenko in [3] established that

$$\sum_{n\leqslant x}\lambda_{\mathrm{sym}^2 f}(n)\ll x^{1/2}\log^2 x.$$

Later, the related sums have been paid more attention by many scholars. Furthermore, Lau and Lü in [10] and Tang and Wu in [16] studied the sums related to symmetric power L-functions of higher degrees. On the other hand, Fomenko in [4] studied the sum of $\lambda^2_{\mathrm{sym}^2 f}(n)$. In fact, he showed that

$$\sum_{n\leqslant x}\lambda^2_{\mathrm{sym}^2 f}(n)=cx+O(x^{\delta}),\quad \delta<1.$$

Later, Tang in [15] improved the error term of the above sum to $O(x^{10/13}\log^9 x)$. In 2013, Zhai in [17] considered the sum

$$\sum_{\substack{n=a^2+b^2\leqslant x\\ a,b\in\mathbb{Z}}}\lambda_f^l(n)$$

for $x\geqslant 1$, $3\leqslant l\leqslant 8$. This provided more interesting result to the average sum of Fourier coefficients. In 2022, Sharma and Sankaranarayanan in [13] proved that

$$\sum_{\substack{n=a^2+b^2+c^2+d^2\leqslant x\\ a,b,c,d\in\mathbb{Z}}}\lambda^2_{\mathrm{sym}^2 f}(n)=c_1x^2+O_f(x^{9/5+\varepsilon})$$

for any $\varepsilon>0$ and sufficiently large $x\geqslant x_0$, where $c_f>0$ is a suitable constant depending on f. Later, Hua in [6] improved the error term. In addition, in [14], Sharma and Sankaranarayanan established the asymptotic formula of the sum

$$S_j:=\sum_{\substack{n=a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2\leqslant x\\ (a_1,a_2,a_3,a_4,a_5,a_6)\in\mathbb{Z}^6}}\lambda^2_{\mathrm{sym}^j f}(n).$$

Indeed, they got

$$S_j=c_{f,j}x^3+O(x^{3-6/(3(j+1)^2+1)+\varepsilon}) \tag{1.1}$$

for $j\geqslant 2$ and any $\varepsilon>0$, where $c_{f,j}$ is an effective constant depending on f and j. Our purpose is to improve the error term of (1.1). We have the following result.

Theorem 1.1. *Let $j \geqslant 2$ be any fixed integer. For a sufficiently large x and any small constant $\varepsilon > 0$ we have*

$$S_j = c_{f,j}x^3 + O(x^{3-6/(3(j+1)^2-1)+\varepsilon}),$$

where

$$c_{f,j} = \frac{16}{3}L(3,\chi)\prod_{n=1}^{j} L(1,\mathrm{sym}^{2n} f)L(3,\mathrm{sym}^{2n} f\otimes\chi)\mathcal{H}_j(3),$$

and χ is the nonprincipal Dirichlet character modulo 4 and $\mathcal{H}_j(s)$ is a convergent Dirichlet series in Lemma 2.2.

Remark 1.1. Obviously, the error term in our theorem is better than (1.1). In our proof, we give one different way to move the integral line, which can be used in more sums.

Notation. Throughout this paper, the letter ε represents a sufficiently small positive constant, not necessarily the same at each occurrences. The constants, both explicit and implicit, in Vinogradov symbols may depend on f and ε.

2. Preliminary lemmas

In this section, we introduce some important lemmas for our proof. Let $r_k(n) := \#\{(n_1, n_2, \ldots, n_k) \in \mathbb{Z}^k\colon n_1^2+n_2^2+\ldots+n_k^2 = n\}$. Then we have the following lemma.

Lemma 2.1. *Let χ be the nonprincipal Dirichlet character modulo 4. Then we have*

$$S_j = \sum_{n\leqslant x}\lambda^2_{\mathrm{sym}^j f}(n)r_6(n) = 16\sum_{n\leqslant x}\lambda^2_{\mathrm{sym}^j f}(n)l(n) - 4\sum_{n\leqslant x}\lambda^2_{\mathrm{sym}^j f}(n)v(n),$$

where $l(n) = \sum\limits_{d\mid n}\chi(d)n^2/d^2$ and $v(n) = \sum\limits_{d\mid n}\chi(d)d^2$.

P r o o f. For the details, we refer to [14], Section 2. □

Lemma 2.2. *Let f be a normalized primitive holomorphic cusp form of weight k for $\mathrm{SL}(2,\mathbb{Z})$, and let $\lambda_{\mathrm{sym}^j f}(n)$ be the nth normalized Fourier coefficient of the jth symmetric power L-function associated to f. If*

$$\mathcal{L}_j(s) = \sum_{n=1}^{\infty}\frac{\lambda^2_{\mathrm{sym}^j f}(n)l(n)}{n^s}$$

for $\Re(s) > 3$, *then*

$$\mathcal{L}_j(s) = \mathcal{G}_j(s)\mathcal{H}_j(s),$$

where

$$\mathcal{G}_j(s) := \zeta(s-2)L(s,\chi)\prod_{n=1}^{j} L(s-2,\mathrm{sym}^{2n} f)L(s,\mathrm{sym}^{2n} f\otimes\chi),$$

and χ *is the nonprincipal character modulo* 4. *The function* $\mathcal{H}_j(s)$ *admits a Dirichlet series which converges uniformly and absolutely in the half plane* $\Re(s) > \frac{5}{2}$, *and* $\mathcal{H}_j(s) \neq 0$ *on* $\Re(s) = 3$.

Proof. The author refers the reader to [14], Lemma 2.2. □

Lemma 2.3 ([14], Lemma 2.3). *Let* f *be a normalized primitive holomorphic cusp form of weight* k *for* $\mathrm{SL}(2,\mathbb{Z})$, *and let* $\lambda_{\mathrm{sym}^j f}(n)$ *be the* n*th normalized Fourier coefficient of the* j*th symmetric power* L*-function associated to* f. *If*

$$V_j(s) = \sum_{n=1}^{\infty}\frac{\lambda^2_{\mathrm{sym}^j f}v(n)}{n^s}$$

for $\Re(s) > 3$, *then*

$$V_j(s) = G_j(s)H_j(s),$$

where

$$G_j(s) := \zeta(s)L(s-2,\chi)\prod_{n=1}^{j} L(s,\mathrm{sym}^{2n} f)L(s-2,\mathrm{sym}^{2n} f\otimes\chi),$$

and χ *is the nonprincipal character modulo* 4. *Here* $H_j(s)$ *admits a Dirichlet series which converges uniformly and absolutely in the half plane* $\Re(s) > \frac{5}{2}$, *and* $H_j(s) \neq 0$ *on* $\Re(s) = 3$.

Lemma 2.4. *For* $\frac{41}{60} \leqslant \sigma \leqslant \frac{3}{4}$ *we have*

$$\int_1^T |\zeta(\sigma+\mathrm{i}t)|^{m(\sigma)}\,\mathrm{d}t \ll T^{1+\varepsilon}, \quad m(\sigma) \geqslant \frac{2112}{859-948\sigma}, \tag{2.1}$$

uniformly for $T \geqslant 1$, *and*

$$\zeta(\sigma+\mathrm{i}t) \ll (1+|t|)^{\max\{13(1-\sigma)/42,0\}+\varepsilon}, \tag{2.2}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1+\varepsilon$ *and* $|t| \geqslant 1$.

Proof. The first result is given by Ivić (see [8]), and the second result is due to Bourgain, see [1]. □

Lemma 2.5. *Let $f \in H_k^*$ and $L(s,f)$ be the L-function associated to f. Then for $\frac{1}{2} \leqslant \sigma \leqslant 1+\varepsilon$, it holds uniformly*

$$L(\sigma + \mathrm{i}t, f) \ll (1+|t|)^{(1-\sigma)/3+\varepsilon} \tag{2.3}$$

for $|t| \geqslant t_0$, where t_0 is sufficiently large.

Proof. The result follows from Good (see [5]) and Phragmén-Lindelöf convexity principle for a strip. □

Lemma 2.6. *Let $f \in H_k^*$, and let $L(s, \mathrm{sym}^2 f)$ be the symmetric square L-function attached to f. Then for $\sigma = \frac{27133}{38316}$,*

$$\int_1^T |L(\sigma + \mathrm{i}t, \mathrm{sym}^2 f)|^{12772/5251} \,\mathrm{d}t \ll T^{1+\varepsilon} \tag{2.4}$$

uniformly for $T \geqslant 1$, and

$$L(\sigma + \mathrm{i}t, \mathrm{sym}^2 f) \ll (1+|t|)^{\max\{6(1-\sigma)/5, 0\}+\varepsilon} \tag{2.5}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1$ and $|t| \geqslant 1$.

Proof. The proof of the first result will be given in Section 4, and the second result is due to Lin, Nunes and Qi, see [11]. □

Lemma 2.7. *Let χ be a primitive charater modulo q and $\mathscr{L}_{m,n}^d(s,\chi)$ be a general L-function of degree $2A$. For any $\varepsilon > 0$,*

$$\int_T^{2T} |\mathscr{L}_{m,n}^d(\sigma + \mathrm{i}t, \chi)|^2 \,\mathrm{d}t \ll (qT)^{2A(1-\sigma)+\varepsilon} \tag{2.6}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1+\varepsilon$, and $T \geqslant 1$. Furthermore,

$$\mathscr{L}_{m,n}^d(\sigma + \mathrm{i}t, \chi) \ll (q|t|+1)^{\max\{A(1-\sigma), 0\}+\varepsilon} \tag{2.7}$$

uniformly for $\varepsilon \leqslant \sigma \leqslant 1+\varepsilon$.

Proof. For the proof, see [9]. □

3. Proof of Theorem 1.1

We are now in a position to prove Theorem 1.1. Firstly, we note that

$$S_j = 16\sum_{n\leqslant x}\lambda^2_{\mathrm{sym}^j f}(n)l(n) - 4\sum_{n\leqslant x}\lambda^2_{\mathrm{sym}^j f}(n)v(n)$$

by Lemma 2.1. Thus, we can first estimate the sum $16\sum_{n\leqslant x}\lambda^2_{\mathrm{sym}^j f}(n)l(n)$. By applying Perron's formula (see [12], Theorem 2.1) for $\mathcal{L}_j(s)$ of Lemma 2.2, we have

$$16\sum_{n\leqslant x}\lambda^2_{\mathrm{sym}^j f}(n)l(n) = \frac{16}{2\pi\mathrm{i}}\int_{\eta-\mathrm{i}T}^{\eta+\mathrm{i}T}\mathcal{L}_j(s)\frac{x^s}{s}\,\mathrm{d}s + O\Big(\frac{x^{3+\varepsilon}}{T}\Big), \tag{3.1}$$

where $\eta = 3+\varepsilon$ and $10\leqslant T\leqslant x$ is a parameter to be chosen later. Note that there is only one simple pole at $s=3$ for $\mathcal{L}_j(s)x^s/s$ coming from the factor $\zeta(s-2)$ by Lemma 2.2 in order to use Cauchy's residue theorem, we shift the contour of the integration in (3.1) to the line $\Re s = \kappa := 2 + 27133/38316$. Then we obtain

$$\begin{aligned}16\sum_{n\leqslant x}\lambda^2_{\mathrm{sym}^j f}(n)l(n) =\ & c_{f,j}x^3 + \frac{16}{2\pi\mathrm{i}}\bigg\{\int_{\kappa-\mathrm{i}T}^{\kappa+\mathrm{i}T}\mathcal{L}_j(s)\frac{x^s}{s}\,\mathrm{d}s + \int_{\kappa+\mathrm{i}T}^{\eta+\mathrm{i}T}\mathcal{L}_j(s)\frac{x^s}{s}\,\mathrm{d}s\\ &+\int_{\eta-\mathrm{i}T}^{\kappa-\mathrm{i}T}\mathcal{L}_j(s)\frac{x^s}{s}\,\mathrm{d}s\bigg\} + O\Big(\frac{x^{3+\varepsilon}}{T}\Big)\\ =:\ & c_{f,j}x^3 + \frac{16}{2\pi\mathrm{i}}(J_1+J_2+J_3) + O\Big(\frac{x^{3+\varepsilon}}{T}\Big),\end{aligned}$$

where $c_{f,j}$ is a suitable constant depending on f and various associated L-functions. More precisely,

$$c_{f,j} = 16\lim_{s\to3}(s-3)\frac{\mathcal{L}_j(s)}{s} = \frac{16}{3}L(3,\chi)\prod_{n=1}^{j}L(1,\mathrm{sym}^{2n}f)L(3,\mathrm{sym}^{2n}f\otimes\chi)\mathcal{H}_j(3).$$

For J_1, by (2.1), (2.4), (2.6) and (2.7), along with Hölder's inequality, we have

$$\begin{aligned}J_1 &\ll \int_{-T}^{T}\frac{|\zeta(\kappa-2+\mathrm{i}t)\prod_{n=1}^{j}L(\kappa-2+\mathrm{i}t,\mathrm{sym}^{2n}f)|}{|t|}x^{\kappa}\,\mathrm{d}t\\ &\ll x^{\kappa+\varepsilon} + x^{\kappa+\varepsilon}\log T\max_{1\leqslant T_1\leqslant T}\frac{1}{T_1}\bigg\{\bigg(\int_{T_1/2}^{T_1}|\zeta(\kappa-2+\mathrm{i}t)|^{12772/1135}\,\mathrm{d}t\bigg)^{1135/12772}\\ &\quad\times\bigg(\int_{T_1/2}^{T_1}|L(\kappa-2+\mathrm{i}t,\mathrm{sym}^2 f)|^{12772/5251}\,\mathrm{d}t\bigg)^{5251/12772}\\ &\quad\times\bigg(\int_{T_1/2}^{T_1}\bigg|\prod_{n=2}^{j}L(\kappa-2+\mathrm{i}t,\mathrm{sym}^{2n}f)\bigg|^2\,\mathrm{d}t\bigg)^{1/2}\bigg\}\\ &\ll x^{\kappa+\varepsilon} + x^{\kappa+\varepsilon}T^{(((j+1)^2-4)/2)(3-\kappa)-1/2+\varepsilon} \ll x^{\kappa+\varepsilon}T^{(((j+1)^2-4)/2)(3-\kappa)-1/2+\varepsilon}.\end{aligned}$$

For the integrals over the horizontal segments J_2 and J_3, by (2.2), (2.5) and (2.7), we have

$$\begin{aligned}
J_2 + J_3 &\ll \int_{\kappa}^{\eta} \left| \frac{\zeta(\sigma - 2 + \mathrm{i}T) \prod_{n=1}^{j} L(\sigma - 2 + \mathrm{i}T, \mathrm{sym}^{2n} f)}{T} x^{\sigma} \right| \mathrm{d}\sigma \\
&\ll \int_{\kappa-2}^{1+\varepsilon} \left| \frac{\zeta(\sigma + \mathrm{i}T) \prod_{n=1}^{j} L(\sigma + \mathrm{i}T, \mathrm{sym}^{2n} f)}{T} x^{\sigma+2} \right| \mathrm{d}\sigma \\
&\ll \frac{x^2}{T} \max_{\kappa-2 \leqslant \sigma \leqslant 1+\varepsilon} x^{\sigma} T^{(13/42+6/5+((j+1)^2-4)/2)(1-\sigma)+\varepsilon} \\
&\ll \frac{x^2}{T} \max_{\kappa-2 \leqslant \sigma \leqslant 1+\varepsilon} x^{\sigma} T^{((j+1)^2/2-103/210)(1-\sigma)+\varepsilon} \\
&\ll \frac{x^{3+\varepsilon}}{T} + x^{\kappa+\varepsilon} T^{((j+1)^2/2-103/210)(3-\kappa)-1+\varepsilon}.
\end{aligned}$$

By comparing the bound $J_2 + J_3$ with the bound J_1, we get

$$16 \sum_{n \leqslant x} \lambda^2_{\mathrm{sym}^j f}(n) l(n) = c_{f,j} x^3 + O(x^{\kappa+\varepsilon} T^{(((j+1)^2-4)/2)(3-\kappa)-1/2+\varepsilon}) + O\Big(\frac{x^{3+\varepsilon}}{T}\Big). \tag{3.2}$$

On taking $x^{3+\varepsilon}/T = x^{\kappa+\varepsilon} T^{(((j+1)^2-4)/2)(3-\kappa)-1/2+\varepsilon}$, i.e., $T = x^{22366/(11183(j+1)^2-6416)}$, we have

$$16 \sum_{n \leqslant x} \lambda^2_{\mathrm{sym}^j f}(n) l(n) = c_{f,j} x^3 + O(x^{3-22366/(11183(j+1)^2-6416)+\varepsilon}). \tag{3.3}$$

Similarly, we apply Perron's formula to $V_j(s)$ with $\eta = 3 + \varepsilon$ and $10 \leqslant T \leqslant x$. Thus,

$$4 \sum_{n \leqslant x} \lambda^2_{\mathrm{sym}^j f}(n) v(n) = \frac{4}{2\pi i} \int_{\eta-\mathrm{i}T}^{\eta+\mathrm{i}T} V_j(s) \frac{x^s}{s} \, \mathrm{d}s + O\Big(\frac{x^{3+\varepsilon}}{T}\Big).$$

We move the line of integration to $\Re(s) = \frac{8}{3}$. Note that there is no singularity in the rectangle for the L-function $V_j(s)$ obtained by Lemma 2.3 and for $V_j(s)x^s/s$ also is. Consequently, by Cauchy's theorem, we get

$$\begin{aligned}
&4 \sum_{n \leqslant x} \lambda^2_{\mathrm{sym}^j f}(n) v(n) \\
&\quad = \frac{4}{2\pi\mathrm{i}} \Big\{ \int_{8/3-\mathrm{i}T}^{8/3+\mathrm{i}T} V_j(s) \frac{x^s}{s} \, \mathrm{d}s + \int_{8/3+\mathrm{i}T}^{\eta+\mathrm{i}T} V_j(s) \frac{x^s}{s} \, \mathrm{d}s + \int_{\eta-\mathrm{i}T}^{8/3-\mathrm{i}T} V_j(s) \frac{x^s}{s} \, \mathrm{d}s \Big\} \\
&\qquad + O\Big(\frac{x^{3+\varepsilon}}{T}\Big) \\
&\quad =: \frac{4}{2\pi\mathrm{i}} (I_1 + I_2 + I_3) + O\Big(\frac{x^{3+\varepsilon}}{T}\Big).
\end{aligned}$$

The contribution of I_1 in absolute value (by using Lemma 2.3, 2.5, 2.6, 2.7 and Cauchy's inequality) is

$$\begin{aligned}
I_1 \ll & \int_{-T}^{T} \frac{|L(2/3+\mathrm{i}t,\chi)\prod_{n=1}^{j} L(2/3+\mathrm{i}t,\mathrm{sym}^{2n} f\otimes\chi)|}{|t|} x^{8/3}\,\mathrm{d}t \\
\ll & x^{8/3+\varepsilon} + x^{8/3+\varepsilon}\log T \\
& \times \max_{1\leqslant T_1\leqslant T} \frac{1}{T_1}\Big\{\Big(\int_{T_1/2}^{T_1}\Big|L\Big(\frac{2}{3}+\mathrm{i}t,\chi\Big)L\Big(\frac{2}{3}+\mathrm{i}t,\mathrm{sym}^2 f\Big)\Big|^2\,\mathrm{d}t\Big)^{1/2} \\
& \times \Big(\int_{T_1/2}^{T_1}\Big|\prod_{n=2}^{j} L\Big(\frac{2}{3}+\mathrm{i}t,\mathrm{sym}^{2n} f\Big)\Big|^2\,\mathrm{d}t\Big)^{1/2}\Big\} \\
\ll & x^{8/3+\varepsilon} + x^{8/3+\varepsilon}\log T \max_{1\leqslant T_1\leqslant T}\frac{1}{T_1} \\
& \times \Big\{\Big(\max_{T_1/2\leqslant t\leqslant T_1}\Big|L\Big(\frac{2}{3}+\mathrm{i}t,\chi\Big)\Big|^2\int_{T_1/2}^{T_1}\Big|L\Big(\frac{2}{3}+\mathrm{i}t,\mathrm{sym}^2 f\Big)\Big|^2\,\mathrm{d}t\Big)^{1/2} \\
& \times \Big(\int_{T_1/2}^{T_1}\Big|\prod_{n=2}^{j} L\Big(\frac{2}{3}+\mathrm{i}t,\mathrm{sym}^{2n} f\Big)\Big|^2\,\mathrm{d}t\Big)^{1/2}\Big\} \\
\ll & x^{8/3+\varepsilon} + x^{8/3+\varepsilon}T^{((1/3)\cdot(1/3)\cdot 2+1)/2+(((j+1)^2-4)/3)/2-1+\varepsilon} \\
\ll & x^{8/3+\varepsilon}T^{(j+1)^2/6-19/18}.
\end{aligned}$$

Here we have used
$$\int_{T}^{2T}\Big|L\Big(\frac{2}{3}+\mathrm{i}t,\mathrm{sym}^2 f\Big)\Big|^2\,\mathrm{d}t \ll T^{1+\varepsilon},$$
which follows from [7], Thoerem 1.

The contribution of horizontal line integrals I_2+I_3 in absolute value (by using Lemma 2.3, 2.5, 2.6 and 2.7) is

$$\begin{aligned}
I_2+I_3 \ll & \int_{8/3}^{\eta}\Big|\frac{L(\sigma-2+\mathrm{i}T,\chi)\prod_{n=1}^{j} L(\sigma-2+\mathrm{i}T,\mathrm{sym}^{2n} f\otimes\chi)}{T}x^{\sigma}\Big|\,\mathrm{d}\sigma \\
\ll & \int_{2/3}^{1+\varepsilon}\Big|\frac{L(\sigma+\mathrm{i}T,\chi)\prod_{n=1}^{j} L(\sigma+\mathrm{i}T,\mathrm{sym}^{2n} f)}{T}x^{\sigma+2}\Big|\,\mathrm{d}\sigma \\
\ll & \frac{x^2}{T}\max_{2/3\leqslant\sigma\leqslant 1+\varepsilon} x^{\sigma}T^{(1/3+6/5+((j+1)^2-4)/2)(1-\sigma)+\varepsilon} \\
\ll & \frac{x^2}{T}\max_{2/3\leqslant\sigma\leqslant 1+\varepsilon} x^{\sigma}T^{((j+1)^2/2-7/15)(1-\sigma)+\varepsilon} \\
\ll & \frac{x^{3+\varepsilon}}{T} + x^{8/3+\varepsilon}T^{((j+1)^2/2-7/15)/3-1+\varepsilon}.
\end{aligned}$$

Comparing two bounds, we get

$$4\sum_{n\leqslant x}\lambda^2_{\mathrm{sym}^j f}(n)v(n) = O(x^{8/3+\varepsilon}T^{(j+1)^2/6-19/18}) + O\Big(\frac{x^{3+\varepsilon}}{T}\Big).$$

We choose T such that $x^{8/3+\varepsilon}T^{(j+1)^2/6-19/18} = x^{(3+\varepsilon)}/T$, i.e., $T = x^{6/(3(j+1)^2-1)}$, thus

$$4\sum_{n\leqslant x}\lambda^2_{\mathrm{sym}^j f}(n)v(n) = O(x^{3-6/(3(j+1)^2-1)+\varepsilon}). \tag{3.4}$$

Combining (3.3) and (3.4), we have

$$S_j = c_{f,j}x^3 + O(x^{3-6/(3(j+1)^2-1)+\varepsilon}).$$

This proves the theorem. □

4. Proof of (2.4)

It is necessary to mention that Huang, Liu and Zhang in [7] considered the moment of Maass forms for $\mathrm{SL}(3,\mathbb{Z})$ in 2021. In fact, denote $m(\sigma)$ as the supremum of all numbers m such that

$$\int_1^T |L(\sigma+\mathrm{i}t,\mathrm{sym}^2 f)|^m\,\mathrm{d}t \ll_f T^{1+\varepsilon}.$$

They got

$$m(\sigma) \geqslant \frac{4(3-2\sigma)}{5(4-3\sigma)(1-\sigma)}, \quad \frac{2}{3}\leqslant\sigma\leqslant 1-\varepsilon.$$

Unfortunately, their result is not suitable to our purpose. In this section, we prove (2.4), which improves the above result at $\sigma = 27133/38316$. In addition, we need some lemmas to prove it.

Lemma 4.1 ([7], Lemma 2.2). *For $\frac{1}{2}<\sigma<1$,*

$$\int_1^T |L(\sigma+\mathrm{i}t,\mathrm{sym}^2 f)|^{m(\sigma)}\,\mathrm{d}t \ll T^{1+\varepsilon}$$

is equivalent to

$$\sum_{r\leqslant R}|L(\sigma+\mathrm{i}t_r,\mathrm{sym}^2 f)|^{m(\sigma)} \ll T^{1+\varepsilon},$$

where

$$T\leqslant t_r\leqslant 2T \quad \text{for } r=1,\dots,R; \quad |t_r-t_s|\geqslant \log^C T \quad \text{for } 1\leqslant r\neq s\leqslant R. \tag{4.1}$$

Lemma 4.2 ([7], Lemma 2.3). *Suppose that $\frac{1}{2} < \sigma < 1$ is fixed. Then*

$$R \ll T^{1+\varepsilon} V^{-m(\sigma)}$$

is equivalent to

$$\sum_{r \leqslant R} |L(\sigma + \mathrm{i}t_r, \mathrm{sym}^2 f)|^{m(\sigma)} \ll T^{1+\varepsilon},$$

where t_r is defined by (4.1) *and*

$$|L(\sigma + \mathrm{i}t_r, \mathrm{sym}^2 f)| \geqslant V \geqslant T^{\varepsilon}.$$

Lemma 4.3 ([8], Lemma 8.2). *Let $t_1 < \ldots < t_R$ be real numbers such that $T \leqslant t_r \leqslant 2T$ for $r = 1, 2, \ldots, R$ and $|t_r - t_s| \geqslant \log^4 T$ for $1 \leqslant r \neq s \leqslant R$. If*

$$T^{\varepsilon} < V \leqslant \Biggl| \sum_{M \leqslant n \leqslant 2M} a(n) n^{\sigma - \mathrm{i}t_r} \Biggr|,$$

where $a(n) \ll M^{\varepsilon}$ for $M \leqslant n \leqslant 2M$, $1 \ll M \ll T^C$ $(C > 0)$, then

$$R \ll T^{\varepsilon}(M^{2-2\sigma} V^{-2} + T V^{-f(\sigma)}).$$

Here

$$f(\sigma) = \frac{10}{7 - 8\sigma} \quad \text{for} \quad \frac{2}{3} < \sigma \leqslant \frac{11}{14}.$$

Now it is time to prove (2.4), indeed, it suffices to prove

$$R \ll T^{1+\varepsilon} V^{-12772/5251} \tag{4.2}$$

for $\sigma = 27133/38316 =: c$. Standardly, Mellin's transform implies that

$$\sum_{n \geqslant 1} \lambda_{\mathrm{sym}^2 f}(n) \mathrm{e}^{-n/Y} n^{-s} = \frac{1}{2\pi \mathrm{i}} \int_{(2)} Y^w \Gamma(w) L(s + w, \mathrm{sym}^2 f) \, \mathrm{d}w,$$

where Y is a parameter which will be chosen later and $s = c + \mathrm{i}t_r$ with t_r satisfying condition (4.1). Moving the integral line to $\Re w = \frac{1}{2} - c$, by the Cauchy's residue theorem, we get

$$\sum_{n \leqslant Y} \lambda_{\mathrm{sym}^2 f}(n) \mathrm{e}^{-n/Y} n^{-s} = L(s, \mathrm{sym}^2 f) + \frac{1}{2\pi \mathrm{i}} \int_{(1/2 - c)} Y^w \Gamma(w) L(s + w, \mathrm{sym}^2 f) \, \mathrm{d}w.$$

For $T \to \infty$, Stirling's formula implies that

$$L(s, \mathrm{sym}^2 f) \ll \Big|\sum_{n \leqslant Y} \lambda_{\mathrm{sym}^2 f}(n) \mathrm{e}^{-n/Y} n^{-s}\Big| \\ + \Big|\int_{-\log^2 T}^{\log^2 T} Y^{(1/2-c+\mathrm{i}v)} \Gamma\Big(\Big(\frac{1}{2}-c\Big)+\mathrm{i}v\Big) L\Big(\frac{1}{2}+\mathrm{i}t_r+\mathrm{i}v, \mathrm{sym}^2 f\Big)\,\mathrm{d}v\Big|,$$

which implies that either

$$V \ll \log T \max_{M \leqslant Y/2} \Big|\sum_{M \leqslant n \leqslant 2M} \lambda_{\mathrm{sym}^2 f}(n) \mathrm{e}^{-n/Y} n^{-s}\Big|$$

or

$$V \ll Y^{(1/2-c)} \log^2 T \Big|L\Big(\frac{1}{2}+\mathrm{i}t_r', \mathrm{sym}^2 f\Big)\Big|.$$

Here

$$\Big|L\Big(\frac{1}{2}+\mathrm{i}t_r', \mathrm{sym}^2 f\Big)\Big| = \max_{-\log^2 T \leqslant v \leqslant \log^2 T} \Big|L\Big(\frac{1}{2}+\mathrm{i}t_r+\mathrm{i}v, \mathrm{sym}^2 f\Big)\Big|.$$

We divide the set $\{t_r\}$ into two subsets S_1, S_2 according to whether $V \leqslant T^{36757/127720}$ or not. For S_1 we get

$$R_1 = |S_1| \ll T^{\varepsilon}(Y_1^{2-2c} V^{-2} + T V^{-10/(7-8c)}) + Y_1^{1/2-c} V^{-1} \sum_{r \leqslant R_1} \Big|L\Big(\frac{1}{2}+\mathrm{i}t_r', \mathrm{sym}^2 f\Big)\Big|,$$

which follows from Lemma 4.3. Cauchy's inequality implies that

$$R_1 \ll T^{\varepsilon}(Y_1^{2-2c} V^{-2} + T V^{-10/(7-8c)}) + {Y_1}^{1-2c} V^{-2} T^{3/2+\varepsilon}.$$

Here we have used

$$\sum_{r \leqslant R} \Big|L\Big(\frac{1}{2}+\mathrm{i}t_r', \mathrm{sym}^2 f\Big)\Big|^2 \ll R T^{\varepsilon} + \log T \int_T^{2T+\log^2 T} \Big|L\Big(\frac{1}{2}+\mathrm{i}t, \mathrm{sym}^2 f\Big)\Big|^2 \,\mathrm{d}t \\ \ll T^{3/2+\varepsilon}. \tag{4.3}$$

Taking $Y_1 = T^{3/2}$ and in view of $V \leqslant T^{36757/127720}$, we have

$$R_1 \ll T^{3-3c+\varepsilon} V^{-2} + T^{1+\varepsilon} V^{-10/(7-8c)} \ll T^{1+\varepsilon} V^{-12772/5251}.$$

To bound R_2, Hölder's inequality implies that

$$R_2 \ll T^{\varepsilon}(Y_2^{2-2c} V^{-2} + T V^{-10/(7-8c)}) \\ + Y_2^{1/2-c} V^{-1} R_2^{3/4} \Big(\sum_{r \leqslant R} \Big|L\Big(\frac{1}{2}+\mathrm{i}t_r', \mathrm{sym}^2 f\Big)\Big|^4\Big)^{1/4},$$

where we have used the subconvex bound of $L(s, \mathrm{sym}^2 f)$ and (4.3). Simplifying it, we have

$$R_2 \ll T^{\varepsilon}(Y_2^{2-2c}V^{-2} + TV^{-10/(7-8c)}) + Y_2^{2-4c}V^{-4}T^{27/10+\varepsilon}.$$

Choosing $Y_2 = T^{27/20c}V^{-1/c}$,

$$R_2 \ll T^{(27/10)(1-c)/c+\varepsilon}V^{-2/c} + T^{1+\varepsilon}V^{-10/(7-8c)} \ll T^{1+\varepsilon}V^{-12772/5251},$$

provided that

$$V \geqslant T^{36757/127720}.$$

Hence,

$$R = R_1 + R_2 \ll T^{1+\varepsilon}V^{-12772/5251}.$$

This completes the proof of (2.4). $\square$

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