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Commentationes Mathematicae Universitatis Carolinae, Vol. 64 (2023), No. 4, 395–410

Persistent URL: `http://dml.cz/dmlcz/152622`

Comment.Math.Univ.Carolin. 64,4 (2023) 395–410

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Attila Nagy, Csaba Tóth

Abstract. We deal with the following problem: how does the structure of a finite semigroup S depend on the probability that two elements selected at random from S, with replacement, define the same inner right translation of S. We solve a subcase of this problem. As the main result of the paper, we show how to construct not necessarily finite medial semigroups in which the index of the kernel of the right regular representation equals two.

Keywords: semigroup; regular representation of semigroups; medial semigroup

Classification: 20M10, 20M15

1. Introduction and motivation

There are many papers in the mathematical literature which use probabilistic methods to study special algebraic structures [3], [4], [5], [6], [8], [11], [12], [13], [20], [21], [23]. In [20], the following problem is examined. If two elements, a and b, are selected at random from a finite semigroup S, with replacement, what is the probability $P_{\theta_S}(S)$ that a and b define the same inner right translations of S, i.e., $(a,b)\in\theta_S$, where θ_S denotes the kernel of the right regular representation of S. It is also investigated how does the structure of a finite semigroup S depend on the probability $P_{\theta_S}(S)$. It is shown that, for an arbitrary finite semigroup S, $P_{\theta_S}(S)\geq 1/|S/\theta_S|$, where $|S/\theta_S|$ is the index of θ_S. Equality is satisfied if and only if each θ_S-class of S contains the same number of elements. In two cases, a solution is given to the problem of how to construct finite semigroups S satisfying the condition $P_{\theta_S}(S)=1/|S/\theta_S|$. These two cases are: S is an arbitrary semigroup and the index of θ_S is 1; S is a commutative semigroup and the index of θ_S is 2. In our present paper we answer the problem in that case when S is a finite medial semigroup and the index of θ_S is 2. We actually deal with a more general problem. Our main result is Theorem 3.5, in which we show how to construct not necessarily finite medial semigroups S in which the index of θ_S equals 2 (not necessarily fulfilling that each θ_S-class of S contains the same number of elements). We construct four medial semigroups and show that a semigroup S is

DOI 10.14712/1213-7243.2024.006

a medial semigroup such that the index of θ_S is 2 if and only if S is isomorphic to one of these four semigroups.

2. Preliminaries

By a *semigroup* we mean a multiplicative semigroup, that is, a nonempty set together with an associative multiplication. A transformation of a semigroup S (acting from the right) is called a *right translation* of S, if $(xy)\varrho = x(y\varrho)$ for all $x, y \in S$. The set of all right translations of a semigroup S is a subsemigroup in the semigroup of all transformations of S. For an arbitrary element a of a semigroup S, let ϱ_a denote the *inner right translation* $x \mapsto xa$ of S. It is known that $\Phi_S\colon a \mapsto \varrho_a$ is a homomorphism of the semigroup S into the semigroup of all right translations of S. The homomorphism Φ_S is called the (*right*) *regular representation* of a semigroup S. Let θ_S denote the kernel of the right regular representation of a semigroup S. It is obvious that $(a, b) \in \theta_S$ for elements $a, b \in S$ if and only if $sa = sb$ for all $s \in S$.

Remark 2.1. If A is a θ_S-class of a semigroup S, then $|sA| = 1$ for every $s \in S$, because $sa_1 = sa_2$ for every $a_1, a_2 \in A$.

A nonempty subset I of a semigroup S is called an *ideal* of S if $as, sa \in I$ for every $a \in I$ and $s \in S$. If I is an ideal of a semigroup S, then the relation ϱ_I on S defined by $(a, b) \in \varrho_I$ if and only if $a = b$ or $a, b \in I$ is a congruence on S which is called the *Rees congruence on S determined by I*. The equivalence classes of S mod ϱ_I are I itself and every one-element set $\{a\}$ with $a \in S \setminus I$. The factor semigroup S/ϱ_I is called the *Rees factor semigroup of S modulo I*. We shall write S/I instead of S/ϱ_I. We may describe S/I as the result of collapsing I into a single (zero) element, while the elements of S outside of I retain their identity.

A homomorphism φ of a semigroup S onto an ideal $I \subseteq S$ is called a *retract homomorphism* if φ leaves the elements of I fixed. An ideal I of a semigroup S is called a *retract ideal* if there is a retract homomorphism of S onto I. In this case, we say that S is a *retract (ideal) extension of I by the Rees factor semigroup S/I*.

An element e of a semigroup is called an *idempotent element* if $e^2 = e$. If every element of a semigroup S is idempotent, then S is called a *band*. A semigroup satisfying the identity $ab = a$ ($ab = b$, respectively) is called a *left zero* (*right zero*, respectively) *semigroup*. It is clear that every left (right, respectively) zero semigroup is a band. A commutative band is called a *semilattice*. A semigroup with a zero element 0 is called a *zero semigroup* if it satisfies the identity $ab = 0$.

For a semigroup S, let ω_S denote the universal relation on S. The next lemma is a characterization of semigroups in which $\theta_S = \omega_S$.

Lemma 2.2 ([20, Theorem 3.2]). *Semigroup S is a semigroup with $\theta_S = \omega_S$ if and only if S is a retract ideal extension of a left zero semigroup by a zero semigroup.*

Remark 2.3. If A is a θ_S-class of a semigroup S such that A is a subsemigroup of S, then $aa_1 = aa_2$ for every $a, a_1, a_2 \in A$, and hence $\theta_A = \omega_A$. Then, using Lemma 2.2, A is a retract ideal extension of a left zero semigroup by a zero semigroup.

Remark 2.4. If a semigroup A is a retract ideal extension of a left zero semigroup E_A by a zero semigroup, then $A^2 = E_A$ and the set of all idempotent elements of A is E_A. If φ_A is a retract homomorphism of A onto E_A (acting from the left), then $a_1a_2 = \varphi_A(a_1a_2) = \varphi_A(a_1)\varphi_A(a_2) = \varphi_A(a_1)$ for every $a_1, a_2 \in A$.

A semigroup S is called a *left*, (*right*, respectively) *commutative semigroup* if it satisfies the identity $xya = yxa$ ($axy = ayx$, respectively). A semigroup is said to be a *medial semigroup* if it satisfies the identity $axyb = ayxb$. It is clear that every left commutative semigroup and every right commutative semigroup is medial. Left commutative, right commutative and medial semigroups are studied in many papers, see, for example, [1], [7], [9], [10], [14], [15], [17], [18], [19], [25], [24], and the books [16], [22]. In this paper we also focus on them. The next lemma characterizes medial semigroups S with the help of the factor semigroup S/θ_S.

Lemma 2.5 ([19, Lemma 3.1]). *A semigroup S is a medial semigroup if and only if the factor semigroup S/θ_S is left commutative.*

Remark 2.6. It is easy to see that a left zero semigroup containing at least two elements is not left commutative. Thus a left commutative semigroup has two elements if and only if it is either a two-element semilattice, a two-element zero semigroup, a two-element group or a two-element right zero semigroup.

For notions not defined but used in this paper, we refer the reader to books [2], [16], [22].

3. Results

At the beginning of this section we construct four medial semigroups which play an important role in the proof of our main theorem. All mappings in this section act from the left.

Construction 3.1. Let A be a semigroup which is a retract ideal extension of a left zero semigroup E_A by a zero semigroup. Let φ_A denote a retract homomorphism of A onto E_A. By Remark 2.4, the ideal E_A of A is the set of all idempotent elements of A, $A^2 = E_A$, and $a_1a_2 = \varphi_A(a_1)$ for every $a_1, a_2 \in A$. Let B be a semigroup such that $A \cap B = \emptyset$. Assume that B is also a retract ideal extension of a left zero semigroup E_B by a zero semigroup. Let φ_B denote a retract homomorphism of B onto E_B. By Remark 2.4, the ideal E_B of B is the set of all idempotent elements of B, $B^2 = E_B$, and $b_1b_2 = \varphi_B(b_1)$ for every $b_1, b_2 \in B$. Let α be a homomorphism of A into E_B and β be a homomorphism of B into itself which leaves the elements of E_B fixed. Assume that the equations

$$\alpha \circ \varphi_A = \alpha, \tag{1}$$

$$\beta \circ \beta = \beta, \tag{2}$$

$$\varphi_B \circ \beta = \varphi_B \tag{3}$$

are satisfied. Since α maps A into E_B, and the mappings φ_B and β leave the elements of E_B fixed, we have

$$\varphi_B \circ \alpha = \alpha = \beta \circ \alpha, \tag{4}$$

$$\beta \circ \varphi_B = \varphi_B. \tag{5}$$

Let $S = A \cup B$. We define an operation "$\cdot$" on S. For every $x, y \in S$, let

$$x \cdot y = \begin{cases} \varphi_A(x) & \text{if } x, y \in A, \\ \alpha(x) & \text{if } x \in A,\ y \in B, \\ \beta(x) & \text{if } x \in B,\ y \in A, \\ \varphi_B(x) & \text{if } x, y \in B. \end{cases}$$

We show that the operation "$\cdot$" is associative. Let $a \in A$ and $x, y \in S$ be arbitrary elements. Using equations (1) and (4), we have

$$(a \cdot x) \cdot y = \begin{cases} \varphi_A(a) & \text{if } x, y \in A, \\ \alpha(a) & \text{otherwise.} \end{cases}$$

By the definition of the operation "$\cdot$", we have

$$a \cdot (x \cdot y) = \begin{cases} \varphi_A(a) & \text{if } x, y \in A, \\ \alpha(a) & \text{otherwise.} \end{cases}$$

Thus

$$(a \cdot x) \cdot y = a \cdot (x \cdot y).$$

Let $b \in B$ and $x, y \in S$ be arbitrary elements. Using (2), (3) and (5), we have

$$(b \cdot x) \cdot y = \begin{cases} \beta(b) & \text{if } x, y \in A, \\ \varphi_B(b) & \text{otherwise.} \end{cases}$$

By the definition of the operation "$\cdot$", we have

$$b \cdot (x \cdot y) = \begin{cases} \beta(b) & \text{if } x, y \in A, \\ \varphi_B(b) & \text{otherwise.} \end{cases}$$

Thus

$$(b \cdot x) \cdot y = b \cdot (x \cdot y).$$

Consequently the operation "$\cdot$" is associative, and hence the algebraic structure $(S; \cdot)$ is a semigroup. It is clear that $(S; \cdot)$ is a right commutative semigroup, and hence a medial semigroup. It follows from the definition of the operation "$\cdot$" that $(a_1, a_2) \in \theta_S$ and $(b_1, b_2) \in \theta_S$ for every $a_1, a_2 \in A$ and $b_1, b_2 \in B$. For every $a \in A, b \in B$, we have $a \cdot a \in A$ and $a \cdot b \in B$, and hence $(a, b) \notin \theta_S$. Thus the θ_S-classes of S are A and B. Since $A^2 \subseteq A$, $B^2 \subseteq B$, $A \cdot B$, $B \cdot A \subseteq B$, the factor semigroup S/θ_S is a two-element semilattice.

To illustrate Construction 3.1, consider the following example. Let $A = \{a, 0\}$ be a two-element zero semigroup (0 is the zero of A, and so $E_A = \{0\}$) and $B = \{e, f\}$ be a two-element left zero semigroup (and hence φ_B is the identity mapping of B). Let α be a mapping of A into B such that $\alpha(x) = e$ for every $x \in A$. It is clear that α is a homomorphism. Let β be the identity mapping of B (that is, $\beta = \varphi_B$). Then β leaves the elements of E_B fixed. The equations (1), (2), and (3) are satisfied. The Cayley-table of the semigroup $(S; \cdot)$ is Table 1.

$\cdot$	a	0	e	f
a	0	0	e	e
0	0	0	e	e
e	e	e	e	e
f	f	f	f	f

TABLE 1.

Construction 3.2. Let B be a semigroup which is a retract ideal extension of a left zero semigroup E_B by a zero semigroup. Let φ_B denote a retract homomorphism of B onto E_B. By Remark 2.4, the ideal E_B of B is the set of all idempotent elements of B, $B^2 = E_B$, and $b_1 b_2 = \varphi_B(b_1)$ for every $b_1, b_2 \in B$. Let A be a nonempty set such that $A \cap B = \emptyset$. Let α be a mapping of A into B, β be a mapping of A into E_B and γ be a homomorphism of B into itself which leaves the elements of E_B fixed. Assume $\alpha \neq \beta$ or $\gamma \neq \varphi_B$, and

$$\varphi_B \circ \alpha = \beta = \gamma \circ \alpha, \tag{6}$$

$$\varphi_B \circ \gamma = \varphi_B = \gamma \circ \gamma. \tag{7}$$

Since $\beta(A) \subseteq E_B$ and γ leaves the element of E_B fixed, we have

$$\varphi_B \circ \beta = \beta = \gamma \circ \beta, \tag{8}$$

$$\gamma \circ \varphi_B = \varphi_B. \tag{9}$$

Let $S = A \cup B$. We define an operation "$\bullet$" on S. For every $x, y \in S$, let

$$x \bullet y = \begin{cases} \alpha(x) & \text{if } x, y \in A, \\ \beta(x) & \text{if } x \in A \text{ and } y \in B, \\ \gamma(x) & \text{if } x \in B,\ y \in A, \\ \varphi_B(x) & \text{if } x, y \in B. \end{cases}$$

We show that the operation "$\bullet$" is associative. Let $a \in A$ and $x, y \in S$ be arbitrary elements. Using (6) and (8), we have

$$(a \bullet x) \bullet y = \beta(a).$$

By the definition of the operation "$\bullet$", we have

$$a \bullet (x \bullet y) = \beta(a).$$

Thus

$$(a \bullet x) \bullet y = a \bullet (x \bullet y).$$

Let $b \in B$ and $x, y \in S$ be an arbitrary elements. Using (7), (9) and the fact that φ_B is a retract homomorphism, we have

$$(b \bullet x) \bullet y = \varphi_B(b).$$

By the definition of the operation "$\bullet$", we have

$$b \bullet (x \bullet y) = \varphi_B(b).$$

Thus

$$(b \bullet x) \bullet y = b \bullet (x \bullet y).$$

Consequently the operation "$\bullet$" is associative, and hence the algebraic structure $(S; \bullet)$ is a semigroup. It is clear that $(S; \bullet)$ is a right commutative semigroup, and hence a medial semigroup. It follows from the definition of the operation "$\bullet$" that $(a_1, a_2) \in \theta_S$ and $(b_1, b_2) \in \theta_S$ for every $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Let $a \in A$ and $b \in B$ be arbitrary elements. Assume $(a, b) \in \theta_S$. Then, for every $a' \in A$ and every $b' \in B$, we have $\alpha(a') = a' \bullet a = a' \bullet b = \beta(a')$ and $\gamma(b') = b' \bullet a = b' \bullet b = \varphi_B(b')$. Thus $\alpha = \beta$ and $\gamma = \varphi_B$. This contradicts the assumption that $\alpha \neq \beta$ or

$\gamma \neq \varphi_B$. Consequently the θ_S-classes of S are A and B. Since $A^2 \subseteq B$, $B^2 \subseteq B$, $A \bullet B$, $B \bullet A \subseteq B$, the factor semigroup S/θ_S is a two-element zero semigroup.

To illustrate Construction 3.2, consider the following example. Let $B = \{b, 0\}$ be a two-element zero semigroup (0 is the zero of B) and $A = \{a\}$ be a singleton. Let $\alpha(a) = b$, $\beta(a) = 0$, and $\gamma = \varphi_B$. Then $\beta(A) = \{0\} = E_B$, and γ leaves the elements of E_B fixed. Since $b \neq 0$, we have $\alpha \neq \beta$. Equations (6) and (7) are also satisfied. The Cayley-table of the semigroup $(S; \bullet)$ is Table 2.

$\bullet$	a	b	0
a	b	0	0
b	0	0	0
0	0	0	0

TABLE 2.

Construction 3.3. Let A be a semigroup which is a retract ideal extension of a left zero semigroup E_A by a zero semigroup. Let φ_A denote a retract homomorphism of A onto E_A. By Remark 2.4, the ideal E_A of A is the set of all idempotent elements of A, $A^2 = E_A$, and $a_1 a_2 = \varphi_A(a_1)$ for every $a_1, a_2 \in A$. Let B be a nonempty set. Let α be a mapping of A into B and γ be a mapping of B into E_A such that the equations

$$\alpha = \alpha \circ \varphi_A, \tag{10}$$

$$\gamma \circ \alpha = \varphi_A, \tag{11}$$

are satisfied. Let

$$\beta = \alpha \circ \gamma. \tag{12}$$

From equations (10) and (11), it follows that

$$\beta \circ \beta = \beta, \tag{13}$$

$$\beta \circ \alpha = \alpha \circ \gamma \circ \alpha = \alpha \circ \varphi_A = \alpha. \tag{14}$$

Using the fact that $\gamma(b) \in E_A$ and φ_A leaves the elements of E_A fixed, the equations (12) and (11) imply

$$\gamma \circ \beta = \gamma \circ \alpha \circ \gamma = \varphi_A \circ \gamma = \gamma. \tag{15}$$

Let $S = A \cup B$. We define an operation "$*$" on S. For every $x, y \in S$, let

$$x * y = \begin{cases} \varphi_A(x) & \text{if } x, y \in A, \\ \alpha(x) & \text{if } x \in A \text{ and } y \in B, \\ \beta(x) & \text{if } x \in B,\ y \in A, \\ \gamma(x) & \text{if } x, y \in B. \end{cases}$$

We show that the operation "$*$" is associative. Let $a \in A$ and $x, y \in S$ be arbitrary elements. Using equations (10) and (14), we have

$$(a * x) * y = \begin{cases} \varphi_A(a) & \text{if } x, y \in A \text{ or } x, y \in B, \\ \alpha(a) & \text{otherwise.} \end{cases}$$

By the definition of the operation "$*$", we have

$$a * (x * y) = \begin{cases} \varphi_A(a) & \text{if } x, y \in A \text{ or } x, y \in B, \\ \alpha(a) & \text{otherwise.} \end{cases}$$

Thus

$$(a * x) * y = a * (x * y).$$

Let $b \in B$ and $x, y \in S$ be an arbitrary elements. Using the equations (12) and (15), we have

$$(b * x) * y = \begin{cases} \beta(b) & \text{if } x, y \in A \text{ or } x, y \in B, \\ \gamma(b) & \text{otherwise.} \end{cases}$$

By the definition of the operation "$*$", we have

$$b * (x * y) = \begin{cases} \beta(b) & \text{if } x, y \in A \text{ or } x, y \in B, \\ \gamma(b) & \text{otherwise.} \end{cases}$$

Thus

$$(b * x) * y = b * (x * y).$$

Consequently the operation "$*$" is associative, and hence the algebraic structure $(S; *)$ is a semigroup. It is clear that $(S; *)$ is a right commutative semigroup, and hence a medial semigroup. It follows from the definition of the operation "$*$" that $(a_1, a_2) \in \theta_S$ and $(b_1, b_2) \in \theta_S$ for every $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Let $a \in A$ and $b \in B$ be arbitrary elements. Assume $(a, b) \in \theta_S$. Then, for every $a' \in A$, we have

$$A \ni a' * a = a' * b \in B$$

which is a contradiction. Thus the θ_S-classes of S are A and B. Since $A^2 \subseteq A$, $A * B$, $B * A \subseteq B$ and $B^2 \subseteq A$, the factor semigroup S/θ_S is a two-element group.

To illustrate Construction 3.3, consider the following example. Let $A = \{e\}$ be a one-element semigroup and $B = \{x, y\}$ be a two-element set. Let α be the mapping of A onto the subset $\{x\}$. Let γ be the only possible mapping of B onto A. We note that $\beta(b) = x$ for every $b \in B$. It is easy to see that the above conditions for α and γ are satisfied. The Cayley-table of the semigroup $(S; *)$ is Table 3.

$*$	e	x	y
e	e	x	x
x	x	e	e
y	x	e	e

Table 3.

Construction 3.4. Let A be a semigroup which is a retract ideal extension of a left zero semigroup E_A by a zero semigroup. Let B be a semigroup such that $A \cap B = \emptyset$, and B is also a retract ideal extension of a left zero semigroup E_B by a zero semigroup. Let φ_A and φ_B denote a retract homomorphism of A onto E_A and B onto E_B, respectively. By Remark 2.4, the ideal E_A of A is the set of all idempotent elements of A, $A^2 = E_A$, and $a_1 a_2 = \varphi_A(a_1)$ for every $a_1, a_2 \in A$. Similarly, the ideal E_B of B is the set of all idempotent elements of B, $B^2 = E_B$, and $b_1 b_2 = \varphi_B(b_1)$ for every $b_1, b_2 \in B$. Let α be a homomorphism of A into E_B and β be a homomorphism of B into E_A. Assume that the equations

$$\alpha \circ \varphi_A = \alpha = \varphi_B \circ \alpha, \tag{16}$$

$$\beta \circ \varphi_B = \beta = \varphi_A \circ \beta, \tag{17}$$

$$\alpha \circ \beta = \varphi_B, \tag{18}$$

$$\beta \circ \alpha = \varphi_A \tag{19}$$

are satisfied. Let $S = A \cup B$. We define an operation "$\star$" on S. For every $x, y \in S$, let

$$x \star y = \begin{cases} \varphi_A(x) & \text{if } x, y \in A, \\ \alpha(x) & \text{if } x \in A,\ y \in B, \\ \beta(x) & \text{if } x \in B,\ y \in A, \\ \varphi_B(x) & \text{if } x, y \in B. \end{cases}$$

We show that the operation "$\star$" is associative. Let $a \in A$ and $x, y \in S$ be arbitrary elements. Using equations (16) and (19), we have

$$(a \star x) \star y = \begin{cases} \varphi_A(a) & \text{if } y \in A, \\ \alpha(a) & \text{otherwise.} \end{cases}$$

By the definition of the operation "$\cdot$", we have

$$a \star (x \star y) = \begin{cases} \varphi_A(a) & \text{if } y \in A, \\ \alpha(a) & \text{otherwise.} \end{cases}$$

Thus

$$(a \star x) \star y = a \star (x \star y).$$

Let $b \in B$ and $x, y \in S$ be arbitrary elements. Using (17) and (18), we have

$$(b \star x) \star y = \begin{cases} \beta(b) & \text{if } y \in A, \\ \varphi_B(b) & \text{if } y \in B. \end{cases}$$

By the definition of the operation "$\star$", we have

$$b \star (x \star y) = \begin{cases} \beta(b) & \text{if } y \in A, \\ \varphi_B(b) & \text{if } y \in B. \end{cases}$$

Thus

$$(b \star x) \star y = b \star (x \star y).$$

Consequently the operation "$\star$" is associative. It is clear that the semigroup $(S;\star)$ is a medial semigroup. It follows from the definition of the operation "$\star$" that $(a_1, a_2) \in \theta_S$ and $(b_1, b_2) \in \theta_S$ for every $a_1, a_2 \in A$ and $b_1, b_2 \in B$. For every $a \in A$, $b \in B$, we have $a \star a \in A$ and $a \star b \in B$, and hence $(a, b) \notin \theta_S$. Thus the θ_S-classes of S are A and B. Since $A^2 \subseteq A$, $B^2 \subseteq B$, $A \star B \subseteq B$, $B \star A \subseteq A$, the factor semigroup S/θ_S is a two-element right zero semigroup.

To illustrate Construction 3.4, consider the following example. Let $A = \{e, f\}$ and $B = \{g, h\}$ be disjoint two-element left zero semigroups. Let α be a mapping of A into B such that $\alpha(e) = g$ and $\alpha(f) = h$. Let β be the mapping of B into A such that $\beta(g) = e$ and $\beta(h) = f$ (that is, β and α are inverses of each other). Since A and B are left zero semigroups, both of α and β are homomorphisms. Since $E_A = A$ and $E_B = B$, α maps A into E_B and β maps B into E_A. In this example, φ_A and φ_B are the identities on the sets A and B, respectively, which fact together with the note that the mappings α and β are mutually inverse, readily yields the validity of equations (16)–(19). The Cayley-table of the semigroup $(S;\star)$ is Table 4.

$\star$	e	f	g	h
e	e	e	g	g
f	f	f	h	h
g	e	e	g	g
h	f	f	h	h

Table 4.

Theorem 3.5. *A semigroup S is a medial semigroup such that the index of θ_S is 2 if and only if S is isomorphic to one of the semigroups defined in Constructions 3.1– 3.4.*

PROOF: Semigroups S defined in Constructions 3.1– 3.4 are medial semigroups such that the index of θ_S is 2.

Conversely, let S be a medial semigroup such that the index of θ_S is 2. By Lemma 2.5 and Remark 2.6, S/θ_S is either a two-element semilattice, a two-element zero semigroup, a two-element group or a two-element right zero semigroup.

First consider the case when S/θ_S is a two-element semilattice. Let A and B denote the θ_S-classes of S. Then A and B are subsemigroups of S such that one of them is an ideal of S. Assume that B is an ideal of S. By Remark 2.3, $\theta_A = \omega_A$ and $\theta_B = \omega_B$. Thus, by Lemma 2.2 and Remark 2.4, A is a retract ideal extension of the set E_A of all idempotent elements of A by a zero semigroup, and B is a retract ideal extension of the set of all idempotent elements E_B of B by a zero semigroup. Let φ_A denote a retract homomorphism of A onto E_A and φ_B denote a retract homomorphism of B onto E_B. By Remark 2.4, $a_1a_2 = \varphi_A(a_1)$ for every $a_1, a_2 \in A$, and $b_1b_2 = \varphi_B(b_1)$ for every $b_1, b_2 \in B$. By Remark 2.1, $|aB| = 1$ for every $a \in A$. Let α be a mapping of A into B defined by

$$\alpha(a) = aB, \qquad a \in A.$$

For every $a_1, a_2 \in A$, we have

$$\alpha(a_1a_2) = a_1a_2B = a_1Ba_2B = \alpha(a_1)\alpha(a_2),$$

because B is a θ_S-class, $a_2B, Ba_2B \subseteq B$, and hence $a_1a_2B = a_1Ba_2B$. Thus α is a homomorphism. For every $a \in A$, we have

$$\alpha(a)\alpha(a) = aBaB = aB = \alpha(a),$$

because $BaB \subseteq B$, and hence $aBaB = aB$. Thus α maps A into the set E_B of all idempotent elements of B.

By Remark 2.1, $|bA| = 1$ for every $b \in B$. Let β be a mapping of B into itself defined by

$$\beta(b) = bA, \qquad b \in B.$$

Let e be an idempotent element of A. Since S is a medial semigroup, we have, for every $b_1, b_2 \in B$,

$$\beta(b_1b_2) = b_1b_2A = b_1b_2e = b_1b_2ee = b_1eb_2e = (b_1A)(b_2A) = \beta(b_1)\beta(b_2),$$

that is, β is a homomorphism. If $f \in E_B$, then

$$\beta(f) = fA = ffA = ff = f,$$

because B is a θ_S-class of S and $fA \subseteq B$. Thus β leaves the elements of E_B fixed. For every $a, a^* \in A$ and $b \in B$, we have

$$(\alpha \circ \varphi_A)(a) = \alpha(\varphi_A(a)) = \alpha(aa^*) = (aa^*)b = a(a^*b) = \alpha(a),$$

and hence (1) is satisfied. For every $a, a^* \in A$ and $b \in B$, we have

$$(\beta \circ \beta)(b) = \beta(\beta(b)) = \beta(b)a = (ba^*)a = b(a^*a) = \beta(b).$$

Thus (2) is satisfied. For every $b, b^* \in B$ and $a \in A$, we have

$$(\varphi_B \circ \beta)(b) = \varphi_B(\beta(b)) = \beta(b)b^* = (ba)b^* = b(ab^*) = \varphi_B(b),$$

and hence (3) is satisfied. We can consider the semigroup $(S; \cdot)$ defined in Construction 3.1. It is clear that the semigroup S is isomorphic to the semigroup $(S; \cdot)$.

In the second step of the proof, suppose that S/θ_S is a two-element zero semigroup. Let A and B denote the θ_S-classes of S. We can suppose that B is an ideal of S. Then $A^2 \subseteq B$. By Remark 2.3, $\theta_B = \omega_B$. Then, by Lemma 2.2 and Remark 2.4, B is a retract ideal extension of a left zero semigroup E_B by a zero semigroup, where E_B is the set of all idempotent elements of B. Moreover, for arbitrary $b_1, b_2 \in B$, we have $b_1b_2 = \varphi_B(b_1)$. By Remark 2.1, $|aA| = |aB| = |bA| = 1$ for every $a \in A$ and $b \in B$. For arbitrary $a \in A$ and $b \in B$, let

$$\alpha(a) = aA, \quad \beta(a) = aB, \quad \gamma(b) = bA.$$

Since $bab \in B$ for every $a \in A$ and $b \in B$ (and hence $(bab, b) \in \theta_S$), we have

$$(ab)^2 = a(bab) = ab.$$

Thus $\beta(a) \in E_B$ for every $a \in A$. For every $b_1, b_2 \in B$, we have

$$\gamma(b_1b_2) = (b_1b_2)A = b_1(b_2A) = b_1(Ab_2A) = (b_1A)(b_2A) = \gamma(b_1)\gamma(b_2),$$

because $b_2A, Ab_2A \subseteq B$ and, by Remark 2.1, $|b_1B| = 1$. Thus γ is a homomorphism. For every $f \in E_B$ and $a \in A$, we have $(fa, f) \in \theta_S$, and hence

$$fa = f(fa) = ff = f.$$

Thus

$$\gamma(f) = fA = f,$$

that is, γ leaves the elements of E_B fixed.

Since $AB, AA \subseteq B$, we have

$$a(AB) = aB = a(AA)$$

for every $a \in A$. Then

$$\varphi_B \circ \alpha = \beta = \gamma \circ \alpha,$$

that is, (6) is satisfied. Using also the inclusions $AB, AA \subseteq B$, we get

$$bAB = bB = b(AA)$$

for every $b \in B$. Then

$$\varphi_B \circ \gamma = \varphi_B = \gamma \circ \gamma,$$

that is, (7) is satisfied. Thus we can consider the semigroup $(S; \bullet)$ defined in Construction 3.2. It is clear that S is isomorphic to $(S; \bullet)$.

In the third step of the proof, suppose that S/θ_S is a two-element group. Let A and B denote the θ_S-classes of S. We can suppose that A is the identity element of S/θ_S. Then A is a subsemigroup of S, $AB, BA \subseteq B$, and $B^2 \subseteq A$. By Remark 2.3, $\theta_A = \omega_A$. Then, by Lemma 2.2 and Remark 2.4, A is a retract extension of a left zero semigroup E_A by a zero semigroup where E_A is the set of all idempotent elements of A. Moreover, for arbitrary $a_1, a_2 \in A$, we have $a_1a_2 = \varphi_A(a_1)$. By Remark 2.1, $|aB| = |bA| = |bB| = 1$ for every $a \in A$ and $b \in B$. For arbitrary $a \in A$ and $b \in B$, let

$$\alpha(a) = aB, \qquad \beta(b) = bA, \qquad \gamma(b) = bB.$$

Since $b_2b_1b_2 \in B$ for every $b_1, b_2 \in B$ (and so $(b_2b_1b_2, b_2) \in \theta_S$), we have

$$(b_1b_2)^2 = b_1(b_2b_1b_2) = b_1b_2.$$

Thus γ maps B into E_A.

Let $b \in B$ be an arbitrary element. Since $B^2 \subseteq A$, we have

$$\beta(b) = bA = bB^2 = (bB)B = \gamma(b)B = \alpha(\gamma(b)) = (\alpha \circ \gamma)(b).$$

Thus

$$\beta = \alpha \circ \gamma,$$

and hence (12) is satisfied. For arbitrary $a \in A$, we have

$$(\alpha \circ \varphi_A)(a) = \alpha(\varphi_A(a)) = \varphi_A(a)B = (aa)B = a(aB) = \alpha(a)$$

and

$$(\gamma \circ \alpha)(a) = \gamma(\alpha(a)) = \alpha(a)B = (aB)B = a(BB) = \varphi_A(a).$$

Thus conditions (10) and (11) are satisfied. Hence we can consider the semigroup $(S; *)$ defined in Construction 3.3. It is clear that S is isomorphic to $(S; *)$.

In the fourth step of the proof, suppose that S/θ_S is a two-element right zero semigroup. Let A and B denote the θ_S-classes of S. Then A and B are subsemigroups of S such that $AB \subseteq B$ and $BA \subseteq A$. By Remark 2.3, $\theta_A = \omega_A$ and $\theta_B = \omega_B$. Thus, by Lemma 2.2 and Remark 2.4, A is a retract ideal extension of a left zero semigroup E_A by a zero semigroup, where E_A is the set of all idempotent elements of A. Moreover, for arbitrary $a_1, a_2 \in A$, we have $a_1a_2 = \varphi_A(a_1)$. Similarly, B is a retract ideal extension of a left zero semigroup E_B by a zero semigroup, where E_B is the set of all idempotent elements of B. Moreover, for arbitrary $b_1, b_2 \in B$, we have $b_1b_2 = \varphi_B(b_1)$.

For every $a \in A$ and $b \in B$, $|aB| = |bA| = 1$ by Remark 2.1. Let α be a mapping of A into B defined by

$$\alpha(a) = aB, \qquad a \in A,$$

and β be a mapping of B into A defined by

$$\beta(b) = bA, \qquad b \in B.$$

For every $a_1, a_2 \in A$, we have $a_2B, Ba_2B \subseteq B$, and hence $a_1a_2B = a_1Ba_2B$, because $|a_1B| = 1$ by Remark 2.1. Then

$$\alpha(a_1a_2) = a_1a_2B = a_1Ba_2B = \alpha(a_1)\alpha(a_2).$$

Thus α is a homomorphism. For every $a \in A$, we have $BaB \subseteq B$, and hence $a(BaB) = aB$, because $|aB| = 1$ by Remark 2.1. Then

$$(\alpha(a))^2 = \alpha(a)\alpha(a) = aBaB = aB = \alpha(a),$$

and hence α maps A into E_B. It can be similarly proved that β is a homomorphism which maps B into E_A.

For arbitrary $a \in A$ and $b \in B$,

$$(\alpha \circ \varphi_A)(a) = \alpha(\varphi_A(a)) = \varphi_A(a)b = (aa)b = a(ab) = \alpha(a)$$

and

$$(\varphi_B \circ \alpha)(a) = \varphi_B(\alpha(a)) = \alpha(a)b = (ab)b = a(bb) = \alpha(a).$$

Thus (16) is satisfied. We can prove in a similar way that (17) is satisfied. For arbitrary $a \in A$ and $b \in B$,

$$(\alpha \circ \beta)(b) = \alpha(\beta(b)) = \beta(b)b = (ba)b = b(ab) = \varphi_B(b).$$

Thus (18) is satisfied and (19) can be similarly proved. Consider the semigroup $(S; \star)$ defined as in Construction 3.4. It is clear that S is isomorphic to $(S; \star)$. □

By [19, Dual of Lemma 3.2], a semigroup S is right commutative if and only if the factor semigroup S/θ_S is commutative. This result and the proof of Theorem 3.5 together imply the following corollary.

Corollary 3.6. *A semigroup S is a right commutative semigroup such that the index of θ_S is 2 if and only if S is isomorphic to one of the semigroups defined in Constructions* 3.1–3.3*.*

By [20], in a finite semigroup S both of the conditions that the index of θ_S is m and $P_{\theta_S}(S) = 1/m$ are satisfied if and only if each θ_S-class contains the same number of elements. Thus the following result is a consequence of Theorem 3.5.

Theorem 3.7. *A semigroup S is a finite medial semigroup such that the index of θ_S is* 2 *and $P_{\theta_S}(S) = 1/2$ if and only if S is isomorphic to one of the semigroups defined in Constructions* 3.1–3.4 *satisfying the condition $|A| = |B| < \infty$ in each of the four cases.*

Corollary 3.6 and Theorem 3.7 imply the following corollary.

Corollary 3.8. *A semigroup S is a finite right commutative semigroup such that the index of θ_S equals* 2 *and $P_{\theta_S}(S) = 1/2$ if and only if S is isomorphic to one of the semigroups defined in Constructions* 3.1–3.3 *satisfying the condition $|A| = |B| < \infty$ in each of the three cases.*

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(*Received* February 15, 2023, *revised* August 10, 2023)