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Ivaylo Kortezov

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Counting paths between points on a circle

Ivaylo Kortezov

Abstract. The paper deals with counting sets of given magnitude whose elements are self-avoiding paths with nodes from a fixed set of points on a circle. Some of the obtained formulae provide new properties of entries in “The On-line Encyclopaedia of Integer Sequences”, while others generate new entries therein.

Keywords: enumerative combinatorics; self-avoiding path; convex polygon

Classification: 05A15

Below we will prove formulae for counting the p-element sets consisting of self-avoiding paths whose sets of nodes are disjoint subsets of a set of n fixed points on a circle. The formulae depend on whether all n points need to be used as nodes and whether one-node paths are allowed. Although each path is self-avoiding, different paths are allowed to intersect. The formulae proven here generalize some results from [1] and [2]. Some special cases of these formulae correspond to new properties of existing sequences in The On-line Encyclopaedia of Integer Sequences (OEIS), see [3], [4], while other ones generate new sequences therein, see [5]–[14]. Also, for $n \geq 3$, these formulae count the p-element sets consisting of self-avoiding paths whose sets of nodes are disjoint subsets of a set of vertices of a given convex n-gon (this last statement follows in a natural way and will not be mentioned further).

All variables in this paper denote positive integers.

Definition 1. Let $A_1, A_2, \ldots, A_k$ be different points in the plane such that no three of them are collinear. If the segments $A_1A_2, A_2A_3, \ldots, A_{k-1}A_k$ have no common internal points then the union of these segments is called a *self-avoiding path* (SAP); A_1, A_2, $\ldots$, A_k are called *nodes* of the SAP.

Note that, according to the definition, the SAP is direction-independent—e.g. $A_1A_2A_4A_3$ and $A_3A_4A_2A_1$ is the same SAP. Also, the definition allows a SAP to have just one node (and zero segments); in this case we will call it a *singleton*. It is not immediately clear whether it is reasonable to include the singletons among

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the SAPs, so below we will calculate the results both with and without them, obtaining outcomes of comparable compactness.

Definition 2. Let n, p be positive integers. Denote by:

- $\operatorname{sap}(n, p)$ the number of p-sets of non-singleton SAPs whose sets of nodes form a partition of a given set of n points on a circle;
- $\operatorname{sap}'(n, p)$ the number of p-sets of (possibly singleton) SAPs whose sets of nodes form a partition of a given set of n points on a circle;
- $\operatorname{SAP}(n, p)$ the number of p-sets of non-singleton SAPs whose sets of nodes are disjoint subsets of a given set of n points on a circle;
- $\operatorname{SAP}'(n, p)$ the number of p-sets of (possibly singleton) SAPs whose sets of nodes are disjoint subsets of a given set of n points on a circle.

The result of the next statement for $p = 1$ has been suggested by the author and accepted by `oeis.org` in the list of properties of A001792, see [3]. The results for some other p have been accepted in `oeis.org` as the sequences A332426 for $p = 2$, see [6], A359404 for $p = 3$, see [7], and A360275 for $p = 4$, see [10]. The statement itself has been proved in [1]; here we provide a proof for the sake of synchronizing the notations.

Theorem 1. *Let n, p be positive integers such that $n > p$. Then*

$$\operatorname{sap}(n, p) = 2^{n-3p} \binom{n}{p} \sum_{i=1}^{p} \binom{p}{i} i^{n-p} (-1)^{p-i},$$

or equivalently, $\operatorname{sap}(n, p) = 2^{n-3p} V_n^{(p)} S_{n-p}^{(p)}$, *with* $V_n^{(p)}$ *being the number of variations for n elements of pth class and* $S_{n-p}^{(p)}$ *being the Stirling number of second kind for $n - p$ elements of pth class.*

Proof: Fix one of the end-nodes of each SAP; call it the *head* of that SAP; call the set of the rest of the nodes of the SAP the *body* of that SAP. There are $\binom{n}{p}$ choices for the set of heads among the n given points on the circle. We have to split the set of the remaining $n-p$ points into the p (nonempty) bodies[1]. For each of the $n - p$ points there are p choices for the body (p^{n-p} variants); we have to exclude the variants where a body remains empty $\left(\binom{p}{1}(p-1)^{n-p}\text{ variants}\right)$, then to include back the variants where two of the bodies remain empty $\left(\binom{p}{2}(p-2)^{n-p}\right.$ variants$\left.\right)$, and continue further by the inclusion-exclusion principle to get

$$\binom{n}{p} \sum_{i=1}^{p} \binom{p}{i} i^{n-p} (-1)^{p-i}.$$

[1]Modulo the order, there are $S_{n-p}^{(p)}$ ways to this, where $S_{n-p}^{(p)}$ is the Stirling number of second kind for $n-p$ elements of pth class, but to take into account the order we do the details explicitly.

Let us now count the SAPs with a given head and body: starting from the head, for each subsequent node, except for the last one, there are 2 choices – the leftmost or the rightmost unused point from the set entitled to the body (in all other cases part of the points remain separated from the rest and there is no way to conclude without self-intersection). Thus the number of SAPs with a given head and body is 2^{x-1} where x is the magnitude of the body.

Among the n given points there are p used for heads and $n-p$ used for the p bodies; summing up the above result for all these, we conclude that the number of ways to form p SAPs from a given decomposition of the set of points into p heads and bodies is $2^{(n-p)-p} = 2^{n-2p}$.

To conclude it remains to note that there are 2 possible choices for the head of each SAP, so we need to divide by 2^p. Thus

$$\operatorname{sap}(n,p) = 2^{-p} 2^{n-2p} \binom{n}{p} \sum_{i=1}^{p} \binom{p}{i} i^{n-p} (-1)^{p-i}.$$

The statement regarding the Stirling numbers of the second kind is directly seen in the proof, since there are $V_n^{(p)}$ ways to choose the heads (the order is now important, as we plan to connect each head with a specific body), $S_{n-p}^{(p)}$ ways to split the remaining $n-p$ points among the p (nonempty) bodies and 2^{n-2p} ways to form SAPs in the entitled bodies; lastly, each of the p SAPs is direction-independent, which is responsible for a multiplication by 2^{-p}. □

The next statement has been hypothesised in [1]; here we provide a proof. The result of the next statement for $p=1$ has been accepted in the list of properties of A001792, see [3]. The results for some other p have been accepted in `oeis.org` as the sequences A359405 for $p=2$, see [8], A360021 for $p=3$, see [9], and A360276 for $p=4$, see [11].

Theorem 2. *Let n, p be positive integers such that $n > p$. Then*

$$\operatorname{sap}'(n,p) = 2^{n-3p} \binom{n}{p} \sum_{i=1}^{p} \binom{p}{i} i^{n-p} 3^{p-i}.$$

Proof: Let among the SAPs exactly m, $m = 1, 2, \ldots, p$, be non-singletons ($m > 0$ since $n > p$). Then there are $\binom{n}{p-m}$ choices for the points used for the singletons. Then for the SAPs whose nodes are the remaining $n+m-p$ points the number of variants is

$$\operatorname{sap}(n+m-p, m) = 2^{n-2m-p} \binom{n+m-p}{m} \sum_{i=1}^{m} \binom{m}{i} i^{n-p} (-1)^{m-i}.$$

Then

$$
\begin{aligned}
\operatorname{sap}'(n,p) &= \sum_{m=1}^{p} \binom{n}{p-m} 2^{n-2m-p} \binom{n+m-p}{m} \sum_{i=1}^{m} \binom{m}{i} i^{n-p}(-1)^{m-i} \\
&= \sum_{m=1}^{p} \sum_{i=1}^{m} 2^{n-2m-p} \binom{n}{p-m} \binom{n+m-p}{m} \binom{m}{i} i^{n-p}(-1)^{m-i} \\
&= \sum_{i=1}^{p} \sum_{m=i}^{p} 2^{n-2m-p} \frac{n!}{(p-m)!(n+m-p)!} \frac{(n+m-p)!}{m!(n-p)!} \\
&\quad \times \frac{m!}{i!(m-i)!} \, i^{n-p}(-1)^{m-i} \\
&= \sum_{i=1}^{p} \sum_{m=i}^{p} 2^{n-2m-p} \frac{n!}{(n-p)!(p-m)!(m-i)!\, i!} \, i^{n-p}(-1)^{m-i} \\
&= 2^{n-3p} \sum_{i=1}^{p} \sum_{m=i}^{p} 2^{2p-2m} \frac{n!}{(n-p)!\, p!} \frac{p!}{(p-i)!\, i!} \\
&\quad \times \frac{(p-i)!}{(p-m)!(m-i)!} \, i^{n-p}(-1)^{m-i} \\
&= 2^{n-3p} \binom{n}{p} \sum_{i=1}^{p} \binom{p}{i} i^{n-p} \sum_{m=i}^{p} \binom{p-i}{m-i} 4^{p-m}(-1)^{m-i} \\
&= 2^{n-3p} \binom{n}{p} \sum_{i=1}^{p} \binom{p}{i} i^{n-p} \sum_{j=0}^{p-i} \binom{p-i}{j} 4^{p-i-j}(-1)^{j} \\
&= 2^{n-3p} \binom{n}{p} \sum_{i=1}^{p} \binom{p}{i} i^{n-p}(4-1)^{p-i}.
\end{aligned}
$$

□

The next statement has been hypothesised in [2]; here we provide a proof. The result for $p = 1$ has been suggested by the author for publishing in `oeis.org` and accepted as A261064, see [4]. The result for $p = 2$ has been suggested by the author for publishing in `oeis.org` and accepted as A360717, see [13].

Theorem 3. *We have*

$$
\operatorname{SAP}(n,p) = 4^{-p} \binom{n}{p} \sum_{i=0}^{p} \binom{p}{i} (2i+1)^{n-p}(-1)^{p-i}.
$$

Proof: If $n \le p$ then the formula is trivially true: the left-hand side equals 0 for obvious reasons, while on the right-hand side $\binom{n}{p} = 0$ when $n < p$ and $\sum_{i=0}^{p} \binom{p}{i}(-1)^{p-i} = 0$ when $n = p$. Now let $n > p$. If the number of used points is k, for which there are $n!/(k!(n-k)!)$ choices, then necessarily $k > p$ and there are $\mathrm{sap}(k,p) = 2^{k-3p}\binom{k}{p}\sum_{i=0}^{p}\binom{p}{i}i^{k-p}(-1)^{p-i}$ variants for the p-set of SAPs, hence

$$\begin{aligned}
\mathrm{SAP}(n,p) &= \sum_{k=p+1}^{n}\sum_{i=0}^{p}\frac{n!}{k!(n-k)!}\frac{k!}{p!(k-p)!}\binom{p}{i}2^{k-3p}i^{k-p}(-1)^{p-i} \\
&= 2^{-2p}\sum_{i=0}^{p}\sum_{k=p+1}^{n}\frac{n!}{(n-p)!p!}\frac{(n-p)!}{(n-k)!(k-p)!}\binom{p}{i}(2i)^{k-p}(-1)^{p-i} \\
&= 4^{-p}\binom{n}{p}\sum_{i=0}^{p}\binom{p}{i}\sum_{k=p+1}^{n}\binom{n-p}{k-p}(2i)^{k-p}(-1)^{p-i} \\
&= 4^{-p}\binom{n}{p}\sum_{i=0}^{p}\binom{p}{i}(-1)^{p-i}\sum_{j=1}^{n-p}\binom{n-p}{j}(2i)^{j} \\
&= 4^{-p}\binom{n}{p}\sum_{i=0}^{p}\binom{p}{i}(-1)^{p-i}((2i+1)^{n-p}-1) \\
&= 4^{-p}\binom{n}{p}\left(\sum_{i=0}^{p}\binom{p}{i}(-1)^{p-i}(2i+1)^{n-p}-\sum_{i=0}^{p}\binom{p}{i}(-1)^{p-i}\right).
\end{aligned}$$

The result now follows since $\sum_{i=0}^{p}\binom{p}{i}(-1)^{p-i} = 0$. □

The next statement has also been proposed in [2]; here we provide a proof. The result for $p = 1$ has been suggested by the author for publishing in `oeis.org` and accepted as A360715, see [12]. The result for $p = 2$ has been suggested by the author for publishing in `oeis.org` and accepted as A360717, see [14].

Theorem 4. *We have*

$$\mathrm{SAP}'(n,p) = 4^{-p}\binom{n}{p}\sum_{i=0}^{p}\binom{p}{i}(2i+1)^{n-p}3^{p-i}.$$

Proof: If $n < p$ then the formula is trivially true: the left-hand side is clearly 0, while on the right-hand side $\binom{n}{p} = 0$.

If $n = p$ then the left-hand side equals 1 for obvious reasons, while the right-hand side equals $4^{-p}\binom{p}{p}\sum_{i=0}^{p}\binom{p}{i}3^{p-i} = 4^{-p}(1+3)^{p} = 1$.

Now let $n > p$. If the number of used points is k for which there are $\binom{n}{k} = n!/(k!(n-k)!)$ choices, then:

- if $k = p$ then there is 1 possible choice for the p SAPs using these points;

- if $k > p$ then there are $\operatorname{sap}'(k,p) = 2^{k-3p}\binom{k}{p}\sum_{i=0}^{p}\binom{p}{i}i^{k-p}3^{p-i}$ variants for the p-set of SAPs, hence

$$\begin{aligned}
\operatorname{SAP}'(n,p) &= \binom{n}{p} + \sum_{k=p+1}^{n}\sum_{i=0}^{p}\frac{n!}{k!(n-k)!}\frac{k!}{p!(k-p)!}\binom{p}{i}2^{k-3p}i^{k-p}3^{p-i}\\
&= \binom{n}{p} + 2^{-2p}\sum_{i=0}^{p}\sum_{k=p+1}^{n}\frac{n!}{(n-p)!\,p!}\frac{(n-p)!}{(n-k)!(k-p)!}\binom{p}{i}(2i)^{k-p}3^{p-i}\\
&= \binom{n}{p} + 4^{-p}\binom{n}{p}\sum_{i=0}^{p}\binom{p}{i}\sum_{k=p+1}^{n}\binom{n-p}{k-p}(2i)^{k-p}3^{p-i}\\
&= \binom{n}{p} + 4^{-p}\binom{n}{p}\sum_{i=0}^{p}\binom{p}{i}3^{p-i}\sum_{j=1}^{n-p}\binom{n-p}{j}(2i)^{j}\\
&= \binom{n}{p} + 4^{-p}\binom{n}{p}\sum_{i=0}^{p}\binom{p}{i}3^{p-i}((2i+1)^{n-p}-1)\\
&= \binom{n}{p} + 4^{-p}\binom{n}{p}\left(\sum_{i=0}^{p}\binom{p}{i}3^{p-i}(2i+1)^{n-p} - \sum_{i=0}^{p}\binom{p}{i}3^{p-i}\right).
\end{aligned}$$

The result now follows since $\sum_{i=0}^{p}\binom{p}{i}3^{p-i} = 4^p$. □

To conclude, let us wrap up the obtained results for the number of p-sets of self-avoiding paths whose sets of nodes are disjoint subsets of a given set of n points on a circle, depending on whether singletons are allowed and whether all n points need to be used as nodes. The common patterns can be easily pointed out.

singletons	all n points are nodes, $n > p$	not all points need to be nodes
excluded	$2^{n-3p}\binom{n}{p}\sum_{i=1}^{p}\binom{p}{i}i^{n-p}(-1)^{p-i}$	$4^{-p}\binom{n}{p}\sum_{i=0}^{p}\binom{p}{i}(2i+1)^{n-p}(-1)^{p-i}$
included	$2^{n-3p}\binom{n}{p}\sum_{i=1}^{p}\binom{p}{i}i^{n-p}3^{p-i}$	$4^{-p}\binom{n}{p}\sum_{i=0}^{p}\binom{p}{i}(2i+1)^{n-p}3^{p-i}$

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I. Kortezov:
Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Block 8 Acad. Georgi Bonchev Str., 1113 Sofia, Bulgaria

E-mail: kortezov@math.bas.bg

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