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Some results on quasi-t-dual Baer modules

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Abstract. Let R be a ring and let M be an R-module with $S = \mathrm{End_R(M)}$. Consider the preradical $\overline{Z}$ for the category of right R-modules Mod-R introduced by Y. Talebi and N. Vanaja in 2002 and defined by $\overline{Z}(M) = \bigcap\{U \leq M \colon M/U$ is small in its injective hull$\}$. The module M is called quasi-t-dual Baer if $\sum_{\varphi\in\mathfrak{I}} \varphi(\overline{Z}^2(M))$ is a direct summand of M for every two-sided ideal $\mathfrak{I}$ of S, where $\overline{Z}^2(M) = \overline{Z}(\overline{Z}(M))$. In this paper, we show that M is quasi-t-dual Baer if and only if $\overline{Z}^2(M)$ is a direct summand of M and $\overline{Z}^2(M)$ is a quasi-dual Baer module. It is also shown that any direct summand of a quasi-t-dual Baer module inherits the property. The last part of the paper is devoted to the comparison of the notions of quasi-dual Baer modules and quasi-t-dual Baer modules. Also, right quasi-t-dual Baer rings are investigated.

Keywords: fully invariant submodule; quasi-dual Baer module; quasi-dual Baer ring; quasi-t-dual Baer module; quasi-t-dual Baer ring

Classification: 16D10, 16D80

1. Introduction

Throughout this paper, R is an associative ring with identity, and all the modules are unital right R-modules unless stated otherwise. Let M be an R-module. The notation $N \subseteq M$ and $N \leq M$ means that N is a subset of M and N is a submodule of M, respectively. We will write $\mathrm{End}_R(M)$ and $E(M)$ for the endomorphism ring of M and the injective hull of M, respectively. By $\mathbb{Q}$ and $\mathbb{Z}$, we denote the ring of rational numbers and integer numbers, respectively. Also, for any prime number p, the Prüfer p-group will be denoted by $\mathbb{Z}(p^\infty)$.

In 1967 in [5], W. E. Clark introduced the concept of quasi-Baer rings. A ring R is called right quasi-Baer if the right annihilator of any right ideal of R is generated as a right ideal by an idempotent. Recall that a submodule K of M is called fully invariant if $f(K) \subseteq K$ for all $f \in \mathrm{End}_R(M)$. In 2004 in [14], S. T. Rizvi and C. S. Roman generalized the notion of right quasi-Baer rings to a module theoretic version. A module M is called quasi-Baer if the right annihilator in M of any two-sided ideal of $\mathrm{End}_R(M)$ is a direct summand of M. Equivalently, for

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any fully invariant submodule N of M, the left annihilator of N in $\mathrm{End}_R(M)$ is generated by an idempotent of $\mathrm{End}_R(M)$. In 2013 in [1], T. Amouzegar and Y. Talebi introduced a dual notion of quasi-Baer modules. A module M is said to be quasi-dual Baer if for every fully invariant submodule N of M, there exists an idempotent e in $S = \mathrm{End}_R(M)$ such that $\{\phi \in S\colon \mathrm{Im}\,\phi \subseteq N\} = eS$. To introduce the concept studied in this article, recall that the singular submodule $Z(M)$ of an R-module M is the set of $m \in M$ such that $mI = 0$ for some essential right ideal I of R. Dually, Y. Talebi and N. Vanaja introduced in [16] the submodule $\overline{Z}(M)$ of M which is defined by

$$\overline{Z}(M) = \bigcap\{U \le M\colon\ M/U \text{ is small in } E(M/U)\}.$$

The R-module M is said to be *cosingular* (or *noncosingular*) if $\overline{Z}(M) = 0$ (or $\overline{Z}(M) = M$). We write the submodule $\overline{Z}(\overline{Z}(M))$ of M as $\overline{Z}^2_R(M_R) = \overline{Z}^2(M_R)$ and abbreviate to $\overline{Z}^2(M)$ when no confusion can result. Similar notations are used in case M is a left R-module. In [2], the authors introduced and studied t-dual Baer modules. A module M is said to be t-dual Baer if $\sum_{\varphi\in\mathfrak{I}}\varphi(\overline{Z}^2(M))$ is a direct summand of M for every right ideal $\mathfrak{I}$ of $\mathrm{End}_R(M)$. Motivated by this work, we introduce the notion of quasi-t-dual Baer modules. We call a module M quasi-t-dual Baer if $\sum_{\varphi\in\mathfrak{I}}\varphi(\overline{Z}^2(M))$ is a direct summand of M for every two-sided ideal $\mathfrak{I}$ of $\mathrm{End}_R(M)$.

In Section 2, the main result shows that an R-module M is quasi-t-dual Baer if and only if $\overline{Z}^2(M)$ is a direct summand of M and $\overline{Z}^2(M)$ is a quasi-dual Baer module (Theorem 2.4). As a consequence of this result, it turns out that a module M is quasi-t-dual Baer if and only if $M = M_1 \oplus M_2$ such that M_1 is a noncosingular quasi-dual Baer module and $\overline{Z}^2(M_2) = 0$ (Corollary 2.8). We also show that being quasi-t-dual Baer is preserved by taking direct summands (Corollary 2.5). We provide a characterization for an arbitrary direct sum $M = \bigoplus_{i\in I} M_i$ of quasi-t-dual Baer modules M_i, $i \in I$, to be quasi-t-dual Baer when each M_i, $i \in I$, is fully invariant in M (Proposition 2.17).

The investigations in Section 3 focus on the comparison of the notions of (right) quasi-dual Baer modules (rings) and (right) quasi-t-dual Baer modules (rings). We begin by providing some examples to show that the implication

$$\text{quasi-t-dual Baer} \Rightarrow \text{quasi-dual Baer}$$

is not true (Example 3.1). Unfortunately, the converse to this implication is still open. On the other hand, a (necessary and) sufficient condition for a quasi-dual Baer module to be quasi-t-dual Baer is provided (Proposition 3.2 and Theorem 3.4). It is shown that any quasi-dual Baer ring is right and left quasi-t-dual Baer (Corollary 3.8). We also prove a characterization of when a direct product of right quasi-t-dual Baer rings is right quasi-t-dual Baer (Proposition 3.13).

2. Quasi-t-dual Baer modules

Recently, a number of research papers have been devoted to the study of many generalizations of known algebraic properties using the preradical Z^2. The concepts obtained were themselves dualized by using the preradical $\overline{Z}^2$. In this way, many new notions were introduced and studied, namely, we include t-lifting modules, t-dual Baer modules, T-dual Rickart modules, and FI-t-lifting modules, among others, see for example [2], [3], [18], and [19]. According to [2], a module M is said to be *t-dual Baer* if $\mathfrak{I}(\overline{Z}^2(M))$ is a direct summand of M for every right ideal $\mathfrak{I}$ of $\mathrm{End}_R(M)$. Motivated by this, it is natural to introduce and investigate the following notion.

Definition 2.1. We say that a module M is *quasi-t-dual Baer* if $\mathfrak{I}(\overline{Z}^2(M))$ is a direct summand of M for every two-sided ideal $\mathfrak{I}$ of $S = \mathrm{End}_R(M)$.

Let N be a submodule of a module M. Then N is said to be *small* in M if $N + L \neq M$ for every proper submodule L of M. A module M is called *small* if M is a small submodule of its injective hull $E(M)$. A module M is called *lifting* if for every submodule N of M, there exists a direct summand K of M such that N/K is small in M/K, see, for example, [4]. Recall that a ring R is called a (*left*) *right H-ring* if every injective (left) right R-module is lifting. Left H-rings are characterized in [4, 28.10]. Note that every quasi-Frobenius ring is a left and right H-ring. Also, every left and right artinian serial ring is a left and right H-ring, see [4, 29.7].

Example 2.2. Note that every lifting module is amply supplemented i.e., for any two submodules A and B of M with $A + B = M$, B contains a submodule C such that C is minimal with property $A + C = M$, see for example [13, Proposition 4.8]. From [2, Theorems 1 and 4], we infer that every lifting module is t-dual Baer. Moreover, it is clear that every t-dual Baer module is quasi-t-dual Baer. This implies that every lifting module is quasi-t-dual Baer. Now using [13, Corollary 4.42], it follows that the R-module R_R is quasi-t-dual Baer for every semiperfect ring R.

The next lemma which is taken from [11, Lemma 2.7 (3) (b)] and [16, Proposition 2.1] will be used frequently in this paper.

Lemma 2.3. *Let M be an R-module. Then the following hold:*

(i) *$\overline{Z}^2(M)$ is a fully invariant submodule of M.*

(ii) *For any decomposition $M = \bigoplus_{i\in I} M_i$, we have $\overline{Z}^2(M) = \bigoplus_{i\in I} \overline{Z}^2(M_i)$.*

(iii) *For any family $(M_i)_{i\in I}$ of modules, we have $\overline{Z}^2\big(\prod_{i\in I} M_i\big) \subseteq \prod_{i\in I} \overline{Z}^2(M_i)$.*

(iv) *If $R = R_1 \oplus R_2$ where R_i, $i = 1, 2$, are nonzero two-sided ideals of R and M is an R-module, then $\overline{Z}^2_{R_i}(MR_i) = \overline{Z}^2_R(MR_i)$ for $i = 1, 2$.*

Following T. Amouzegar and Y. Talebi in [1], a module M is said to have the *FI-strong summand sum property* (FI-SSSP for short) if the sum of any family of fully invariant direct summands of M is a direct summand of M. Next, we provide a characterization of quasi-t-dual Baer modules which can be considered as the analogue of [2, Theorem 2].

Theorem 2.4. *Let M be an R-module with $S = \mathrm{End}_R(M)$. Then the following are equivalent:*

(i) *M is a quasi-t-dual Baer module;*
(ii) *$\mathfrak{I}(\overline{Z}^2(M))$ is a direct summand of M for every left ideal $\mathfrak{I}$ of S;*
(iii) *$\overline{Z}^2(M)$ is a direct summand of M and $\overline{Z}^2(M)$ is a quasi-dual Baer module;*
(iv) *$\overline{Z}^2(M)$ has the FI-SSSP and $S\varphi(\overline{Z}^2(M))$ is a direct summand of M for every $\varphi \in S$;*
(v) *$\sum_{\varphi\in A} S\varphi(\overline{Z}^2(M))$ is a direct summand of M for every nonempty subset A of S.*

Proof: Throughout this proof, $\mu\colon \overline{Z}^2(M) \to M$ stands for the inclusion map and when $\overline{Z}^2(M)$ is a direct summand of M, $\pi\colon M \to \overline{Z}^2(M)$ stands for the projection map. Moreover, let $T = \mathrm{End}_R(\overline{Z}^2(M))$. Note that $\overline{Z}^2(M)$ is fully invariant in M by Lemma 2.3 (i). This implies that $\overline{Z}^2(M) = S(\overline{Z}^2(M))$.

(i) ⇒ (ii) Let $\mathfrak{I}$ be a left ideal of S. Then $\mathfrak{I}S$ is a two-sided ideal of S. Therefore, $\mathfrak{I}S(\overline{Z}^2(M)) = \mathfrak{I}(\overline{Z}^2(M))$ is a direct summand of M since M is quasi-t-dual Baer.

(ii) ⇒ (iii) Since $\overline{Z}^2(M) = S(\overline{Z}^2(M))$, $\overline{Z}^2(M)$ is a direct summand of M by (ii). Now let I be a left ideal of the ring T. Consider the subset $\mathfrak{I} = \{\mu\phi\pi\colon \phi \in I\}$ of S. Let $f \in S$ and $\phi \in I$. It is easily seen that $f(\mu\phi\pi) = \mu\pi f(\mu\phi\pi) = \mu(\pi f\mu\phi)\pi$. Since $\pi f\mu \in T$, we have $\pi f\mu\phi \in I$ and hence $f(\mu\phi\pi) \in \mathfrak{I}$. So $\mathfrak{I}$ is a left ideal of S. By hypothesis, $\mathfrak{I}(\overline{Z}^2(M))$ is a direct summand of M. But $\mathfrak{I}(\overline{Z}^2(M)) = I(\overline{Z}^2(M)) \subseteq \overline{Z}^2(M)$. Then $I(\overline{Z}^2(M))$ is a direct summand of $\overline{Z}^2(M)$. This shows that $\overline{Z}^2(M)$ is a quasi-dual Baer R-module, see [17, Proposition 2.4].

(iii) ⇒ (iv) By [1, Lemma 2.2], $\overline{Z}^2(M)$ has the FI-SSSP. Now take $\varphi \in S$ and consider the subset $A = \{\pi f\varphi\mu\colon f \in S\}$ of T. Since $\overline{Z}^2(M)$ is fully invariant in M, it follows easily that $S\varphi(\overline{Z}^2(M)) = A(\overline{Z}^2(M))$. Moreover, it is easy to check that $A(\overline{Z}^2(M)) = TA(\overline{Z}^2(M))$. Since $\overline{Z}^2(M)$ is quasi-dual Baer, it follows from [17, Proposition 2.4] that $TA(\overline{Z}^2(M))$ is a direct summand of $\overline{Z}^2(M)$. Hence $S\varphi(\overline{Z}^2(M))$ is a direct summand of M as $\overline{Z}^2(M)$ is a direct summand of M.

(iv) ⇒ (v) Let A be a nonempty subset of S. Note that $S\varphi(\overline{Z}^2(M))$ is a direct summand of M for every $\varphi \in S$ by (iv). In particular, $S1_M(\overline{Z}^2(M)) = \overline{Z}^2(M)$ is a direct summand of M since $\overline{Z}^2(M)$ is fully invariant in M. Therefore, $S\varphi(\overline{Z}^2(M))$ is a direct summand of $\overline{Z}^2(M)$ for every $\varphi \in A$. Moreover, it is easily seen that $S\varphi(\overline{Z}^2(M))$ is a fully invariant submodule of $\overline{Z}^2(M)$ for every $\varphi \in A$. Now the implication follows from the fact that $\overline{Z}^2(M)$ has the FI-SSSP.

(v) $\Rightarrow$ (i) Let $\mathfrak{I}$ be a two-sided ideal of S. Since $\mathfrak{I}(\overline{Z}^2(M)) = \sum_{\varphi \in \mathfrak{I}} S\varphi(\overline{Z}^2(M))$, it follows from (v) that $\mathfrak{I}(\overline{Z}^2(M))$ is a direct summand of M. Consequently, M is a quasi-t-dual Baer module. □

As an application of Theorem 2.4, we obtain the following corollary which shows that being quasi-t-dual Baer is preserved by taking direct summands.

Corollary 2.5. *Let M be a quasi-t-dual Baer module. Then every direct summand of M is also quasi-t-dual Baer.*

Proof: Let N be a direct summand of M. Then $M = N \oplus N'$ for some submodule N' of M. By Lemma 2.3 (ii), we have $\overline{Z}^2(M) = \overline{Z}^2(N) \oplus \overline{Z}^2(N')$. Moreover, using Theorem 2.4, it follows that $\overline{Z}^2(M)$ is a direct summand of M and $\overline{Z}^2(M)$ is a quasi-dual Baer R-module. Therefore $\overline{Z}^2(N)$ is a direct summand of N. Moreover, $\overline{Z}^2(N)$ is a quasi-dual Baer R-module by [17, Corollary 2.5]. Now using again Theorem 2.4, we infer that N is quasi-t-dual Baer. □

In the sequel, the class of quasi-t-dual Baer right (or left) R-modules will be denoted by $\mathcal{C}_R$ (${}_R\mathcal{C}$, respectively). It is worth pointing out two special subclasses of $\mathcal{C}_R$, namely

$$\mathcal{C}_{1r} = \{M \in \text{Mod–}R\colon \overline{Z}(M) = M \text{ and } M \text{ is quasi-dual Baer}\} \text{ and}$$
$$\mathcal{C}_{2r} = \{M \in \text{Mod–}R\colon \overline{Z}^2(M) = 0\}.$$

In fact, it is clear that $\mathcal{C}_{2r} \subseteq \mathcal{C}_R$. In addition, using [17, Proposition 2.4], we infer that a noncosingular module M is quasi-dual Baer if and only if M is quasi-t-dual Baer. Thus $\mathcal{C}_{1r} \subseteq \mathcal{C}_R$. Similarly, we can define $\mathcal{C}_{1l}$ and $\mathcal{C}_{2l}$.

Example 2.6. Let D be a commutative local domain with maximal ideal $\mathfrak{m}$ and quotient field $Q \neq D$. Consider the ring $R = \begin{bmatrix} D & Q \\ 0 & Q \end{bmatrix}$. Then the Jacobson radical of R is $\text{Rad}\,(R) = \begin{bmatrix} \mathfrak{m} & Q \\ 0 & 0 \end{bmatrix}$ and it is easy to check that $R/\text{Rad}\,(R) \cong (D/\mathfrak{m}) \times Q$ (as rings). This implies that R is a semilocal ring. Moreover, we have $\text{Soc}\,(R_R) = \begin{bmatrix} 0 & Q \\ 0 & Q \end{bmatrix}$ and $\text{Soc}\,({}_RR) = 0$.

(i) Using [20, Corollary 2.7], we see that $\overline{Z}(R_R) = \text{Soc}\,({}_RR) = 0$ and hence $R_R \in \mathcal{C}_{2r}$. This implies that R is a right quasi-t-dual Baer ring.

(ii) Now to study if R is left quasi-t-dual Baer, note that ${}_RR = I_1 \oplus I_2$ is the direct sum of the left ideals $I_1 = \begin{bmatrix} 0 & Q \\ 0 & Q \end{bmatrix}$ and $I_2 = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$. Using again

[20, Corollary 2.7], we deduce that $\overline{Z}({}_RI_1) = \operatorname{Soc}(R_R)I_1 = I_1$ and $\overline{Z}({}_RI_2) = \operatorname{Soc}(R_R)I_2 = 0$. Therefore, $\overline{Z}^2({}_RR) = I_1$ by Lemma 2.3 (ii). Moreover, it is easy to check that for every endomorphism φ of ${}_RR$, either $\varphi(I_1) = 0$ or $\varphi(I_1) = I_1$. Taking into account Theorem 2.4, we obtain that ${}_RR \in_R \mathcal{C}$ and ${}_RI_1 \in \mathcal{C}_{1l}$. Also, it is clear that ${}_RI_2 \in \mathcal{C}_{2l}$ and ${}_RR \notin \mathcal{C}_{1l} \cup \mathcal{C}_{2l}$.

Example 2.7. (i) Let R be a right V-ring. Applying [16, Proposition 2.5], it follows that $\overline{Z}(M) = M$ for any R-module M. Thus $\mathcal{C}_R = \mathcal{C}_{1r}$, see [17, Proposition 2.4].

(ii) It is well known that every small module is cosingular. Hence every small module M is quasi-t-dual Baer. This implies that

(a) for every commutative domain which is not a field R, the R-module R_R is quasi-t-dual Baer by [9, Corollary 6]; and
(b) for any family $(M_i)_{i\in I}$ of small modules, $M = \prod_{i\in I} M_i$ is quasi-t-dual Baer since $\overline{Z}^2(M) = 0$ by Lemma 2.3 (iii).

The next result which is another consequence of Theorem 2.4 shows that the class of quasi-t-dual Baer right R-modules is precisely

$$\mathcal{C}_R = \{M_1 \oplus M_2 \colon\ M_1 \in \mathcal{C}_{1r} \text{ and } M_2 \in \mathcal{C}_{2r}\}.$$

Corollary 2.8. *Let M be an R-module. Then the following are equivalent:*

(i) *M is a quasi-t-dual Baer module;*
(ii) *$M = M_1 \oplus M_2$ such that M_1 is a noncosingular quasi-dual Baer submodule and $\overline{Z}^2(M_2) = 0$ (in this case, $\overline{Z}^2(M) = M_1$).*

Proof: (i) $\Rightarrow$ (ii) By Theorem 2.4, there exists a submodule M_2 of M such that $M = M_1 \oplus M_2$, where $M_1 = \overline{Z}^2(M)$. Hence M_1 is noncosingular by [19, Lemma 3.10]. Moreover, M_1 is quasi-t-dual Baer by Corollary 2.5. Therefore M_1 is a quasi-dual Baer module.

(ii) $\Rightarrow$ (i) Note that $\overline{Z}^2(M) = \overline{Z}^2(M_1) \oplus \overline{Z}^2(M_2) = M_1$ is a direct summand of M. Now the result follows from Theorem 2.4. □

Example 2.9. Let R be a right hereditary, right noetherian ring. Let an R-module $M = N \oplus L$ be a direct sum of an injective submodule N and a submodule L with $\overline{Z}^2(L) = 0$ (for example, L may be taken to be small in $E(L)$). Then N is quasi-dual Baer by [12, Corollary 2.30]. Moreover, N is noncosingular by [16, Proposition 2.7]. In addition, it is clear that $\overline{Z}(L) = 0$ and hence $\overline{Z}^2(L) = 0$. From Corollary 2.8, we infer that M is a quasi-t-dual Baer module.

Corollary 2.10. *Let M be an indecomposable R-module. Then the following are equivalent:*

(i) *M is a quasi-t-dual Baer module;*
(ii) (a) $\overline{Z}^2(M) = 0$*; or*
(b) $\overline{Z}(M) = M$ *and M is a quasi-dual Baer module.*

PROOF: This follows directly from Corollary 2.8. □

It is well known that any simple R-module is either a small module or an injective module. Recall that the class of noncosingular modules is closed under homomorphic images, see [16, Proposition 2.4]. Combining Corollary 2.8 and [17, Corollary 3.9], we obtain the following corollary.

Corollary 2.11. *Let M be a nonzero module over a commutative perfect ring R. Then the following are equivalent:*

(i) *M is quasi-t-dual Baer;*
(ii) $M = \left(\bigoplus_{i\in I} S_i\right) \oplus N$ *where each S_i, $i \in I$, is a simple injective R-module and* $\overline{Z}^2(N) = 0$.

Next, we provide two classes of rings over which all modules are quasi-t-dual Baer.

Example 2.12. (i) Let R be a right perfect ring which has no injective simple R-modules (for example, R can be a local right perfect ring which is not a division ring). Using [16, Theorem 3.8 (3)], we conclude that $\overline{Z}^2(M) = 0$ (i.e. $M \in \mathcal{C}_{2r}$) for all R-modules M. In particular, every R-module is quasi-t-dual Baer.

(ii) Let R be a right H-ring and let M be an R-module. By [16, the proof of Theorem 3.8 (1)], $M = M_1 \oplus M_2$ such that $\overline{Z}(M_1) = M_1$ is injective and $\overline{Z}^2(M_2) = 0$. Since R is a right H-ring, M_1 is lifting and hence M_1 is a quasi-t-dual Baer module, see Example 2.2. Therefore M_1 is quasi-dual Baer as $\overline{Z}^2(M_1) = M_1$. Consequently, M is quasi-t-dual Baer by Corollary 2.8.

Proposition 2.13. *Assume that R is a commutative noetherian ring and let M be an R-module which has no nonzero submodules N with* $\operatorname{Rad}(N) = N$. *Then the following are equivalent:*

(i) *M is a quasi-t-dual Baer module;*
(ii) $\overline{Z}^2(M)$ *is a semisimple, injective, and projective module.*

If R is a Dedekind domain, then (i)–(ii) *are also equivalent to:*

(iii) $\overline{Z}^2(M) = 0$*, that is* $M \in \mathcal{C}_{2r}$.

PROOF: (i) ⇒ (ii) By Corollary 2.8, $\overline{Z}(\overline{Z}^2(M)) = \overline{Z}^2(M)$. Applying [21, Satz 2.6], we infer that $\overline{Z}^2(M)$ is semisimple and projective. Hence $\overline{Z}^2(M) = \bigoplus_{i\in I} S_i$ is a direct sum of simple projective submodules S_i, $i \in I$. By Lemma 2.3 (ii),

$\overline{Z}^2(S_i) = S_i$ for all $i \in I$. It follows that each S_i is an injective module. Since R is noetherian, we conclude that $\overline{Z}^2(M)$ is injective.

(ii) $\Rightarrow$ (i) Since $\overline{Z}^2(M)$ is semisimple, it is clear that $\overline{Z}^2(M)$ is a quasi-dual Baer module. Moreover, $\overline{Z}^2(M)$ is a direct summand of M since $\overline{Z}^2(M)$ is injective. Therefore M is quasi-t-dual Baer by Theorem 2.4.

(ii) $\Rightarrow$ (iii) Suppose that R is a Dedekind domain. Then R is a small R-module, see Example 2.7 (ii), and so every simple R-module is a small module. Since $\overline{Z}^2(M)$ is semisimple and injective, it follows that $\overline{Z}^2(M) = 0$.

(iii) $\Rightarrow$ (ii) This is clear. □

The next result should be compared with Example 2.9.

Corollary 2.14. *Let M be a module over a Dedekind domain R. Then the following are equivalent:*

(i) *M is a quasi-t-dual Baer module;*
(ii) *$M = M_1 \oplus M_2$ such that M_1 is an injective module and $\overline{Z}^2(M_2) = 0$. In particular, $\mathcal{C}_{1r}$ is exactly the class of all injective R-modules.*

Proof: (i) $\Rightarrow$ (ii) It is well known that $M = M_1 \oplus M_2$ such that M_1 is injective and M_2 has no nonzero submodules N with $\mathrm{Rad}\,(N) = N$, see for example [10, Theorem 8]. Note that M_2 is quasi-t-dual Baer by Corollary 2.5. Then $\overline{Z}^2(M_2) = 0$ by Proposition 2.13.

(ii) $\Rightarrow$ (i) This follows from Example 2.9. □

Consider the question $\mathbf{Q_1}$: When is the direct sum of two or more quasi-t-dual Baer modules, quasi-t-dual Baer? Note that for any indexed set of modules $(M_i)_{i\in I}$ with $\overline{Z}^2(M_i) = 0$ for all $i \in I$, we have $\overline{Z}^2\big(\bigoplus_{i\in I} M_i\big) = 0$, see Lemma 2.3 (ii). In view of Corollary 2.8, one can observe that the question $\mathbf{Q_1}$ is equivalent to the question $\mathbf{Q_2}$: When is the direct sum of two or more noncosingular quasi-dual Baer modules, quasi-dual Baer?

The next two propositions deal with two special cases of direct sums of quasi-t-dual Baer modules. We first prove the following elementary lemma.

Lemma 2.15. *Let an R-module $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ be a direct sum of quasi-t-dual Baer R-modules M_λ, $\lambda \in \Lambda$. Then $\overline{Z}^2(M)$ is a direct summand of M.*

Proof: Note that $\overline{Z}^2(M) = \bigoplus_{\lambda\in\Lambda}\overline{Z}^2(M_\lambda)$ by Lemma 2.3 (ii). Moreover, $\overline{Z}^2(M_\lambda)$ is a direct summand of M_λ for each $\lambda \in \Lambda$ by Theorem 2.4. Therefore $\overline{Z}^2(M)$ is a direct summand of M. □

Proposition 2.16. *Let M be an R-module. If M is quasi-t-dual Baer, then every direct sum of copies of M is quasi-t-dual Baer.*

Proof: Assume that M is quasi-t-dual Baer and let I be a nonempty index set. By Lemma 2.15, $\overline{Z}^2(M^{(I)}) = (\overline{Z}^2(M))^{(I)}$ is a direct summand of $M^{(I)}$. Moreover, $\overline{Z}^2(M)$ is quasi-dual Baer by Theorem 2.4. Therefore $(\overline{Z}^2(M))^{(I)}$ is quasi-dual Baer by [1, Theorem 2.7]. Now using again Theorem 2.4, we conclude that $M^{(I)}$ is a quasi-t-dual Baer module. □

Proposition 2.17. *Let $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ be such that each M_λ, $\lambda \in \Lambda$, is a fully invariant submodule of M. Then M is quasi-t-dual Baer if and only if M_λ is quasi-t-dual Baer for all $\lambda \in \Lambda$.*

Proof: The necessity is clear by Corollary 2.5. Conversely, suppose that each M_λ is quasi-t-dual Baer. Note first that $\overline{Z}^2(M) = \bigoplus_{\lambda\in\Lambda} \overline{Z}^2(M_\lambda)$ is a direct summand of M by Lemma 2.15. Using Theorem 2.4, we only need to show that $\overline{Z}^2(M)$ is quasi-dual Baer. Fix $\lambda \in \Lambda$. Since $\overline{Z}^2(M_\lambda)$ is a fully invariant submodule of M_λ (Lemma 2.3 (i)) and M_λ is fully invariant in M, it follows that $\overline{Z}^2(M_\lambda)$ is a fully invariant submodule of M. But $\overline{Z}^2(M)$ is a direct summand of M. Then it is not difficult to see that $\overline{Z}^2(M_\lambda)$ is a fully invariant submodule of $\overline{Z}^2(M)$. Moreover, $\overline{Z}^2(M_\lambda)$ is quasi-dual Baer by Theorem 2.4. Now applying [17, Proposition 2.19], we deduce that $\overline{Z}^2(M)$ is a quasi-dual Baer module since $\overline{Z}^2(M) = \bigoplus_{\lambda\in\Lambda} \overline{Z}^2(M_\lambda)$. Therefore, M is a quasi-t-dual Baer module. □

Proposition 2.18. *The following statements are equivalent for a module M:*

(i) *M is a quasi-t-dual Baer module;*
(ii) *$M \oplus \overline{Z}^2(M)$ is a quasi-t-dual Baer module.*

Proof: (i) ⇒ (ii) By Corollary 2.8, $\overline{Z}^2(M)$ is a noncosingular quasi-dual Baer module and $M = \overline{Z}^2(M) \oplus N$ for some submodule N of M with $\overline{Z}^2(N) = 0$. Therefore the R-module $L = M \oplus \overline{Z}^2(M)$ can be written as $L = M_1 \oplus M_2 \oplus M_3$ the direct sum of submodules M_i, $i = 1, 2, 3$, of L with $M_1 \cong M_2 \cong \overline{Z}^2(M)$ and $\overline{Z}^2(M_3) = 0$. It follows from [1, Theorem 2.7] that $M_1 \oplus M_2$ is quasi-dual Baer. Moreover, note that $\overline{Z}^2(L) = M_1 \oplus M_2$ by Lemma 2.3 (ii). Now use Theorem 2.4 to conclude that L is quasi-t-dual Baer.

(ii) ⇒ (i) This is clear by Corollary 2.5. □

3. Quasi-t-dual Baer (rings) modules versus quasi-dual Baer (rings) modules

The investigations in this section focus on the comparison of the notions of quasi-t-dual Baer modules and quasi-dual Baer modules. We begin by exhibiting some quasi-t-dual Baer modules which are not quasi-dual Baer.

Example 3.1. (i) Let p be a prime number. Consider the $\mathbb{Z}$-modules $M_1 = \mathbb{Z}$ and $M_2 = \mathbb{Z}/p^3\mathbb{Z}$. It is clear that $M_i \ll E(M_i)$ for each $i \in \{1, 2\}$. Therefore $\overline{Z}(M_i) = 0$ for each $i \in \{1, 2\}$. This implies that the $\mathbb{Z}$-modules M_1 and M_2 are quasi-t-dual Baer. However, neither of M_1 and M_2 is quasi-dual Baer since they are both indecomposable $\mathbb{Z}$-modules, see [17, Corollary 3.7].

(ii) Let $\mathbb{P}$ denote the set of all prime numbers and consider the $\mathbb{Z}$-module $M = \prod_{p\in\mathbb{P}} \mathbb{Z}/p\mathbb{Z}$. Note that the torsion submodule of M is $T(M) = \bigoplus_{p\in\mathbb{P}} \mathbb{Z}/p\mathbb{Z}$. Then clearly $T(M)$ is not a direct summand of M. It is shown in [17, Example 3.4] that M is not quasi-dual Baer. Indeed, for every fixed prime number q, let π_q be the endomorphism of M defined by $(x_p)_{p\in\mathbb{P}} \mapsto (y_p)_{p\in\mathbb{P}}$ such that $y_p = 0$ for all $p \neq q$ and $y_q = x_q$. Since $\sum_{p\in\mathbb{P}} \pi_p(M) = T(M)$, we have $\mathfrak{I}(M) = T(M)$ where $\mathfrak{I} = \mathfrak{D}(T(M))$. Now use [17, Corollary 2.6]. On the other hand, since each $\mathbb{Z}/p\mathbb{Z}$ is a small $\mathbb{Z}$-module, we have $\overline{Z}^2(M) \subseteq \prod_{p\in\mathbb{P}} \overline{Z}^2(\mathbb{Z}/p\mathbb{Z}) = 0$, see Lemma 2.3 (iii). Therefore M is a quasi-t-dual Baer module.

One may ask whether every quasi-dual Baer module is quasi-t-dual Baer. We do not know the answer to this question. It would be desirable to construct a quasi-dual Baer module which is not quasi-t-dual Baer, but we have not been able to find an example of such a module. In the next result, we give a characterization for a quasi-dual Baer module to be quasi-t-dual Baer.

Proposition 3.2. *Let M be a quasi-dual Baer R-module. Then the following conditions are equivalent:*

(i) *M is a quasi-t-dual Baer module;*
(ii) *$\overline{Z}^2(M)$ is a direct summand of M;*
(iii) *$\mathfrak{I}(M) = \overline{Z}^2(M)$, where $\mathfrak{I} = \mathfrak{D}(\overline{Z}^2(M))$.*

PROOF: (i) $\Rightarrow$ (ii) This follows from Theorem 2.4.

(ii) $\Rightarrow$ (iii) It is clear that for any direct summand K of M, $\mathfrak{I}(M) = K$, where $\mathfrak{I} = \mathfrak{D}(K)$.

(iii) $\Rightarrow$ (i) Using [17, Corollary 2.6] and Lemma 2.3 (i), we conclude that $\overline{Z}^2(M)$ is a direct summand of M. Hence $\overline{Z}^2(M)$ is quasi-dual Baer by [17, Corollary 2.5]. It follows from Theorem 2.4 that M is quasi-t-dual Baer. □

An R-module M is said to be *retractable* if $\mathrm{Hom}_R(M, N) \neq 0$ for any nonzero submodule N of M, see [8] and [15]. For example, any finitely generated module over a commutative ring is retractable by [8, Theorem 2.7]. Recall that a ring R is called *right* (*left*) *semiartinian* if every nonzero right (left) R-module has nonzero socle, and R is called *semiartinian* if it is right and left semiartinian. By [8, Theorem 2.8], every module over a commutative semiartinian ring is retractable. Next, we provide sufficient conditions for a quasi-dual Baer module to be quasi-t-dual Baer. To prove the next theorem, we need the following lemma.

Lemma 3.3. *Let M be a quasi-dual Baer R-module. Then $M = M_1 \oplus M_2$ is a direct sum of submodules M_1 and M_2 such that $\overline{Z}(M_1) = M_1$ and $\mathrm{Hom}_R(M, \overline{Z}^2(M_2)) = 0$.*

Proof: Note that $N = \overline{Z}^2(M)$ is a fully invariant submodule of M. According to [17, Proposition 2.1], we obtain that $M = M_1 \oplus M_2$ for some submodules M_1 and M_2 of M with $M_1 \subseteq N$ and $\mathrm{Hom}_R(M, M_2 \cap N) = 0$. By modularity, we have $N = M_1 \oplus (M_2 \cap N)$. On the other hand, we have $N = \overline{Z}^2(M) = \overline{Z}^2(M_1) \oplus \overline{Z}^2(M_2)$, see Lemma 2.3 (ii). Moreover, $\overline{Z}^2(M_2) \subseteq M_2 \cap N$ by [16, Proposition 2.1 (1)]. Therefore $\overline{Z}^2(M_1) = M_1$ and $\overline{Z}^2(M_2) = M_2 \cap N$. It follows that $\overline{Z}(M_1) = M_1$ and $\mathrm{Hom}_R(M, \overline{Z}^2(M_2)) = 0$. □

Theorem 3.4. *Let M be a quasi-dual Baer nonzero R-module and assume that M is retractable. Then M is quasi-t-dual Baer.*

Proof: By Lemma 3.3, there exists a direct sum decomposition $M = M_1 \oplus M_2$ with $\overline{Z}(M_1) = M_1$ and $\mathrm{Hom}_R(M, \overline{Z}^2(M_2)) = 0$. Since M is retractable, $\overline{Z}^2(M_2)$ must be zero. Hence $\overline{Z}^2(M) = M_1$ is a direct summand of M by Lemma 2.3 (ii). Now the result follows from Proposition 3.2. □

Combining Theorem 3.4 with [8, Theorems 2.7 and 2.8], we obtain the following corollary.

Corollary 3.5. *Let M be a quasi-dual Baer nonzero R-module over a commutative ring R. Suppose that one of the following conditions is fulfilled:*

(i) *R is a semiartinian ring; or*
(ii) *M is a finitely generated R-module.*

Then M is quasi-t-dual Baer.

Proposition 3.6. *Let M be a module such that $M \oplus \overline{Z}^2(M)$ is quasi-dual Baer. Then M and $M \oplus \overline{Z}^2(M)$ are quasi-t-dual Baer.*

Proof: Note that $\overline{Z}^2(M)$ is fully invariant in M. Using [14, Lemma 1.11], it follows that there exists a fully invariant submodule X of $\overline{Z}^2(M)$ such that $N = \overline{Z}^2(M) \oplus X$ is fully invariant in $L = M \oplus \overline{Z}^2(M)$. Consider the two-sided ideal $\mathfrak{I} = \mathrm{Hom}_R(L, N)$ of $\mathrm{End}_R(L)$. It is easily seen that $\overline{Z}^2(M) \oplus 0 \subseteq \mathfrak{I}(L) \subseteq N$. Hence $\overline{Z}^2(M) \oplus 0$ is a direct summand of $\mathfrak{I}(L)$. But $\mathfrak{I}(L)$ is a direct summand of L by [17, Proposition 2.4]. This implies that $\overline{Z}^2(M)$ is a direct summand of M. Moreover, $\overline{Z}^2(M)$ is quasi-dual Baer by [17, Corollary 2.5]. Therefore M is quasi-t-dual Baer by Theorem 2.4. Now use Proposition 2.18 to infer that $M \oplus \overline{Z}^2(M)$ is quasi-t-dual Baer. □

Remark 3.7. If R is a ring such that every R-module is quasi-dual Baer, then every R-module is quasi-t-dual Baer, see Proposition 3.6. But the converse does not hold. To see this, we can take a local right perfect ring which is not a division ring R (e.g. R can be taken to be $\mathbb{Z}/p^n\mathbb{Z}$ for some prime number p and some integer $n \geq 2$). By Example 2.12 (i), every R-module is quasi-t-dual Baer. However, the R-module R_R is not quasi-dual Baer since R is not a simple ring, see [17, Proposition 2.10].

It is shown in [17, Corollary 2.11] that the quasi-dual Baer property is left-right symmetric for any ring. Moreover, a ring R is quasi-dual Baer if and only if R is a finite product of simple rings by [17, Proposition 2.10]. A ring R is called a right quasi-t-dual Baer ring if the right R-module R_R is a quasi-t-dual Baer R-module. Left quasi-t-dual Baer rings are defined similarly. The ring R is called quasi-t-dual Baer in case R is left and right quasi-t-dual Baer. It is well known that for any ring R, the R-module R_R is retractable. The next corollary follows easily from Theorem 3.4.

Corollary 3.8. *Every quasi-dual Baer ring is a quasi-t-dual Baer ring. That is, every finite product of simple rings is a quasi-t-dual Baer ring.*

We next present some right quasi-t-dual Baer rings which are not quasi-dual Baer.

Example 3.9. (i) Let R be a local ring which is not a division ring. Then $\overline{Z}(R_R) \neq R$ since otherwise R will be a right V-ring by [16, Corollary 2.6]. In this case R will be a division ring. Hence $\overline{Z}(R_R) \ll R_R$ and so $\overline{Z}^2(R_R) = 0$. This implies that R is a right quasi-t-dual Baer ring. Similarly, we can see that R is left quasi-t-dual Baer. On the other hand, R is not quasi-dual Baer by [17, Proposition 2.10].

(ii) Let R be a commutative semiperfect ring which is not semisimple. Then R is a quasi-t-dual Baer ring by Example 2.2. However, the ring R is not quasi-dual Baer by [17, Proposition 2.10].

Example 3.10. Consider the ring R of all upper triangular 2×2 matrices with entries in a field F. It is well known that R is a left and right hereditary artinian ring, see for example [7, Example 13.6]. Note that $R = M_1 \oplus M_2$ with $M_1 = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ is an injective indecomposable R-module and $M_2 = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}$ is a simple small R-module. Then R is a right quasi-t-dual Baer ring by Example 2.9. On the other hand, note that R is not quasi-dual Baer, since otherwise $\mathrm{Rad}\,(R)$ is a direct summand of the R-module R_R by [17, Proposition 2.10]. But

$\mathrm{Rad}\,(R)$ is small in R_R. Hence $\mathrm{Rad}\,(R) = 0$. This contradicts the fact that $\mathrm{Rad}\,(R) = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$.

Proposition 3.11. *The following conditions are equivalent for a ring R:*

(i) *R is a right quasi-t-dual Baer ring;*
(ii) *$I\overline{Z}^2(R_R)$ is a direct summand of R_R for any two-sided ideal I of R;*
(iii) *$I\overline{Z}^2(R_R)$ is a direct summand of R_R for any left ideal I of R;*
(iv) *every free right R-module is quasi-t-dual Baer;*
(v) *every projective right R-module is quasi-t-dual Baer.*

PROOF: (i) ⇒ (ii) This follows directly from the definition of a quasi-t-dual Baer module.

(ii) ⇒ (iii) This is obvious.

(iii) ⇒ (i) Let $\mathfrak{I}$ be a left ideal of $\mathrm{End}_R(R_R)$. Put $I = \sum_{f\in\mathfrak{I}} f(R)$. It is easy to check that I is a left ideal of R. Then $I\overline{Z}^2(R_R)$ is a direct summand of R_R. But $\mathfrak{I}(\overline{Z}^2(R_R)) = I\overline{Z}^2(R_R)$. Hence $\mathfrak{I}(\overline{Z}^2(R_R))$ is a direct summand of R_R. It follows from Theorem 2.4 that R is a right quasi-t-dual Baer ring.

(i) ⇒ (iv) By Proposition 2.16.

(iv) ⇒ (v) This follows from Corollary 2.5 and the fact that every projective module is isomorphic to a direct summand of a free module.

(v) ⇒ (i) This is immediate. □

Our next endeavor is to characterize when a product of right quasi-t-dual Baer rings is right quasi-t-dual Baer. The following lemma is needed.

Lemma 3.12.

(i) *Let $(R_i)_{i\in I}$ be an indexed set of rings with $R = \prod_{i\in I} R_i$ and assume that $\overline{Z}^2_{R_i}(R_{i R_i}) = 0$ for all $i \in I$. Then $\overline{Z}^2_R(R_R) = 0$.*
(ii) *For any ring R, $\overline{Z}^2_R(R_R)$ is a two-sided ideal of R.*
(iii) *Let $R = R_1 \oplus R_2$ be a ring decomposition of a ring R. Then R is right quasi-t-dual Baer if and only if both R_1 and R_2 are right quasi-t-dual Baer rings.*

PROOF: (i) Using Lemma 2.3 (iv), we infer that $\overline{Z}^2_R(R_{iR}) = \overline{Z}^2_{R_i}(R_{i R_i}) = 0$ for each $i \in I$. But $\overline{Z}^2_R(R_R) \subseteq \prod_{i\in I} \overline{Z}^2_R(R_{iR})$ by Lemma 2.3 (iii). It follows that $\overline{Z}^2_R(R_R) = 0$.

(ii) It is clear that $\overline{Z}^2_R(R_R)$ is a right ideal of R. Moreover, $\overline{Z}^2_R(R_R)$ is a fully invariant submodule of the right R-module R_R by Lemma 2.3 (i). Thus $a\overline{Z}^2_R(R_R) \subseteq \overline{Z}^2_R(R_R)$ for every $a \in R$, that is, $\overline{Z}^2_R(R_R)$ is a left ideal of R.

(iii) Note that $\overline{Z}^2_R(R_R) = \overline{Z}^2_R(R_{1R}) \oplus \overline{Z}^2_R(R_{2R}) = \overline{Z}^2_{R_1}(R_{1R_1}) \oplus \overline{Z}^2_{R_2}(R_{2R_2})$ by Lemma 2.3 (ii) and (iv). Moreover, $\overline{Z}^2_{R_i}(R_{i R_i})$ is a two-sided ideal of R_i for $i = 1, 2$ by (ii).

($\Rightarrow$) Let A_1 be a two-sided ideal of R_1. Then A_1 is a two-sided ideal of R. Hence $A_1\overline{Z}^2_R(R_R)$ is a direct summand of R_R by Proposition 3.11. It is clear that $A_1\overline{Z}^2_R(R_R) = A_1\overline{Z}^2_{R_1}(R_{1R_1})$. So $A_1\overline{Z}^2_{R_1}(R_{1R_1})$ is a direct summand of R_{1R_1}. From Proposition 3.11, it follows that R_1 is a right quasi-t-dual Baer ring.

($\Leftarrow$) Take a two-sided ideal A of R. Then,

$$A\overline{Z}^2_R(R_R) = A\overline{Z}^2_{R_1}(R_{1R_1}) \oplus A\overline{Z}^2_{R_2}(R_{2R_2}) = (AR_1)\overline{Z}^2_{R_1}(R_{1R_1}) \oplus (AR_2)\overline{Z}^2_{R_2}(R_{2R_2}).$$

Since each AR_i is a two-sided ideal of R_i, it follows from Proposition 3.11 that $(AR_i)\overline{Z}^2_{R_i}(R_{iR_i})$ is a direct summand of R_{iR_i} for $i = 1, 2$. Therefore $A\overline{Z}^2(R_R)$ is a direct summand of R_R. Using again Proposition 3.11, we conclude that R is a right quasi-t-dual Baer ring. □

Proposition 3.13. *Let $(R_i)_{i\in I}$ be an indexed set of rings and let $R = \prod_{i\in I} R_i$. Then the following statements are equivalent:*

(i) *R is a right quasi-t-dual Baer ring;*
(ii) *there exists a finite subset $J \subseteq I$ such that $\overline{Z}^2_{R_i}(R_{iR_i}) = 0$ for every $i \in I \setminus J$ and each R_j, $j \in J$, is a right quasi-t-dual Baer ring.*

Proof: (i) $\Rightarrow$ (ii) Consider the two-sided ideal $A = \bigoplus_{i\in I} R_i$ of R. We claim that $A\overline{Z}^2_R(R_R) = \overline{Z}^2_R(A_R)$. Note that $\overline{Z}^2_R(A_R) = \bigoplus_{i\in I} \overline{Z}^2_R(R_{iR})$ by Lemma 2.3 (ii). Also, we have $\overline{Z}^2_R(R_{iR}) = \overline{Z}^2_{R_i}(R_{iR_i})$, see Lemma 2.3 (iv) for all $i \in I$. Thus $\overline{Z}^2_R(R_{iR})$ is a two-sided ideal of R_i for every $i \in I$. Therefore $A\overline{Z}^2_R(A_R) = \overline{Z}^2_R(A_R)$ and hence $\overline{Z}^2_R(A_R) \subseteq A\overline{Z}^2_R(R_R)$ by [16, Proposition 2.1 (1)]. Moreover, we have $A\overline{Z}^2_R(R_R) \subseteq A\big(\prod_{i\in I} \overline{Z}^2_R(R_{iR})\big)$ by Lemma 2.3 (iii). However, $A\big(\prod_{i\in I} \overline{Z}^2_R(R_{iR})\big) = \bigoplus_{i\in I} \overline{Z}^2_R(R_{iR}) = \overline{Z}^2_R(A_R)$. So $A\overline{Z}^2_R(R_R) \subseteq \overline{Z}^2_R(A_R)$. It follows that $A\overline{Z}^2_R(R_R) = \overline{Z}^2_R(A_R)$ as claimed. Now applying Proposition 3.11, we obtain that $\bigoplus_{i\in I} \overline{Z}^2_R(R_{iR})$ is a direct summand of R_R. Thus $\big(\bigoplus_{i\in I} \overline{Z}^2_R(R_{iR})\big) \oplus B = R_R$ for some right ideal B of R. This implies that $1_R = a + b$ for some $a \in \bigoplus_{i\in I} \overline{Z}^2_R(R_{iR})$ and some $b \in B$. This yields $aR = \bigoplus_{i\in I} \overline{Z}^2_R(R_{iR})$ and $bR = B$. Then there exists a finite subset $J \subseteq I$ such that $\overline{Z}^2_R(R_{iR}) = 0$ for every $i \in I \setminus J$. Therefore $\overline{Z}^2_{R_i}(R_{iR_i}) = 0$ for all $i \in I \setminus J$, see Lemma 2.3 (iv). Note that R_j is a right quasi-t-dual Baer ring for all $j \in J$ by Lemma 3.12 (iii).

(ii) $\Rightarrow$ (i) Set $T = \prod_{i\in I\setminus J} R_i$ and $S = \prod_{j\in J} R_j$. Then $R \cong T \times S$ (as rings). Note that $\overline{Z}^2_T(T_T) = 0$ by Lemma 3.12 (i). Hence T is a right quasi-t-dual Baer ring. Since J is a finite set, the proof is completed by induction and using Lemma 3.12 (iii). □

From the preceding result, it follows easily that an infinite product of right quasi-t-dual Baer rings need not be a right quasi-t-dual Baer ring. Next, we provide some explicit examples.

Example 3.14. Taking a right quasi-t-dual Baer ring R with $\overline{Z}^2_R(R_R) \neq 0$, it follows from Proposition 3.13 that $R^{\mathbb{N}}$ is not a right quasi-t-dual Baer ring, where $\mathbb{N}$ denotes the set of all positive integers. For example, we can take R to be one of the following rings:

(i) Let $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$, where F is a field. By Example 3.10, R is a right quasi-t-dual Baer ring and $R_R = M_1 \oplus M_2$ such that M_1 is a nonzero injective module and M_2 is a small module. Since R is right hereditary, $\overline{Z}^2(M_1) = M_1$, see [16, Proposition 2.7], and hence $\overline{Z}^2_R(R_R) = M_1$ by Lemma 2.3 (ii).

(ii) Let $R = \prod_{i=1}^n R_i$ be the product of simple rings R_i, $1 \leq i \leq n$, such that at least one of them is a right V-ring, see [6]. So $\overline{Z}^2_R(R_R) \neq 0$ by [16, Propositions 2.1 (1) and 2.5]. Moreover, R is a right quasi-t-dual Baer ring by Corollary 3.8.

Remark 3.15. Let F be a field.

(i) Consider the ring $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$. By Example 3.14 (i), $\overline{Z}^2_R(R_R) = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$. Similarly, we can show that $\overline{Z}^2_R({}_RR) = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$. Thus $\overline{Z}^2_R(R_R) \neq \overline{Z}^2_R({}_RR)$. This shows that the preradical $\overline{Z}^2$ is not left-right symmetric.

(ii) Assume that R is a right quasi-t-dual Baer ring. Then $R_R = \overline{Z}^2_R(R_R) \oplus I$ for some right ideal I of R by Theorem 2.4. Note that I need not be a two-sided ideal of R. In fact, for the ring $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$, see Examples 3.10 and 3.14 (i), we have $R_R = \overline{Z}^2_R(R_R) \oplus M_2$ where $M_2 = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}$. On the other hand, it is clear that M_2 is not a left ideal of R.

Proposition 3.16. *Let $R = R_1 \oplus R_2$ be a ring decomposition of a ring R such that R_1 is a right V-ring and $\overline{Z}^2_{R_2}(R_{2R_2}) = 0$. Then R is a right quasi-t-dual Baer ring if and only if R_1 is a finite product of simple rings.*

Proof: We first note that using [16, Corollary 2.6] and Lemma 2.3 (ii) and (iv), we get

$$\overline{Z}^2_R(R_R) = \overline{Z}^2_R(R_{1R}) = \overline{Z}^2_{R_1}(R_{1R_1}) = R_1.$$

For the necessity, assume that R is a right quasi-t-dual Baer ring and take a two-sided ideal I_1 of R_1. Then clearly I_1 is a two-sided ideal of R. Thus $I_1\overline{Z}^2_R(R_R) = I_1R_1 = I_1$ is a direct summand of R_R by Proposition 3.11. So I_1 is a direct summand of R_{1R_1}. According to [17, Proposition 2.10], R_1 is a finite product of simple rings. Conversely, note that $\overline{Z}^2_{R_2}(R_{2R_2}) = 0$ by hypothesis. Moreover,

R_1 is right quasi-t-dual Baer by Corollary 3.8. Now use Proposition 3.13 to deduce that R is a right quasi-t-dual Baer ring. □

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