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A Kalmár-style completeness proof for the logics of the hierarchy $\mathbb{I}^n\mathbb{P}^k$

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Abstract. The logics of the family $\mathbb{I}^n\mathbb{P}^k{:=}\{I^nP^k\}_{(n,k)\in\omega^2}$ are formally defined by means of finite matrices, as a simultaneous generalization of the weakly-intuitionistic logic I^1 and of the paraconsistent logic P^1. It is proved that this family can be naturally ordered, and it is shown a sound and complete axiomatics for each logic of the form I^nP^k. The involved completeness proof showed here is obtained by means of a generalization of the well-known Kalmár's method, usually applied for many-valued logics.

Keywords: mathematical logic; Kalmár's completeness proof; many-valued logic

Classification: 03B50, 03B53

1. Introduction and preliminaries

In informal terms, paraconsistent logics are the deductive systems allowing the existence of theories that are not necessarily trivializable in the presence of contradictions. In other words, a given logic $\mathcal{L}$ having a negation connective $\neg$ is paraconsistent if and only if there are a theory Γ and a formula φ such that, from the set $\Gamma \cup \{\varphi, \neg\varphi\}$, not every formula can be inferred. From this, a paraconsistent logic can deal with contradictory formulas whose consequence relation is not trivial. With this basic and informal ideas in mind, several works establish principles useful for the characterization and classification of different paraconsistent logics, see, for instance, [3], [2] or [5]. Among such principles, we focus on the following ones, taking as a starting point a logic $\mathcal{L}$ whose consequence relation is $\vdash_{\mathcal{L}}$, and with a negation connective $\neg$:

- A theory (that is, a set of formulas) Γ is *contradictory* if and only if there is a formula φ such that $\Gamma \vdash_{\mathcal{L}} \varphi$ and $\Gamma \vdash_{\mathcal{L}} \neg\varphi$. On the other hand, we say that Γ is *trivial* if and only if $\Gamma \vdash_{\mathcal{L}} \varphi$ for every formula φ. Finally, we say that Γ is *explosive* if and only if for every formula φ, it holds that $\Gamma, \varphi, \neg\varphi \vdash_{\mathcal{L}} \psi$ for every formula ψ (in other words, Γ is explosive if and

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only if $\Gamma \cup \{\varphi, \neg\varphi\}$ is a trivial theory for every formula φ). With this in mind, we can consider the following principles which will determine the paraconsistent character of a given logic $\mathcal{L}$:

- The logic $\mathcal{L}$ respects the *Principle of non-contradiction* (PNC) if and only if it allows the existence of non-contradictory theories.
- $\mathcal{L}$ respects the *Principle of nontriviality* (PNT) if it posseses nontrivial theories.
- On the other hand, $\mathcal{L}$ respects the *Principle of explosion* (PE) if every theory is explosive. That is, if $\Gamma, \varphi, \neg\varphi \vdash_{\mathcal{L}} \psi$ for *every set of formulas* $\Gamma \cup \{\varphi, \psi\}$.

With all this background, we say that $\mathcal{L}$ is a paraconsistent logic if it does not validate (PE). This means that $\mathcal{L}$ is paraconsistent if and only if there is (at least) a theory Γ, a formula φ and a formula ψ, such that $\Gamma, \varphi, \neg\varphi \nvdash_{\mathcal{L}} \psi$. So, a paraconsistent logic allows the existence of contradictory theories which are simultaneously nontrivial ones.

One of the more important families of paraconsistent logics is $\{C_n\}_{n\in\omega}$, which was introduced by N. da Costa in [7], see also [6], by means of Hilbert-style Axiomatics. The paraconsistent character of those logics is achieved as follows: simply the formula-schema $\neg(\varphi \wedge \neg\varphi)$ is not established as a theorem in them. So, the coexistence of contradictory formulas is neither forbidden nor obligatory. This would allow the existence of formulas φ where $\varphi \wedge \neg\varphi$ is a true expression, meanwhile there are at the same time formulas ψ where $\psi \wedge \neg\psi$ is not valid. Now, for these ones, the formula ψ° defined as $\psi^\circ := \neg(\psi \wedge \neg\psi)$ contributes to determine trivial theories (by the way, in [7] these formulas are called "well-behaved" ones). For instance, in the case of the logic C_1, $\{\psi^\circ, \psi, \neg\psi\}$ is a trivial theory (trivial theories can be obtained in the other logics of the hierarchy, too). On the other hand, usually the sets of the form $\{\varphi, \neg\varphi\}$ are obviously contradictory and not trivial theories. An interesting point about the C_n-logics is that there are not any mentions about the characterization of the internal structure of a well-behaved formula. Simply, two types of formulas are allowed: the well/not-well behaved ones.

Some years after, the propositional logic P^1 was defined by A. Sette in [20]. It possesses special characteristics that distinguish it from the family $\{C_n\}_{\{0\leq n\leq\omega\}}$. One of them is, precisely, that the well-behavior of a certain formula φ can be determined according to its internal structure. Simply, φ is well-behaved if and only if it is not an atomic formula. Moreover, in P^1 this fact can be explained by its semantics, as we shall see later on.

Among other additional properties, even when P^1 can be defined by means of a Hilbert-style axiomatics, it can also be obtained by means of a finite matrix (meanwhile no one of the C_n-logics, with $n \geq 1$, can be characterized in this way). The matrix semantics for P^1 is built taking as basis a set of three truth-values: T_0 and F_0 (intended as the "classical truth-values") together with T_1, which can be associated to an "intermediate truth". Besides that, P^1 is *maximal* with respect to the propositional classical logic (CL), in the sense that, if any axiom-schema (independent of the original ones) is added to the axiomatics of P^1, then this new axiomatics generates CL. Finally, P^1 is *algebraizable*, as it was shown in [13].

A notion which is dual (in a certain sense) to paraconsistency is *paracompleteness*, see [4] for a extensive analysis. Roughly speaking, from a complete theory Γ of a certain logic $\mathcal{L}$, it would be possible to infer φ or $\neg\varphi$ for every formula φ. Now, if $\mathcal{L}$ has a disjunction connective (with certain properties), Γ is complete if, from it, it is possible to obtain $\varphi \vee \neg\varphi$ for every φ. With this underlying idea, A. Sette and W. Carnielli defined in [21] the logic I^1, which, in general terms, shares with P^1 several properties among the already mentioned (finite axiomatizability, maximality relative to CL and algebraizability). Besides that, it is also defined in I^1 the well-behavior of a formula (with respect to completeness, in this case): the expression φ^* abbreviates $\varphi \vee \neg\varphi$. Moreover, in I^1 it holds that the "well behavior with respect to completeness" of φ can be characterized by its internal structure: φ is well-behaved (with respect to completeness) if and only if φ is not an atomic formula. This fact is similar to the case of P^1, already mentioned. From all this, one of the more remarkable differences between I^1 and P^1 is the following: in P^1, not every formula of the form $\neg(\varphi \wedge \neg\varphi)$ is a P^1-theorem; on the other hand, every formula of the form $\varphi \vee \neg\varphi$ is a P^1-theorem. Now, I^1 behaves exactly in the opposite way: $\varphi \vee \neg\varphi$ is not usually an I^1-theorem, meanwhile $\neg(\varphi \wedge \neg\varphi)$ is always an I^1-theorem.

Continuing with some properties of the logic I^1, it can be defined by means of a 3-valued matrix, too. In this case (and unlike P^1), the "new truth value" is F_1, an "intermediate truth-value of falsehood". Considering all these facts, it was suggested in [21] a generalization of these logics *by the addition of new intermediate truth-values*, in such a way that the "new logics" already obtained constitute a family (which could be ordered in a natural way). Following (and simplifying at some extent) these suggestions, it was defined in [8] the family $\mathbb{I}^n\mathbb{P}^k = \{I^nP^k\}_{(n,k)\in\omega^2}$. Every member of $\mathbb{I}^n\mathbb{P}^k$ (usually mentioned here just as an *I^nP^k-logic*) can be considered as a generalization of I^1 and of P^1 at the same time, by several reasons. First of all, the classical logic CL can be identified simply with I^0P^0. Similarly, P^1 (or I^1) is simply I^0P^1 (I^1P^0, respectively). Moreover, every I^nP^k-logic has $n+k+2$ truth-values (as it will be seen later). In addition, it can be established an order relation within $\mathbb{I}^n\mathbb{P}^k$. The logics of this family fail to validate formulas of the form $\neg(\varphi \wedge \neg\varphi)$ and/or $\varphi \vee \neg\varphi$ (with the obvious exception of I^0P^0, wherein both formulas are tautologies). It is worth commenting that, since I^nP^k-logics are finite-valued and mostly paraconsistent/paracomplete, they can be applied to the study of several interesting properties, see [5], [10] or [19], for example.

However, an open problem referred to this family consists of providing an adequate (i.e. sound and complete) axiomatics for *all* the I^nP^k-logics. This paper is essentially devoted to offer a suitable axiomatics for them. Moreover, the soundness and completeness theorems shown here can be considered *general* in this sense: their proofs are given in such a way that the adequacy of all the logics of $\mathbb{I}^n\mathbb{P}^k$ (with respect to the axiomatics here presented) is demonstrated in a structured mode, common to any pair $(n, k) \in \omega^2$ previously fixed. The technique to prove this result is adapted to the well-known Kalmár's method to prove completeness for CL, see [16].

To avoid unnecesary information or formalism, the notions to be used to prove adequacy will be reduced as much as possible (this entails that this paper will contain some notational abuses). Besides that, the structure of this article is as follows: in the next section the I^nP^k-logics will be defined by means of finite matrices, some simple properties will be shown here, and it will be defined an order relation $\preceq$ in the family $\mathbb{I}^n\mathbb{P}^k$ (this justifies the expression "hierarchy" used for this family). In addition, it will be demonstrated that $I^{n_1}P^{k_1} \preceq I^{n_2}P^{k_2}$ if and only if $(n_2, k_2) \leq_\Pi (n_1, k_1)$, where $\leq_\Pi$ is the order of the product on ω^2 (that is, $(a_1, b_1) \leq_\Pi (a_2, b_2)$ if and only if $a_1 \leq a_2$ and $b_1 \leq b_2$, being $\leq$ the standard order in ω). In Section 3, it will be presented a general axiomatics for all the I^nP^k-logics and it will be proven some properties, which are essential to the proof of adequacy (result developed in Section 4). By the way, the technique to prove completeness consists of an adaptation of Kalmár's method, modifying the premises appearing in the formal proofs involved. This will be clear all along that section. For that, it is assumed that the reader is familiar with the notions of formal proof, schema axioms, inference rules and so on, within the context of Hilbert-style axiomatics. So, the definitions of these concepts (and other related ones) will be omitted. This paper concludes with some comments about future work.

2. Semantic presentation of the hierarchy $\mathbb{I}^n\mathbb{P}^k$

To define a matrix semantics for the logics of the family $\mathbb{I}^n\mathbb{P}^k$, it is necessary to start with the definition of the language $L(C)$, common to all the I^nP^k-logics:

Definition 2.1. The *set of connectives of all the I^nP^k-logics* is $C{:=}\{\neg, \rightarrow\}$, with obvious arities. The *language* $L(C)$ (or *set of formulas*) for the I^nP^k-logics is the absolutely free algebra over C generated by a countable set $\mathcal{V}$, in the usual way.

Along this paper, the uppercase greek letters $\Gamma, \Delta, \Sigma, \ldots$ denote *sets of formulas* of $L(C)$. In addition, the lowercase greek letters φ, ψ, θ are metavariables ranging over the *individual formulas of* $L(C)$. Finally, the letters α, α_1, $\alpha_2, \ldots$ will be used as metavariables referred only to the *atomic formulas* (that is, the elements of $\mathcal{V}$). All these notations can be used with subscripts, when neccesary. On the other hand, the expression $\varphi[\alpha_1, \ldots, \alpha_m]$ indicates that the atomic formulas occurring on φ *are precisely* $\alpha_1, \ldots, \alpha_m$ (this expression will be applied in the development of the completeness proof).

Despite their common language, the difference between each one of the I^nP^k-logics is given by their respective matrix semantics, defined as follows:

Definition 2.2. Let $(n, k) \in \omega^2$, with $\omega = \{0, 1, 2, \ldots\}$. The *matrix* $M_{(n,k)}$ is defined as a pair $M_{(n,k)} = ((A_{(n,k)}, C_{(n,k)}), D_{(n,k)})$, where

a) $(A_{(n,k)}, C_{(n,k)})$ is an algebra, *similar to* $L(C)$, with support

$$A_{(n,k)} := \{F_0, F_1, \ldots, F_n, T_0, T_1, \ldots, T_k\}^1.$$

b) $D_{(n,k)} = \{T_0, T_1, \ldots, T_k\}$.

In addition, the operations $\neg$ and $\rightarrow$ of $C_{(n,k)}$ (also called *truth-functions*)[2] are defined by the truth tables indicated below.

	F_0	F_r	T_i	T_0
$\neg$	T_0	F_{r-1}	T_{i-1}	F_0

$\rightarrow$	F_0	F_s	T_j	T_0
F_0	T_0	T_0	T_0	T_0
F_r	T_0	T_0	T_0	T_0
T_i	F_0	F_0	T_0	T_0
T_0	F_0	F_0	T_0	T_0

With $1 \leq r, s \leq n$; $1 \leq i, j \leq k$.

Remark 2.3. Realize that the truth-values $F_1, \ldots, F_n$ can be considered informally as *intermediate values of falsehood*, meanwhile $T_1, \ldots, T_k$ are *intermediate values of truth.* In addition, every application of $\neg$ to a "non classical value", approximates more and more the value to the "classical ones", F_0 and T_0. Note that

[1] Every algebra $(A_{(n,k)}, C_{(n,k)})$ will be identified with its support, if there is no risk of confusion.

[2] Strictly speaking, the operations of $C_{(n,k)}$ *are not* the connectives of C, of course. However, they will be denoted in the same way for the sake of simplicity.

there are needed n negations at most to pass from F_r to F_0. Similarly, the values of the form T_i "become" T_0 after k negations at most. On the other hand the implication $\rightarrow$ cannot distinguish between classical or intermediate truth-values: it just considers every value of the form F_i as being F_0, and every value of the form T_j as being T_0.

Taking into account the previous truth-tables, some secondary (and useful) truth-functions can be defined. As a motivation, it would be desirable that disjunction ($\vee$) and conjunction ($\wedge$) behave as $\rightarrow$ in this aspect: they cannot distinguish classical from intermediate truth-values. For that, it is taken as starting point the unary function of "classicalization" © (the meaning of this neologism is obvious), defined by $©(A) := (A \rightarrow A) \rightarrow A$ for every $A \in A_{(n,k)}$. So, the truth-table associated to it is

	T_0	T_i	F_r	F_0
©	T_0	T_0	F_0	F_0

From © it is defined the truth-function $\sim$, of *strong* (also called *classical*) negation, as $\sim A := \neg(©A)$. So, its associated truth-table is

	F_0	F_r	T_i	T_0
$\sim$	T_0	T_0	F_0	F_0

It is possible to define $\vee$ and $\wedge$ now, adapting the usual definition for CL: $A \vee B := \sim A \rightarrow B$, meanwhile $A \wedge B := \sim (A \rightarrow \sim B)$. For these connectives, their associated truth-functions are:

$\vee$	F_0	F_s	T_j	T_0
F_0	F_0	F_0	T_0	T_0
F_r	F_0	F_0	T_0	T_0
T_i	T_0	T_0	T_0	T_0
T_0	T_0	T_0	T_0	T_0

$\wedge$	F_0	F_s	T_j	T_0
F_0	F_0	F_0	F_0	F_0
F_r	F_0	F_0	F_0	F_0
T_i	F_0	F_0	T_0	T_0
T_0	F_0	F_0	T_0	T_0

With $1 \leq i, j \leq k$; $1 \leq r, s \leq n$.

From the previous definitions, it is clear that all the binary truth-functions consider all the non-designated values F_j as behaving as F_0, and similarly for all the values T_i. The same fact holds for $\sim$. In the case of $\neg$, however, all the truth-values can be differentiated. This is the main difference of $\neg$ and $\sim$, and justifies the definition and the study of the I^nP^k-logics. For example, when $n \geq 1$, the formula $\varphi \vee \neg\varphi$ is not an I^nP^k-tautology (it is enough to consider a valuation v such that $v(\varphi) = T_i$ with $i \geq 1$), meanwhile this formula is valid if $\neg$ is replaced by $\sim$. That is, $\models_{(n,k)} \varphi \vee \sim \varphi$ for any I^nP^k-logic. On the other hand, when $k \geq 1$, $\sim (\varphi \wedge \sim \varphi)$ is a I^nP^k-tautology (for every $(n,k) \in \omega^2$), but $\neg (\varphi \wedge \neg \varphi)$ is not valid in all the I^nP^k-logics. Indeed, $\neg (\varphi \wedge \neg \varphi)$ is only valid in the I^nP^0-logics.

After a deeper analysis it is possible to see the following fact, using the convention that $\neg^t\varphi$ indicates $\neg (\ldots (\neg \varphi)) \ldots)$ (t times) and $\neg^0\varphi$ is φ: given a fixed logic I^nP^k, $\models_{(n,k)} \varphi \vee \neg \varphi$ if and only if $\varphi = \neg^t(\psi \to \theta)$ (with $t \geq 0$), or $\varphi = \neg^t\alpha$, with $\alpha \in \mathcal{V}$ and $t \geq n$: otherwise (when $\varphi = \neg^t\alpha$ with $t < n$) $\not\models_{(n,k)} \varphi \vee \neg \varphi$. In a similar way, $\models_{(n,k)} \neg (\varphi \wedge \neg \varphi)$ if and only if $\varphi = \neg^t(\psi \to \theta)$ with $t \geq 0$, or $\varphi = \neg^t\alpha$, with $t \geq k$, $\alpha \in \mathcal{V}$. From these comments we can see that $\neg (\varphi \wedge \neg \varphi)$ and $\varphi \vee \neg \varphi$ are not I^nP^k-tautologies in general terms. So, it is natural to distinguish between "well-behaved" formulas and "not well-behaved" ones. This distinction is formalized by means of the unary "well-behavior" truth-functions, defined in the obvious way: $A^* := A \vee \neg A$; $A^\circ := \neg (A \wedge \neg A)$ for every $A \in A_{(n,k)}$. Its respective truth-tables are

	F_0	F_r	T_i	T_0
$*$	T_0	F_0	T_0	T_0

	F_0	F_r	T_i	T_0
$\circ$	T_0	T_0	F_0	T_0

Remark 2.4. The secondary truth-function associated to the connective $^\circ$, which is motivated by (PNC), was defined in this paper following the ideas developed in [7]. In that work, that principle characterizes consistency (and, laterally, it defines certain explosive theories, as it was previously commented). Note however that in some recent papers the notion of consistency is considered as being a *primitive* concept, see [3], [2]. In these logics, known as *logics of formal inconsistency*, it is possible to deal with consistency (and with inconsistency, triviality and explosiveness) in such a way that two essential notions can be separated: consistency and non contradiction. On the other hand, the distinction between both concepts cannot be done in the I^nP^k-logics: the truth-table of $^\circ$ is, *by definition*, the one applied to non contradiction. With the same spirit, the truth-table of * is secondary, having in mind the validity/not validity of the formula $\varphi \vee \neg \varphi$ (which is related with (PEM), the principle of excluded middle, in this case).

Besides the behavior of the mentioned truth-functions in each matrix $M_{(n,k)}$, recall that its definition is motivated by the definition of a *consequence relation on* $L(C)$ (and therefore of a *logic*), in the usual way:

Definition 2.5. An $M_{(n,k)}$*-valuation* is any homomorphism $v \colon L(C) \to A_{(n,k)}$ (this notion makes sense because $L(C)$ and $A_{(n,k)}$ are similar algebras). Recall here that every $M_{(n,k)}$-valuation can be defined just considering functions $v :$ $\mathcal{V} \to A_{(n,k)}$ and extending it to all $L(C)$. The *logic* I^nP^k is the pair $I^nP^k :=$

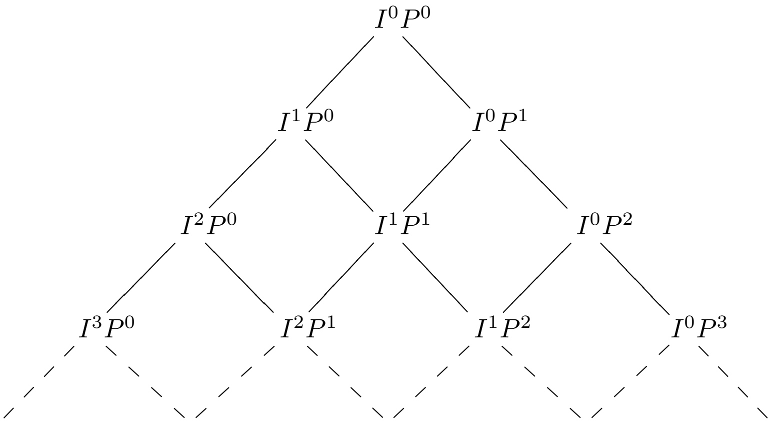

Figure 1. Ordered structure of the Hierarchy $\mathbb{I}^n\mathbb{P}^k$.

$(C, \models_{(n,k)})$, being $\models_{(n,k)} \subseteq \wp(L(C)) \times L(C)$ defined as usual, i.e., $\Gamma \models_{(n,k)} \varphi$ if and only if for very $M_{(n,k)}$-valuation v, $v(\Gamma) \subseteq D_{(n,k)}$ implies $v(\varphi) \in D_{(n,k)}$. In this context, φ is an *I^nP^k-tautology* if and only if $\emptyset \models_{(n,k)} \varphi$ (this fact will be denoted by $\models_{(n,k)} \varphi$, as usual). The family $\{I^nP^k\}_{(n,k)\in\omega^2}$ will be denoted by $\mathbb{I}^n\mathbb{P}^k$.

Remark 2.6. The family $\mathbb{I}^n\mathbb{P}^k$ includes some well-known logics. Indeed, I^0P^0 is just the classical logic CL. On the other hand, the logic I^1P^0 is I^1, meanwhile I^0P^1 is just P^1. In addition, all the I^nP^k-logics can be "naturally ordered", taking into account the following definition.

Definition 2.7. The *order relation* $\preceq\ \subseteq (\mathbb{I}^n\mathbb{P}^k)^2$ is defined as follows: $I^{n_1}P^{k_1} \preceq I^{n_2}P^{k_2}$ if and only if for every $\Gamma \cup \{\varphi\} \subseteq L(C)$, $\Gamma \models_{(n_1,k_1)} \varphi$ implies $\Gamma \models_{(n_2,k_2)} \varphi$.

It will be shown in the sequel that the order $\preceq$ in $\mathbb{I}^n\mathbb{P}^k$ can be depicted as the Figure 1 shows. This claim is based on an essential fact that can be checked by analysis of cases (among the truth-values of any I^nP^k-logic).

Proposition 2.8. *In the logic I^nP^k (for n, k fixed), the following formulas are tautologies:*

a) $\neg^n\varphi \vee \neg^{n+1}\varphi$ $\quad$ (*$(n+1)$-generalization of PEM*),
b) $\neg(\neg^k\varphi \wedge \neg^{k+1}\varphi)$ $\quad$ (*$(k+1)$-generalization of PNC*).

Proposition 2.9. *We have $I^{n_1}P^{k_1} \preceq I^{n_2}P^{k_2}$ if and only if $(n_2, k_2) \leq_\Pi (n_1, k_1)$ (being $\leq_\Pi$ the order of the product on ω^2). Therefore, the Hierarchy $(\mathbb{I}^n\mathbb{P}^k, \preceq)$ is a lattice.*

Proof: If $(n_2, k_2) \leq_\Pi (n_1, k_1)$, then $A_{(n_2,k_2)} \subseteq A_{(n_1,k_1)}$, and $D_{(n_2,k_2)} \subseteq D_{(n_1,k_1)}$. Now suppose that $\Gamma_0 \not\models_{(n_2,k_2)} \varphi_0$ for some $\Gamma_0 \cup \{\varphi_0\} \subseteq L(C)$. So, there exists a valuation $v\colon \mathcal{V} \to A_{(n_2,k_2)}$ such that $v(\Gamma_0) \subseteq D_{(n_2,k_2)}$, $v(\varphi_0) \notin D_{(n_2,k_2)}$. Define the valuation $w\colon \mathcal{V} \to A_{(n_1,k_1)}$ as $w(\alpha) = v(\alpha)$ for every $\alpha \in \mathcal{V}$. It can be proved that for every $\psi \in L(C)$, $w(\psi) = v(\psi)$. Thus, $w(\Gamma_0) \subseteq D_{(n_2,k_2)} \subseteq D_{(n_1,k_1)}$ and $w(\varphi_0) \in \{F_0, \ldots, F_{n_2}\} \subseteq \{F_0, \ldots, F_{n_1}\}$. That is, $\Gamma_0 \not\models_{(n_1,k_1)} \varphi_0$. The previous argument shows that $I^{n_1}P^{k_1} \preceq I^{n_2}P^{k_2}$.

For the converse, suppose $(n_2, k_2) \not\leq_\Pi (n_1, k_1)$. There are two cases that must be analyzed in different ways. First, if $n_2 > n_1$ consider any formula $\varphi_1 := \neg^{n_1}\alpha \vee \neg^{n_1+1}\alpha$, with $\alpha \in \mathcal{V}$. So, $\models_{(n_1,k_1)} \varphi_1$, by Proposition 2.8 a). Now, defining the valuation $v_1\colon \mathcal{V} \to A_{(n_2,k_2)}$ by $v_1(\alpha) := F_{n_2}$, it holds $v_1(\varphi_1) = \neg^{n_1}F_{n_2} \vee \neg^{n_1+1}F_{n_2} = F_{n_2-n_1} \vee F_{n_2-(n_1+1)} = F_0$ (since $n_1 + 1 \leq n_2$). Thus, $\not\models_{(n_2,k_2)} \varphi_1$. On the other hand, if $k_2 > k_1$, let $\varphi_2 := \neg(\neg^{k_1}\alpha \wedge \neg^{k_1+1}\alpha)$. As in the first case, $\models_{(n_1,k_1)} \varphi_2$, by Proposition 2.8 b). Now, if it is defined the valuation $v_2\colon \mathcal{V} \to A_{(n_2,k_2)}$ such that $v_2(\alpha) = T_{k_2}$, then $\not\models_{(n_1,k_1)} \varphi_2$ (note here that $k_1 + 1 \leq k_2$). So, for both possibilities it holds $I^{n_1}P^{k_1} \not\preceq I^{n_2}P^{k_2}$. This concludes the proof. □

Some consequences of the previous result, useful to visualize $\preceq$ (actually, its underlying strict order $\prec$) are the following:

Corollary 2.10. *In I^nP^k it holds that:*

a) $I^{n+1}P^k \prec I^nP^k$.
b) $I^nP^{k+1} \prec I^nP^k$.
c) I^nP^{k+1} *and* $I^{n+1}P^k$ *are not comparable.*

This section concludes with the mention of the following result that will be applied at the end of this paper.

Proposition 2.11. *The consequence relation $\models_{(n,k)}$ verifies:*
a) $\Gamma \models_{(n,k)} \varphi$ *implies* $\Gamma \cup \{\psi\} \models_{(n,k)} \varphi$ *[Monotonicity]*
b) $\Gamma, \varphi \models_{(n,k)} \psi$ *if and only if* $\Gamma \models_{(n,k)} \varphi \to \psi$ *[Semantic deduction theorem]*
c) *If* $\Gamma \models_{(n,k)} \varphi$, *then* $\Gamma' \models_{(n,k)} \varphi$ *for some finite set* $\Gamma' \subseteq \Gamma$ *[Finitariness]*

Proof: Obviously, a) holds. The claim b) arises from the truth-table of $\to$. With respect to c), $\models_{(n,k)}$ is finitary because it is naturally defined by means of a single finite matrix, see [22]. □

3. A Hilbert-style axiomatics for the I^nP^k-logics

From now on, consider an I^nP^k-logic fixed, with $(n,k) \in \omega^2$. To obtain the desired axiomatics, the secondary truth-functions $\sim$, ©, $\vee$ and $\wedge$ from the previous section will be reflected by means of the definition of secondary connectives in $L(C)$. Formally:

Definition 3.1. The secondary connectives ©, $\sim$, $\vee$, $\wedge$, * and $^\circ$ are defined in $L(C)$ in the following way:
$\copyright\,\varphi := (\varphi \to \varphi) \to \varphi$,
$\sim \varphi := \neg\,(\copyright\,\varphi)$,
$\varphi \vee \psi := \sim \varphi \to \psi$,
$\varphi \wedge \psi := \sim (\varphi \to \sim \psi)$,
$\varphi^* := \varphi \vee \neg\,\varphi$,
$\varphi^\circ := \neg\,(\varphi \wedge \neg\,\varphi)$.
In addition, the connectives $\vee_{\text{CL}}$ and $\wedge_{\text{CL}}$ are defined by:
$\varphi \vee_{\text{CL}} \psi := \neg\,\varphi \to \psi$,
$\varphi \wedge_{\text{CL}} \psi := \neg\,(\varphi \to \neg\,\psi)$ [3].
Finally, recall that the expression $\neg^t\varphi$ indicates $\neg\,(\ldots(\neg\,\varphi))\ldots)$, t times, and that $\neg^0\varphi$ is merely φ, as we said in the previous section.

Taking into account the previous conventions, the axiomatics for the I^nP^k-logics will be presented in the sequel. For that consider, from now on, an arbitrary (fixed) pair $(n,k) \in \omega^2$.

Definition 3.2. The *consequence relation* $\vdash_{(n,k)} \subseteq \wp(L(C)) \times L(C)$ is defined by means of the following Hilbert-style axiomatics, considering these schema axioms:
Ax_1 $\varphi \to (\psi \to \varphi)$,
Ax_2 $(\varphi \to (\psi \to \theta)) \to ((\varphi \to \psi) \to (\varphi \to \theta))$,
Ax_3 $(\varphi \to \psi)^*$,
Ax_4 $(\varphi \to \psi)^\circ$,
Ax_5 $(\neg^n\varphi)^*$,
Ax_6 $(\neg^k\varphi)^\circ$,
Ax_7 $\varphi^* \to [\psi^\circ \to ((\neg\,\varphi \to \neg\,\psi) \to ((\neg\,\varphi \to \psi) \to \varphi))]$,
Ax_8 $\varphi^* \to [\psi^\circ \to ((\varphi \to \neg\,\psi) \to ((\varphi \to \psi) \to \neg\,\varphi))]$,
Ax_9 $\varphi^* \to (\neg\,\neg\,\varphi \to \varphi)$,
Ax_{10} $\varphi^\circ \to (\varphi \to \neg\,\neg\,\varphi)$,
Ax_{11} $\varphi^* \to (\neg\,\varphi)^*$,
Ax_{12} $\varphi^\circ \to (\neg\,\varphi)^\circ$.

[3] The "classical" connectives $\wedge_{CL}$ and $\vee_{CL}$ are not essential in the proof of Completeness. However, they are indicated here for a better explanation of the comparison between these connectives with respect to $\wedge$ and $\vee$, as it will be remarked later.

The only inference rule for this axiomatics is modus ponens (MP): $\dfrac{\varphi,\ \ \varphi \to \psi}{\psi}$.

From this definition, the well-known notions of formal proof (with or without hypotheses), formal theorem, etc. are the usual ones. Because of this, $\vdash_{(n,k)}$ is *monotonic*, i.e., $\Gamma \vdash_{(n,k)} \varphi$ implies $\Gamma \cup \{\psi\} \vdash_{(n,k)} \varphi$. This fact will be widely used.

Remark 3.3. It is well known that the inclusion of Ax_1, Ax_2 and MP entail that it is valid $\vdash_{(n,k)} \varphi \to \varphi$. Moreover:

Theorem 3.4. *We have that $\vdash_{(n,k)}$ satisfies the (syntactic) deduction theorem (DT). That is, $\Gamma, \varphi \vdash_{(n,k)} \psi$ if and only if $\Gamma \vdash_{(n,k)} \varphi \to \psi$.*

Proof: This result holds because the inclusion of axioms Ax_1 and Ax_2 too, and considering that the only (primitive) inference rule is modus ponens. See [16] for a detailed proof. □

Ax_1 and Ax_2 allow to obtain some useful rules in relation to $\vdash_{(n,k)}$, too:

Proposition 3.5. *Given the logic I^nP^k, the following secondary rules are valid:*

Permutation (Perm): $\dfrac{\varphi \to (\psi \to \theta)}{\psi \to (\varphi \to \theta)}$.

Transitivity (Trans): $\dfrac{\varphi \to \psi, \qquad \psi \to \theta}{\varphi \to \theta}$.

Reduction (Red): $\dfrac{(\varphi \to \psi) \to \theta}{\psi \to \theta}$.

The following two results involve formulas of the form φ^* or φ°:

Proposition 3.6. *For every $\varphi \in L(C)$, for every $(n,k) \in \omega^2$, it holds:*
$\vdash_{(n,k)} (\varphi^*)^*$; $\vdash_{(n,k)} (\varphi^*)^\circ$; $\vdash_{(n,k)} (\copyright\,\varphi)^*$; $\vdash_{(n,k)} (\copyright\,\varphi)^\circ$.

This result is valid since $\varphi^* := \sim \varphi \to \neg\varphi$ and $\copyright\,\varphi = (\varphi \to \varphi) \to \varphi$, and considering axioms Ax_3 and Ax_4 from Definition 3.2.

Proposition 3.7. *If $\models_{(n,k)} \varphi$, then $\vdash_{(n,k)} \varphi^*$ and $\vdash_{(n,k)} \varphi^\circ$.*

Proof: Note that no formula of the form $\neg^t\alpha$, with $\alpha \in \mathcal{V}$, $t \geq 0$, is a tautology. Then, $\models_{(n,k)} \varphi$ implies that φ is necessarily of the form $\neg^q(\psi \to \theta)$, with $q \geq 0$. From this, apply Ax_3, Ax_4 (and, eventually, Ax_{11} and Ax_{12}). □

The next result shows some basic I^nP^k*-theorems*:

Proposition 3.8. *The following formulas of $L(C)$ are theorems with respect to $\vdash_{(n,k)}$:*

a) $\varphi \to \copyright\,\varphi$; a′) $\copyright\,\varphi \to \varphi$,

b) $\varphi^* \to (\sim\varphi \to \neg\varphi)$,
c) $\varphi^* \to [\psi^\circ \to ((\neg\varphi \to \neg\psi) \to (\psi \to \varphi))]$,
d) $\varphi^* \to [\psi^\circ \to ((\varphi \to \psi) \to (\neg\psi \to \neg\varphi))]$,
e) $(\sim\varphi \to \sim\psi) \to ((\sim\varphi \to \copyright\,\psi) \to \copyright\,\varphi)$,
f) $(\sim\varphi \to \sim\psi) \to ((\sim\varphi \to \psi) \to \varphi))$.

Proof: The following are schematic formal proofs (in the context of $\vdash_{(n,k)}$) for every formula above indicated. Sometimes it will be applied Theorem 3.4 or Proposition 3.5 without explicit mention.

For a): $\varphi \to \copyright\,\varphi = \varphi \to ((\varphi \to \varphi) \to \varphi)$ is a particular case of Ax_1. For the case of a'):

1) $(\varphi \to \varphi) \to \varphi$ [Hyp.; Def. $\copyright\,\varphi$]
2) $\varphi \to \varphi$ [Rem. 3.3]
3) φ [1), 2), MP]

So, it is valid $\copyright\,\varphi \vdash_{(n,k)} \varphi$.

For b):

1) φ^* [Hyp.]
2) $(\copyright\,\varphi)^\circ$ [Prop. 3.6]
3) $\varphi^* \to [(\copyright\,\varphi)^\circ \to ((\varphi \to \sim\varphi) \to ((\varphi \to \copyright\,\varphi) \to \neg\varphi)]$ [Ax_8, Def. 3.1 (of $\sim$)]
4) $(\varphi \to \sim\varphi) \to ((\varphi \to \copyright\,\varphi) \to \neg\varphi)$ [1), 2), 3), MP]
5) $(\varphi \to \copyright\,\varphi) \to ((\varphi \to \sim\varphi) \to \neg\varphi)$ [4), Perm.]
6) $\varphi \to \copyright\,\varphi$ [a)]
7) $(\varphi \to \sim\varphi) \to \neg\varphi$ [5), 6), MP]
8) $\sim\varphi \to (\varphi \to \sim\varphi)$ [Ax_1]
9) $\sim\varphi \to \neg\varphi$ [7), 8), Trans.]

That is, $\varphi^* \vdash_{(n,k)} \sim\varphi \to \neg\varphi$.

For c):

1) φ^* [Hyp.]
2) ψ° [Hyp.]
3) $\neg\varphi \to \neg\psi$ [Hyp.]
4) ψ [Hyp.]
5) $\psi \to (\neg\varphi \to \psi)$ [Ax_1]
6) $\neg\varphi \to \psi$ [4), 5), MP]
7) $\varphi^* \to [\psi^\circ \to ((\neg\varphi \to \neg\psi) \to ((\neg\varphi \to \psi) \to \varphi))]$ [Ax_7]
8) $(\neg\varphi \to \neg\psi) \to ((\neg\varphi \to \psi) \to \varphi)$ [7), 1), 2), MP]
9) $(\neg\varphi \to \psi) \to \varphi$ [8), 3), MP]
10) φ [9), 6), MP]

Thus, $\varphi^*, \psi^\circ, \neg\varphi \to \neg\psi, \psi \vdash_{(n,k)} \varphi$.

For d): Adapt the proof of c), using Ax_8 instead of Ax_7. Then, it will be valid $\varphi^*, \psi^\circ, \varphi \to \psi, \neg\psi \vdash_{(n,k)} \neg\varphi$.

For e):

1) $(© \varphi)^*$ [Prop. 3.6]

2) $(© \psi)^\circ$ [Prop. 3.6]

3) $(© \varphi)^* \to [(© \psi)^\circ \to ((\sim\varphi \to \sim\psi) \to ((\sim\varphi \to © \psi) \to © \varphi))]$ [Def. 3.1 (of $\sim$), Ax_7]

4) $(\sim\varphi \to \sim\psi) \to ((\sim\varphi \to © \psi) \to © \varphi)$ [1), 2), 3), MP]

So, $\vdash_{(n,k)} (\sim\varphi \to \sim\psi) \to ((\sim\varphi \to © \psi) \to © \varphi)$.

For f):

1) $\sim\varphi \to \sim\psi$ [Hyp.]

2) $\sim\varphi \to \psi$ [Hyp.]

3) $\psi \to © \psi$ [a)]

4) $\sim\varphi \to © \psi$ [2), 3), Trans.]

5) $(\sim\varphi \to \sim\psi) \to ((\sim\varphi \to © \psi) \to © \varphi)$ [e)]

6) $© \varphi$ [1), 4), 5), MP]

7) $© \varphi \to \varphi$ [a')]

8) φ [6), 7), MP]

From all this, $\sim\varphi \to \sim\psi, \sim\varphi \to \psi \vdash_{(n,k)} \psi$. Now, apply Theorem 3.4, as in the previous results. This concludes the proof. □

Remark 3.9. Now it is convenient to relate the axiomatics given in Definition 3.2 with a well-known axiomatics for CL $= I^0P^0$. According to [16], CL can be axiomatized by MP joined with the following three schema axioms:

$Bx_1 = Ax_1$,

$Bx_2 = Ax_2$,

$Bx_3 = (\neg\varphi \to \neg\psi) \to ((\neg\varphi \to \psi) \to \varphi)$.

Note that, cf. Definition 3.2, fixed an arbitrary consequence relation $\vdash_{(n,k)}$, the axiom Bx_3 of the previous axiomatics is replaced by a weaker version (Ax_7). Anyway, since in the particular case of $\vdash_{(0,0)}$, axioms Ax_5 and Ax_6 establish that, for every formula $\varphi \in L(C)$, $\vdash_{(0,0)} \varphi^*$ and $\vdash_{(0,0)} \varphi^\circ$, it is possible to recover the axiomatics determined by Bx_1, Bx_2 and Bx_3, actually. Moreover:

Proposition 3.10. *Let $\varphi \in L(C)$, in such a way that φ is a formal theorem of CL (that is, $\vdash_{(0,0)} \varphi$), and let $\varphi' \in L(C)$, obtained by φ replacing all the occurrences of the symbol $\neg$ in φ by $\sim$. Then $\vdash_{(n,k)} \varphi'$.*

Proof: Consider the axiomatics for I^0P^0 indicated in Remark 3.9, and compare it with the general axiomatics given in Definition 3.2. First of all note that neither $Bx_1(= Ax_1)$ nor $Bx_2(= Ax_2)$ have occurrences of $\neg$. Besides, since $Bx_3 = (\neg\varphi \to \neg\psi) \to ((\neg\varphi \to \psi) \to \varphi))$, and considering Proposition 3.8 f), it holds $\vdash_{(n,k)} (\sim\varphi \to \sim\psi) \to ((\sim\varphi \to \psi) \to \varphi)$ $(= Bx_3{}')$. From these facts, it can be easily proved by induction on the length of the formal proof of φ (with respect to $\vdash_{(0,0)}$) that $\vdash_{(0,0)} \varphi$ implies $\vdash_{(n,k)} \varphi'$. □

Corollary 3.11. *Suppose $\varphi \in L(C)$, and let the formula $\varphi'' \in L(C)$ built by replacing the eventual occurrences of $\neg$ in φ by $\sim$, and by replacing every occurrence of $\vee_{CPL}$ ($\wedge_{CPL}$, respectively), understood as an abbreviation (cf. Definition* 3.1*), by $\vee$ ($\wedge$, respectively). Then, $\vdash_{(0,0)} \varphi$ implies $\vdash_{(n,k)} \varphi''$.*

For instance, since $\vdash_{(0,0)} \varphi \vee_{\mathrm{CL}} \neg\varphi$, then $\vdash_{(n,k)} \varphi \vee \sim\varphi$. However, it is not generally valid that $\vdash_{(n,k)} \varphi \vee_{\mathrm{CL}} \neg\varphi$, obviously. The next result collects some particular cases of the previous corollary:

Corollary 3.12. *The relation $\vdash_{(n,k)}$ verifies, given $(n,k) \in \omega^2$:*

a) $\vdash_{(n,k)} \varphi \to \varphi \vee \psi$; a′) $\vdash_{(n,k)} \psi \to \varphi \vee \psi$,
b) $\vdash_{(n,k)} \varphi \wedge \psi \to \varphi$; b′) $\vdash_{(n,k)} \varphi \wedge \psi \to \psi$,
c) $\vdash_{(n,k)} (\varphi \to \theta) \to ((\psi \to \theta) \to (\varphi \vee \psi \to \theta))$,
d) $\vdash_{(n,k)} \varphi \to (\psi \to (\varphi \wedge \psi))$,
e) $\vdash_{(n,k)} \varphi \wedge \psi \to \varphi \vee \psi$.

Finally, to prove completeness, it will be necessary to show:

Proposition 3.13. *The following are I^nP^k-theorems:*

a) $\varphi \to \varphi^*$,
b) $\varphi^\circ \to (\neg\varphi \to (\varphi \to \psi))$,
c) $(\varphi^\circ)^\circ$,
d) $\neg(\varphi^*) \to \varphi^\circ$,
e) $\sim\varphi \to \varphi^\circ$,
f) $\varphi^* \to (\neg(\varphi \vee \psi) \to \neg\varphi)$,
g) $\psi^\circ \to \big(\varphi \to (\neg\psi \to \neg(\varphi \to \psi))\big)$,
h) $(\neg(\varphi \to \psi))^*$; $(\neg(\varphi \to \psi))^\circ$,
i) $(\neg(\varphi^*))^\circ$,
j) $\neg(\varphi^*) \to (\varphi \to \psi)$.

Proof: These formal theorems are demonstrated as in Proposition 3.8, applying DT and Proposition 3.5 if it were necessary:

For a): Taking into account Corollary 3.12 a'), it is valid $\vdash_{(n,k)} \varphi \to \varphi \vee \neg\varphi$. That is, $\vdash_{(n,k)} \varphi \to \varphi^*$.

For b):

1) φ° [Hyp.]
2) $\neg\varphi$ [Hyp.]
3) $(\neg\varphi \to \psi)^*$ [Ax_3]

4) $(\neg\varphi \to \psi)^* \to \big[\varphi^\circ \to \big((\neg(\neg\varphi \to \psi) \to \neg\varphi) \to (\varphi \to (\neg\varphi \to \psi))\big)\big]$ [Prop. 3.8 c)]
5) $(\neg(\neg\varphi \to \psi) \to \neg\varphi) \to (\varphi \to (\neg\varphi \to \psi))$ [1), 3), 4), MP]
6) $\neg\varphi \to (\neg(\neg\varphi \to \psi) \to \neg\varphi)$ [Ax_1]
7) $\neg(\neg\varphi \to \psi) \to \neg\varphi$ [2), 6), MP]
8) $\varphi \to (\neg\varphi \to \psi)$ [5), 7), MP]
9) $\neg\varphi \to (\varphi \to \psi)$ [8), Perm.]
10) $\varphi \to \psi$ [2), 9), MP]
Thus, it holds $\varphi^\circ \vdash_{(n,k)} \neg\varphi \to (\varphi \to \psi)$, as was desired.

For c):
1) $(\copyright(\neg\varphi \to \sim\varphi))^\circ$ [Prop. 3.6]
2) $(\sim(\neg\varphi \to \sim\varphi))^\circ$ [1), Ax_{12}]
3) $(\varphi \wedge \neg\varphi)^\circ$ [2), Def. 3.1 (of $\wedge$)]
4) $(\neg(\varphi \wedge \neg\varphi))^\circ$ [3), Ax_{12}]
5) $(\varphi^\circ)^\circ$ [4), Def. 3.1 (of $\circ$)]

For d):
1) $(\varphi^*)^\circ$ [Prop. 3.6]
2) $(\copyright(\neg\varphi \to \sim\varphi))^*$ [Prop. 3.6]
3) $(\varphi \wedge \neg\varphi)^*$ [2), Def. 3.1 (of $\wedge$), Ax_{11}]
4) $(\varphi \wedge \neg\varphi)^* \to \big[(\varphi^*)^\circ \to \big((\varphi \wedge \neg\varphi \to \varphi^*) \to ((\neg(\varphi^*) \to \varphi^\circ))\big)\big]$ [Prop. 3.8 d)]
5) $(\varphi \wedge \neg\varphi \to \varphi \vee \neg\varphi) \to ((\neg(\varphi^*) \to \varphi^\circ)$ [1), 3), 4), MP]
6) $\varphi \wedge \neg\varphi \to \varphi \vee \neg\varphi$ [Cor. 3.12 e)]
7) $\neg(\varphi^*) \to \varphi^\circ$ [5), 6), MP]
So, $\vdash_{(n,k)} \neg(\varphi^*) \to \varphi^\circ$.

For e):
1) $(\varphi \wedge \neg\varphi)^* \to [(\copyright\varphi)^\circ \to ((\varphi \wedge \neg\varphi \to \copyright\varphi) \to (\sim\varphi \to \varphi^\circ))]$ [Prop. 3.8 d), Def. 3.1 (of $\sim$ and $\circ$)]
2) $\varphi \wedge \neg\varphi \to \varphi$ [Cor. 3.12 b')]
3) $\varphi \to \copyright\varphi$ [Prop. 3.8 a)]
4) $\varphi \wedge \neg\varphi \to \copyright\varphi$ [2), 3), Trans.]
5) $(\copyright\varphi)^\circ$ [Prop. 3.6]
6) $(\copyright(\neg\varphi \to \sim\varphi))^*$ [Prop. 3.6]
7) $(\sim(\neg\varphi \to \sim\varphi))^*$ [6), Def. 3.1 (of $\sim$), Ax_{11}]
8) $(\varphi \wedge \neg\varphi)^*$ [7), Def. 3.1 (of $\wedge$)]
9) $\sim\varphi \to \varphi^\circ$ [1), 4), 5), 8), MP]
Therefore, $\vdash_{(n,k)} \sim\varphi \to \varphi^\circ$.

For f):
1) φ^* [Hyp.]
2) $(\varphi \vee \psi)^\circ$ [Ax_4, Def. 3.1 (of $\vee$)]

3) $\varphi^* \to [(\varphi \vee \psi)^\circ \to ((\varphi \to \varphi \vee \psi) \to (\neg(\varphi \vee \psi) \to \neg\varphi)]$ [Prop. 3.8 d)]
4) $(\varphi \to \varphi \vee \psi) \to (\neg(\varphi \vee \psi) \to \neg\varphi)$ [1), 2), 3), MP]
5) $\varphi \to \varphi \vee \psi$ [Cor. 3.12 a)]
6) $\neg(\varphi \vee \psi) \to \neg\varphi$ [4), 5), MP]
Thus, $\varphi^* \vdash_{(n,k)} \neg(\varphi \vee \psi) \to \neg\varphi$.

For g):
1) ψ° [Hyp.]
2) $(\varphi \to \psi)^*$ [Ax_3]
3) $(\varphi \to \psi)^* \to \big[\psi^\circ \to \big(((\varphi \to \psi) \to \psi) \to (\neg\psi \to \neg(\varphi \to \psi))\big)\big]$ [Prop. 3.8 d)]
4) $((\varphi \to \psi) \to \psi) \to (\neg\psi \to \neg(\varphi \to \psi))$ [1), 2), 3), MP]
5) $(\varphi \to \psi) \to (\varphi \to \psi)$ [Rem. 3.3]
6) $\varphi \to ((\varphi \to \psi) \to \psi)$ [5), Perm.]
7) $\varphi \to (\neg\psi \to (\varphi \to \psi))$ [4), 6), Trans.]
So, $\psi^\circ \vdash_{(n,k)} \varphi \to (\neg\psi \to \neg(\varphi \to \psi))$ is obtained.

For h): By Ax_3 and Ax_{11} it holds $\vdash_{(n,k)} (\neg(\varphi \to \psi))^*$. By Ax_4 and Ax_{12} it holds $\vdash_{(n,k)} (\neg(\varphi \to \psi))^\circ$.

For i): It is a particular case of h).

For j):
1) $\neg(\varphi^*)$ [Hyp.]
2) φ [Hyp.]
3) $\varphi \to \varphi^*$ [a)]
4) φ^* [2), 3), MP]
5) $(\varphi^*)^\circ$ [Prop. 3.6]
6) $(\varphi^*)^\circ \to (\neg(\varphi^*) \to (\varphi^* \to \psi))$ [b)]
7) ψ [1), 4), 5), 6), MP]
That is, it holds $\neg(\varphi^*), \varphi \vdash_{(n,k)} \psi$. Then, apply Theorem 3.4. This last result completes the proof. □

4. General soundness and completeness

It is easy to check that the axioms given in Definition 3.2 are I^nP^k-tautologies. So, taking into account that MP preserves I^nP^k-tautologies, it holds:

Theorem 4.1 (Weak soundness). *If* $\vdash_{(n,k)} \varphi$*, then* $\models_{(n,k)} \varphi$.

A theorem of (weak) completeness arises as an adaptation of the well-known Kalmár's proof for classical logic CL, cf. [16]. The main difference with respect to Kalmár's original formulation is the following: meanwhile traditionally every valuation v determines a formal proof with a fixed set of premises, in this version we will deal with *sets of premises of different cardinality*, according to the properties of v. More specifically, we have

Definition 4.2. For every formula $\varphi[\alpha_1, \alpha_2, \dots, \alpha_m] \in L(C)$, for every I^nP^k-valuation v, for every atomic formula α_p, $1 \leq p \leq m$, let Q_p^v be the set associated to α_p and to v, defined by:

- If $v(\alpha_p) = F_r$ (with $1 \leq r \leq n$), then: $Q_p^v = \{\neg(\alpha_p{}^*), \neg((\neg\alpha_p)^*), \dots, \neg((\neg^{r-1}\alpha_p)^*), (\neg^r\alpha_p)^*\}$.
- If $v(\alpha_p) = T_i$ (with $1 \leq i \leq k$), then: $Q_p^v = \{\alpha_p \wedge \neg\alpha_p, \neg\alpha_p \wedge \neg^2\alpha_p, \dots, \neg^{i-1}\alpha_p \wedge \neg^i\alpha_p, (\neg^i\alpha_p)^\circ\}$.
- If $v(\alpha_p) = F_0$, then $Q_p^v = \{\sim\alpha_p, (\alpha_p)^*\}$.
- If $v(\alpha_p) = T_0$, then $Q_p^v = \{©\,\alpha_p, (\alpha_p)^\circ\}$.

In addition, let the set $\Delta_\varphi^v := Q_1^v \cup Q_2^v \cup \dots \cup Q_m^v$.

On the other hand, for every I^nP^k-valuation v indicated above, the *formula* φ^v (determined by φ and v) is defined as follows:

- If $v(\varphi) = F_r$ (with $0 \leq r \leq n$), then $\varphi^v = \neg^{r+1}\varphi$.
- If $v(\varphi) = T_i$ (with $0 \leq i \leq k$), then $\varphi^v = \varphi$.

For the next technical (and essential) result, the following obvious fact will be applied without explicit mention. According to the previous definition, if $\varphi \in L(C)$ and ψ is a subformula of φ then, for every valuation v, $\Delta_\psi^v \subseteq \Delta_\varphi^v$. Bearing this in mind it is possible to demonstrate:

Lemma 4.3. *For every formula $\varphi = \varphi[\alpha_1, \dots, \alpha_m] \in L(C)$, for every I^nP^k-valuation v, it holds that $\Delta_\varphi^v \vdash_{(n,k)} \varphi^v$.*

Proof: By induction on the complexity of φ. The analysis is divided in the following cases:

Case 1: Let $\varphi \in \mathcal{V}$ (without losing generality, $\varphi = \alpha_1$, which implies $\Delta_\varphi^v = Q_1^v$). Then: If $v(\varphi) = F_0$, then $\varphi^v = \neg\varphi$ and $\Delta_\varphi^v = \{\sim\varphi, \varphi^*\}$. So, $\Delta_\varphi^v \vdash_{(n,k)} \varphi^v$ by Proposition 3.8 b).

If $v(\varphi) = F_r$, $1 \leq r \leq n$, then

$$\varphi^v = \neg^{r+1}\varphi \qquad \text{and} \qquad \{(\neg^r\varphi)^*, \neg(\neg^{r-1}\varphi \vee \neg^r\varphi)\} \subseteq \Delta_\varphi^v.$$

Now, by Proposition 3.13 f), $\vdash_{(n,k)} (\neg^r\varphi)^* \to (\neg(\neg^{r-1}\varphi \vee \neg^r\varphi) \to \neg^{r+1}\varphi)$. From all this, $\Delta_\varphi^v \vdash_{(n,k)} \neg^{r+1}\varphi (= \varphi^v)$.

If $v(\varphi) = T_i$, $1 \leq i \leq k$, then $\varphi \wedge \neg\varphi \in \Delta_\varphi^v$. By Corollary 3.12 b'), $\Delta_\varphi^v \vdash_{(n,k)} \varphi$ $(= \varphi^v)$.

If $v(\varphi) = T_0$, then $\Delta_\varphi^v = \{©\,\varphi, (\varphi)^\circ\}$. Thus, $\Delta_\varphi^v \vdash_{(n,k)} ©\,\varphi (= (\varphi \to \varphi) \to \varphi))$. So, since $\vdash_{(n,k)} \varphi \to \varphi$, it holds $\Delta_\varphi^v \vdash_{(n,k)} \varphi (= \varphi^v)$.

The proof of Case 1 is completed.

Case 2: When φ is of the form $\neg\psi$ consider the following subcases:
2.1: Let $v(\psi) = F_0$. By (I.H), $\Delta^v_\psi \vdash_{(n,k)} \psi^v (= \neg\psi = \varphi^v)$. Hence, $\Delta^v_\varphi \vdash_{(n,k)} \varphi^v$.
2.2: Let $v(\psi) = F_r$, with $1 \leq r \leq n$. Note that $v(\varphi) = F_{r-1}$, which implies $\varphi^v = \neg^r\varphi = \neg^{r+1}\psi = \psi^v$. So, $\Delta^v_\varphi \vdash_{(n,k)} \varphi^v$, by (I.H).
2.3: Let $v(\psi) = T_1$. From Definition 2.2, $\psi = \neg^{q-1}\alpha$, with $1 \leq q \leq k$, $\alpha \in \mathcal{V}$, and $v(\alpha) = T_q$. Thus, $\Delta^v_\psi = Q^v_\alpha = \{(\alpha \wedge \neg\alpha), \ldots, (\neg^{q-1}\alpha \wedge \neg^q\alpha), (\neg^q\alpha)^\circ\}$. From this, using Corollary 3.12 b), it holds $\Delta^v_\varphi \vdash_{(n,k)} \neg^q\alpha\ (= \varphi^v)$.
2.4: Let $v(\psi) = T_i$, with $2 \leq i \leq k$. So, $v(\varphi) = T_{i-1}$, $1 \leq i-1 \leq k-1$. In addition, $\psi = \neg^q\alpha$, with $0 \leq q \leq k-i$, $v(\alpha) = T_{i+q}$ and $\alpha \in \mathcal{V}$. From this, $\varphi \wedge \psi = \neg^q\alpha \wedge \neg^{q+1}\alpha \in Q^v_\alpha = \Delta^v_\psi = \Delta^v_\varphi$ (since $q+1 \leq k$). So, applying Corollary 3.12 b), $\Delta^v_\varphi \vdash_{(n,k)} \varphi = \varphi^v$.
2.5: Let $v(\psi) = T_0$. Then, $v(\varphi) = F_0$. To prove that $\Delta^v_\varphi \vdash_{(n,k)} \neg\varphi (= \neg\neg\psi)$ it suffices to demonstrate

$$(\star) \qquad \Delta^v_\psi \vdash_{(n,k)} \psi^\circ.$$

Indeed, if this fact holds, from Ax_{10}, then it would be verified $\Delta^v_\psi \vdash_{(n,k)} \psi \to \neg\neg\psi$. And, since it holds $\Delta^v_\psi \vdash_{(n,k)} \psi$ (by (I.H)), it will be obtained $\Delta^v_\varphi \vdash_{(n,k)} \varphi^v$. Now, to prove $(\star)$, consider the following possibilities:
2.5.1: Assume that ψ is of the form $\neg^q(\theta_1 \to \theta_2)$ (with $0 \leq q$). Applying Ax_4 and (if necessary) Ax_{12}, it holds $\vdash_{(n,k)} \psi^\circ$, and therefore $\Delta^v_\varphi \vdash_{(n,k)} \psi^\circ$.
2.5.2: Assume that ψ is of the form $\neg^q\alpha$, $\alpha \in \mathcal{V}$, $0 \leq q$. In this case, $Q^v_\alpha = \Delta^v_\psi$. Consider here the different possibilities for $v(\alpha)$:
2.5.2.1: Let $v(\alpha) = F_0$. Then, $\Delta^v_\alpha = \{\sim \alpha, \alpha^*\}$. By Proposition 3.13 e), $\Delta^v_\psi \vdash_{(n,k)} \alpha^\circ$. Then, apply Ax_{12} (q times).
2.5.2.2: Let $v(\alpha) = F_r$ (with $1 \leq r \leq n$). Since $\neg(\alpha^*) \in \Delta^v_\alpha$, it holds $\Delta^v_\alpha \vdash_{(n,k)} \alpha^\circ$, because Proposition 3.13 d). From this, $\Delta^v_\psi \vdash_{(n,k)} \psi^\circ$, by Ax_{12} again.
2.5.2.3: Let $v(\alpha) = T_i$ (with $1 \leq i \leq k$). So, $i \leq q$ (in fact: if $i > q$, then $v(\psi) = T_{i-q}$, $i-q \geq 1$, contradicting $v(\psi) = T_0$). Besides, note that $(\neg^i\alpha)^\circ \in \Delta^v_\alpha$, which implies $\Delta^v_\psi \vdash_{(n,k)} (\neg^i\alpha)^\circ$. So, since $i \leq q$, $\Delta_\psi \vdash_{(n,k)} (\psi)^\circ$, again by Ax_{12}.
2.5.2.4: Let $v(\alpha) = T_0$. Then, $\Delta^v_\psi \vdash_{(n,k)} \alpha^\circ$, since $\alpha^\circ \in Q^v_\alpha = \Delta^v_\psi$. Then, applying Ax_{12} one more time, $(\star)$ is valid. The proof of Case 2 is concluded.

Case 3: Assume that φ is of the form $\psi \to \theta$. There exist the following possibilities[4]:

[4]The first three subcases indicate the possibilities for $v(\varphi) = T_0$. The last two cases correspond to $v(\varphi) = F_0$.

3.1: Let $v(\psi) = F_0$ (and so, $\varphi^v = \psi \to \theta$). By (I.H), $\Delta^v_\psi \vdash_{(n,k)} \neg\psi$ ($\star$). In addition, it can be proved that

$$(\star\star) \qquad \Delta^v_\psi \vdash_{(n,k)} \psi^\circ$$

(such a proof runs as follows, according the internal structure of ψ):
3.1.1: Let $\psi = \alpha \in \mathcal{V}$. So, $\Delta^v_\psi \vdash_{(n,k)} \sim \psi$. Applying Proposition 3.13 e), it holds ($\star\star$).
3.1.2: Let $\psi = \neg^q\alpha$, $1 \leq q$, $\alpha \in \mathcal{V}$. Consider the following possibilities for $v(\alpha)$:
3.1.2.1: Let $v(\alpha) = F_0$. Then, by Proposition 3.13 e), $Q^v_\alpha \vdash_{(n,k)} \alpha^\circ$.
3.1.2.2: Let $v(\alpha) = F_r$, $1 \leq r \leq n$. Then, $q \geq r$ (while $q < r$ implies $v(\psi) = v(\neg^q\alpha) = \neg^q(v(\alpha)) = F_{r-q} \neq F_0$, which is absurd). Besides that, $\neg((\neg^{r-1}\alpha)^*) \in Q^v_\alpha$. Therefore $Q^v_\alpha \vdash_{(n,k)} (\neg^{r-1}\alpha)^\circ$, because Proposition 3.13 d).
3.1.2.3: Let $v(\alpha) = T_i$, $1 \leq i \leq k$. So, $q \geq i$ (by similar reasons to 3.1.2.2). In addition, $(\neg^i\alpha)^\circ \in Q^v_\alpha$, and so $Q^v_\alpha \vdash_{(n,k)} (\neg^i\alpha)^\circ$.
3.1.2.4: Let $v(\alpha) = T_0$. Obviously, $Q^v_\alpha \vdash_{(n,k)} (\alpha)^\circ$, from Definition 4.2.

Now note that Ax_{12} can be applied in all the subcases 3.1.2.1–3.1.2.4, in such a way to obtain $\Delta^v_\psi \vdash_{(n,k)} \psi^\circ$, completing the proof ($\star\star$) for Subcase 3.1.2.
3.1.3: Let $\psi = \neg^q(\theta_1 \to \theta_2)$, with $0 \leq q$. By Ax_3, $\vdash_{(n,k)} (\theta_1 \to \theta_2)^\circ$. Now, apply Ax_{12} q times.

So, it was proven ($\star\star$) for all the possibilities of Subcase 3.1. From this, ($\star$) and Proposition 3.13 b), it holds $\Delta^v_\varphi \vdash_{(n,k)} \psi \to \theta$ $(= \varphi^v)$.
3.2: Let $v(\psi) = F_r$, $1 \leq r \leq n$. Again, $v(\varphi) = T_0$ and so $\varphi^v = \psi \to \theta$. Note that, since $v(\psi) = F_r$, $\psi = \neg^q\alpha$, with $q \geq 0$, $\alpha \in \mathcal{V}$. Thus, $v(\alpha) = F_{r+q}$, with $r+q \leq n$, which implies $Q^v_\alpha = \{\neg(\alpha^*), \neg((\neg\alpha)^*), \ldots, \neg((\neg^{r+q-1}\alpha)^*), (\neg^{r+q}\alpha)^*\}$. So, $\neg((\neg^q\alpha)^*) \in Q^v_\alpha$, because $r \geq 1$. Thus, $\Delta^v_\varphi \vdash_{(n,k)} \neg(\psi^*)$. From this and Proposition 3.13 j), $\Delta^v_\varphi \vdash_{(n,k)} \varphi^v$.
3.3: Let $v(\theta) = T_i$, $0 \leq i \leq k$. So, $\varphi^v = \psi \to \theta$ one more time. By (I.H), $\Delta^v_\theta \vdash_{(n,k)} \theta$. Now apply Ax_1.
3.4: Let $v(\psi) = T_i$, $v(\theta) = F_r$, $0 \leq i \leq k$, $1 \leq r \leq n$. Using (I.H), $\Delta^v_\varphi \vdash_{(n,k)} \psi$, which implies $\Delta^v_\varphi \vdash_{(n,k)} (\psi \to \theta) \to \theta$, because $\vdash_{(n,k)} \psi \to ((\psi \to \theta) \to \theta)$. Hence, $\Delta^v_\varphi \vdash_{(n,k)} (\psi \to \theta) \to \theta^*$, by Proposition 3.13 a). Now, considering that $\Delta^v_\varphi \vdash_{(n,k)} (\psi \to \theta)^*$ and $\Delta^v_\varphi \vdash_{(n,k)} (\theta^*)^\circ$ (because Ax_3 and Proposition 3.6, respectively), it is valid that $\Delta^v_\varphi \vdash_{(n,k)} \neg(\theta^*) \to \neg(\psi \to \theta)$, by Proposition 3.8 d). In addition, reasoning as in Subcase 3.2 (with respect to θ), $\Delta^v_\varphi \vdash_{(n,k)} \neg(\theta)^*$. Thus, $\Delta^v_\varphi \vdash_{(n,k)} \neg(\psi \to \theta)(= \varphi^v)$, as it is desired.
3.5: Let $v(\psi) = T_i$, $0 \leq i \leq k$, $v(\theta) = F_0$. Adapting ($\star\star$) of Subcase 3.1 to θ it can be obtained $\Delta^v_\varphi \vdash_{(n,k)} \theta^\circ$. Besides that, by (I.H), $\Delta^v_\varphi \vdash_{(n,k)} \psi$ and $\Delta^v_\varphi \vdash_{(n,k)} \neg\theta$. Considering Proposition 3.13 g) now, it holds $\Delta^v_\varphi \vdash_{(n,k)} \neg(\psi \to \theta) = \varphi^v$.
The analysis of this last subcase finishes the proof. □

Lemma 4.4. *Let $\Delta \cup \{\psi, \theta\}$ be a subset of $L(C)$. If the following $n+k+4$ syntactic consequences are valid:*

1)	$\Delta, \neg(\psi^*), (\neg\psi)^* \vdash_{(n,k)} \theta,$
2)	$\Delta, \neg(\psi^*), \neg((\neg\psi)^*), (\neg^2\psi)^* \vdash_{(n,k)} \theta,$
$\vdots$	$\vdots$
$n-1)$	$\Delta, \neg(\psi^*), \ldots, \neg((\neg^{n-2}\psi)^*), (\neg^{n-1}\psi)^* \vdash_{(n,k)} \theta,$
$n)$	$\Delta, \neg(\psi^*), \ldots, \neg((\neg^{n-1}\psi)^*), (\neg^{n}\psi)^* \vdash_{(n,k))} \theta,$
$n+1)$	$\Delta,\ \psi \wedge \neg\psi,\ (\neg\psi)^\circ \vdash_{(n,k)} \theta,$
$n+2)$	$\Delta,\ \psi \wedge \neg\psi,\ \neg\psi \wedge \neg^2\psi,\ (\neg^2\psi)^\circ \vdash_{(n,k)} \theta,$
$\vdots$	$\vdots$
$n+k-1)$	$\Delta,\ \psi \wedge \neg\psi,\ \ldots,\ \neg^{k-2}\psi \wedge \neg^{k-1}\psi,\ (\neg^{k-1}\psi)^\circ \vdash_{(n,k)} \theta,$
$n+k)$	$\Delta, \psi \wedge \neg\psi,\ \ldots, \neg^{k-1}\psi \wedge \neg^{k}\psi,\ (\neg^{k}\psi)^\circ \vdash_{(n,k)} \theta,$
$n+k+1)$	$\Delta,\ \sim\psi, \psi^* \vdash_{(n,k)} \theta,$
$n+k+2)$	$\Delta, ©\,\psi, \psi^\circ \vdash_{(n,k)} \theta,$
$n+k+3)$	$\vdash_{(n,k)} \theta^*,$
$n+k+4)$	$\vdash_{(n,k)} \theta^\circ.$

Then it is valid that $\Delta \vdash_{(n,k)} \theta$.

Proof: First, by Hypothesis 1) to n) can be obtained

$$(\star) \qquad \Delta, \neg(\psi^*) \vdash_{(n,k)} \theta.$$

Indeed, by $n-1$), $\Delta, \neg(\psi^*), \ldots, \neg((\neg^{n-2}\psi)^*) \vdash_{(n,k)} (\neg^{n-1}\psi)^* \to \theta$. Besides that, it holds that $\vdash_{(n,k)} ((\neg^{n-1}\psi)^*)^*$ (by Proposition 3.6), and $\vdash_{(n,k)} \theta^\circ$ (by Hypothesis $n+k+4$)). Applying all this to Proposition 3.8 d) it is verified:

$$(\dagger) \qquad \Delta, \neg(\psi^*), \ldots, \neg((\neg^{n-2}\psi)^*) \vdash_{(n,k)} \neg\theta \to \neg((\neg^{n-1}\psi)^*).$$

It is also valid $\Delta, \neg(\psi^*), \ldots, \neg((\neg^{n-2}\psi)^*) \vdash_{(n,k))} \neg((\neg^{n-1}\psi)^*) \to \theta$, because of Ax_5, n) and DT. In addition, $\vdash_{(n,k)} (\neg((\neg^{n-1}\psi)^*))^*$, because of Proposition 3.6 and Ax_{11}. Thus, considering Proposition 3.8 d) and Hypothesis $n+k+4$) again, it holds:

$$(\dagger\dagger) \qquad \Delta, \neg(\psi^*), \ldots, \neg((\neg^{n-2}\psi)^*) \vdash_{(n,k)} \neg\theta \to \neg^2((\neg^{n-1}\psi)^*).$$

Hence, from ($\dagger$) and ($\dagger\dagger$) and Corollary 3.12 d):

$$(\diamond) \qquad \Delta, \neg(\psi^*), \ldots, \neg((\neg^{n-2}\psi)^*) \vdash \neg\theta \to [\neg((\neg^{n-1}\psi)^*) \wedge \neg^2((\neg^{n-1}\psi)^*)].$$

On the other hand, it holds $\vdash_{(n,k)} (\neg\theta)^*$, because of Hypothesis $n+k+3$) and Ax_{11}. And, of course, $\vdash_{(n,k)} [\neg((\neg^{n-1}\psi)^*) \wedge \neg^2((\neg^{n-1}\psi)^*)]^\circ$. So, by Proposition 3.8 d), $\Delta, \neg(\psi^*), \ldots, \neg((\neg^{n-2}\psi)^*) \vdash_{(n,k)} \neg[\neg((\neg^{n-1}\psi)^*) \wedge \neg^2((\neg^{n-1}\psi)^*)] \to \neg\neg\theta$. That is, $\Delta, \neg(\psi^*), \ldots, \neg((\neg^{n-2}\psi)^*) \vdash_{(n,k)} (\neg((\neg^{n-1}\psi)^*))^\circ \to \neg\neg\theta$. Thus, from Proposition 3.13 i), $\Delta, \neg(\psi^*), \ldots, \neg((\neg^{n-2}\psi)^*) \vdash_{(n,k)} \neg\neg\theta$. Thus (by Hypothesis $n+k+3$) and Ax_9),

$$(\Diamond\Diamond) \qquad \Delta, \neg(\psi^*), \ldots, \neg((\neg^{n-2}\psi)^*) \vdash_{(n,k)} \theta.$$

The procedure used above to prove $(\Diamond\Diamond)$ can be applied using (in decreasing order) the Hypotheses 1), $\ldots$, $n-1$), proving $(\Diamond)$ (note that the formula $\neg(\psi^*)$ cannot be "suppressed" yet).

From $(\star)$ (and monotonicity), $\Delta, \sim\psi \vdash \neg(\psi^*) \to \theta$. Moreover, from Hypothese $n+k+1$), it holds $\Delta, \sim\psi \vdash \psi^* \to \theta$. From these facts and Corollary 3.12 c), it is valid $\Delta, \sim\psi \vdash_{(n,k)} \psi^* \vee \neg(\psi^*) \to \theta$. Now realizing that $\vdash_{(n,k)} \psi^* \vee \neg(\psi^*)$ (because of Proposition 3.6), it is obtained

$$(\mathbf{I}) \qquad \Delta, \sim\psi \vdash_{(n,k)} \theta.$$

On the other hand, from $n+1$) to $n+k$) it is valid

$$(\star\star) \qquad \Delta, \psi \wedge \neg\psi \vdash_{(n,k)} \theta.$$

The reasoning is as follows: using $n+k$) and Ax_6, it holds: $\Delta, \psi \wedge \neg\psi, \ldots, \neg^{k-2}\psi \wedge \neg^{k-1}\psi \vdash_{(n,k)} \neg^{k-1}\psi \wedge \neg^k\psi \to \theta$. It is also valid $\Delta, \psi \wedge \neg\psi, \ldots, \neg^{k-2}\psi \wedge \neg^{k-1}\psi \vdash_{(n,k)} (\neg^{k-1}\psi)^\circ \to \theta$, because of Hypothese $n+k-1$). Hence, by Corollary 3.12 c) and recalling Definition 3.1, $\Delta, \ldots, \neg^{k-2}\psi \wedge \neg^{k-1}\psi \vdash_{(n,k)} (\neg^{k-1}\psi \wedge \neg^k\psi) \vee \neg(\neg^{k-1}\psi \wedge \neg^k\psi) \to \theta$. That is, $\Delta, \ldots, \neg^{k-2}\psi \wedge \neg^{k-1}\psi \vdash_{(n,k)} (\neg^{k-1}\psi \wedge \neg^k\psi)^* \to \theta$. In addition, it holds $\vdash_{(n,k)} (\neg^{k-1}\psi \wedge \neg^k\psi)^*$, by Definition 3.1, Ax_3 and Ax_{11}. From these two facts, it holds

$$(\Diamond\Diamond\Diamond) \qquad \Delta, \ldots, \neg^{k-2}\psi \wedge \neg^{k-1}\psi \vdash_{(n,k)} \theta.$$

Adapting the reasoning applied in $(\Diamond\Diamond\Diamond)$ to the Hypotheses $n+k-2$), $\ldots$, $n+1$) (in a decreasing order, as before), it is obtained $(\star\star)$, as desired.

From $(\star\star)$ and monotonicity it is valid $\Delta, ©\psi \vdash_{(n,k)} \psi \wedge \neg\psi \to \theta$. So (by Hypothese $n+k+4$), Proposition 3.6, Ax_{11} and Proposition 3.8 d)), $\Delta, ©\psi \vdash_{(n,k)} \neg\theta \to \psi^\circ$. On the other hand, by Hypothese $n+k+2$), it holds $\Delta, ©\psi \vdash_{(n,k)} \psi^\circ \to \theta$. So, $\Delta, ©\psi \vdash_{(n,k)} \neg\theta \to \neg(\psi^\circ)$ (again, by Hypothese $n+k+4$), Proposition 3.6, Ax_{11} and Proposition 3.8 d)). Thus, $\Delta, ©\psi \vdash_{(n,k)} \neg\theta \to (\psi^\circ \wedge \neg(\psi^\circ))$, by Corollary 3.12 d). Therefore, $\Delta, ©\psi \vdash_{(n,k)} \neg(\psi^\circ \wedge \neg(\psi^\circ)) \to \neg\neg\theta$ (because of Hypothese $n+k+3$), Ax_{11} and Proposition 3.8 d)). That is, $\Delta, ©\psi \vdash_{(n,k)} (\psi^\circ)^\circ \to \neg\neg\theta$. Hence, $\Delta, ©\psi \vdash_{(n,k)} \neg\neg\theta$, because of Proposition 3.13 c). Now, taking into account Hypothese $n+k+3$) and Ax_9, it is valid

$$(\mathbf{II}) \qquad \Delta, ©\psi \vdash_{(n,k)} \theta.$$

From (**I**), (**II**) and Corollary 3.12 c), it is verified $\Delta \vdash_{(n,k)} (\copyright\,\psi)^* \to \theta$. Hence, it is valid $\Delta \vdash_{(n,k)} \theta$, by Proposition 3.6. □

Using Lemmas 4.3 and 4.4 it is possible to demonstrate:

Theorem 4.5 (Weak completeness). *$\models_{(n,k)} \varphi$ implies $\vdash_{(n,k)} \varphi$.*

PROOF: Suppose $\models_{(n,k)} \varphi$, with $\varphi = \varphi[\alpha_1, \dots, \alpha_m]$, and consider the set $VAL_\varphi := \{v_t\}_{1 \leq t \leq (n+k+2)^m}$ (the set of all the $I^n P^k$-valuations effectively used to evaluate φ). Define in VAL_φ the equivalence relation $\equiv_1$, as follows: for every $v_{t_1}, v_{t_2} \in VAL_\varphi$, $v_{t_1} \equiv_1 v_{t_2}$ if and only if for every α_p, with $2 \leq p \leq m$, $v_{t_1}(\alpha_p) = v_{t_2}(\alpha_p)$. This relation has $(n+k+2)^{m-1}$ equivalence classes (indicated, in a general way, by $\|v\|$). Besides that, taking into account Definition 4.2, it holds that (given a fixed equivalence class $\|v\|$) $Q_p^{v_{t_1}} = Q_p^{v_{t_2}}$, for every $2 \leq p \leq m$, for every pair $v_{t_1}, v_{t_2} \in \|v\|$. This allows to define the set $\Delta_1^{\|v\|} := Q_2^{v_t} \cup \dots \cup Q_m^{v_t}$, being v_t any element of $\|v\|$. In addition, note that every class $\|v\|$ has exactly $(n+k+2)$ valuations and verifies that, for every $v_{t_1}, v_{t_2} \in \|v\|$, $v_{t_1} \neq v_{t_2}$ implies $v_{t_1}(\alpha_1) \neq v_{t_2}(\alpha_1)$. Finally, note that, since $\models_{(n,k)} \varphi$, for every $v \in VAL_\varphi$, $\varphi^v = \varphi$. All these facts (together with Lemma 4.3) imply that (for every $\|v\|$) the following formal proofs can be built:

1) $\Delta_1^{\|v\|}, \neg(\alpha_1^*), (\neg\alpha_1)^* \vdash_{(n,k)} \varphi$,

1.2) $\Delta_1^{\|v\|}, \neg(\alpha_1^*), \neg((\neg\alpha_1)^*), (\neg^2\alpha_1)^* \vdash_{(n,k)} \varphi$,

$\vdots \qquad \vdots$

1.n) $\Delta_1^{\|v\|}, \neg(\alpha_1^*), \dots, \neg((\neg^{n-1}\alpha_1)^*), (\neg^n\alpha_1)^* \vdash_{(n,k))} \varphi$,

1.$(n+1)$) $\Delta_1^{\|v\|}, \alpha_1 \wedge \neg\alpha_1, (\neg\alpha_1)^\circ \vdash_{(n,k)} \varphi$,

1.$(n+2)$) $\Delta_1^{\|v\|}, \alpha_1 \wedge \neg\alpha_1, \neg\alpha_1 \wedge \neg^2\alpha_1, (\neg^2\alpha_1)^\circ \vdash_{(n,k)} \varphi$,

$\vdots \qquad \vdots$

1.$(n+k)$) $\Delta_1^{\|v\|}, \alpha_1 \wedge \neg\alpha_1, \dots, \neg^{k-1}\alpha_1 \wedge \neg^k\alpha_1, (\neg^k\alpha_1)^\circ \vdash_{(n,k)} \varphi$,

1.$(n+k+1)$) $\Delta_1^{\|v\|}, \sim\alpha_1, \alpha_1^* \vdash_{(n,k)} \varphi$,

1.$(n+k+2)$) $\Delta_1^{\|v\|}, \copyright\,\alpha_1, \alpha_1^\circ \vdash_{(n,k)} \varphi$.

Moreover, by Proposition 3.7, it is valid:

1.$(n+k+3)$) $\Delta_1^{\|v\|} \vdash_{(n,k)} \varphi^*$,

1.$(n+k+4)$) $\Delta_1^{\|v\|} \vdash_{(n,k)} \varphi^\circ$.

All the previous facts allow to apply Lemma 4.4 in such a way that for every $\|v\|$ it holds $\Delta_1^{\|v\|} \vdash_{(n,k)} \varphi$ (there are $(n+k+2)^{m-1}$ formal proofs of this type). That is, it is possible "to eliminate" any reference to formulas of the form α_1^v in every formal proof obtained, by means of an adequate subdivision of the set VAL_φ, and by the application of Lemma 4.4. Note here that this process can be applied one more time, reagrouping the formal proofs already obtained.

So, by a new application of Lemma 4.4 and of Proposition 3.7, any reference to formulas of the form α_2^v can be suppressed. The same procedure can be applied by a finite number of times, until obtaining the following formal proofs:

$m.1)$	$\neg(\alpha_m^*), (\neg\alpha_m)^* \vdash_{(n,k)} \varphi$,
$m.2)$	$\neg(\alpha_m^*), \neg((\neg\alpha_m)^*), (\neg^2\alpha_m)^* \vdash_{(n,k)} \varphi$,
$\vdots$	$\vdots$
$m.n)$	$\neg(\alpha_m^*), \ldots, \neg((\neg^{n-1}\alpha_m)^*), (\neg^n\alpha_m)^* \vdash_{(n,k))} \varphi$,
$m.(n+1))$	$\alpha_m \wedge \neg\alpha_m,\ (\neg\alpha_m)^\circ \vdash_{(n,k)} \varphi$,
$m.(n+2))$	$\alpha_m \wedge \neg\alpha_m,\ \neg\alpha_m \wedge \neg^2\alpha_m,\ (\neg^2\alpha_m)^\circ \vdash_{(n,k)} \varphi$,
$\vdots$	$\vdots$
$m.(n+k))$	$\alpha_m \wedge \neg\alpha_m,\ \ldots, \neg^{k-1}\alpha_m \wedge \neg^k\alpha_m,\ (\neg^k\alpha_m)^\circ \vdash_{(n,k)} \varphi$,
$m.(n+k+1))$	$\sim\alpha_m, \alpha_m^* \vdash_{(n,k)} \varphi$,
$m.(n+k+2))$	$\copyright\,\alpha_m, \alpha_m^\circ \vdash_{(n,k)} \varphi$,
$m.(n+k+3))$	$\vdash_{(n,k)} \varphi^*$,
$m.(n+k+4))$	$\vdash_{(n,k)} \varphi^\circ$.

Applying Lemma 4.4 and Proposition 3.7 for a last time, $\vdash_{(n,k)} \varphi$. □

Note that, in the proof developed above, *all the* $(n+k+2)^m$ *valuations of* VAL_φ are needed to obtain the formal proofs that allow to demonstrate $\vdash_{(n,k)} \varphi$.

Theorems 4.1 and 4.5 prove *weak adequacy*: $\models_{(n,k)} \varphi$ if and only if $\vdash_{(n,k)} \varphi$. This result can be improved:

Theorem 4.6 (Strong adequacy)**.** *For every* $\Gamma \cup \{\varphi\} \subseteq L(C)$, $\Gamma \models_{(n,k)} \varphi$ *if and only if* $\Gamma \vdash_{(n,k)} \varphi$.

PROOF: By Proposition 2.11, $\models_{(n,k)}$ verifies semantics deduction theorem and is finitary. Moreover, by the definition of formal proof used in this paper, $\vdash_{(n,k)}$ is finitary, and (by Theorem 3.4) it verifies syntactic deduction theorem, as it was already mentioned. From all these facts, and taking into account that both $\models_{(n,k)}$ and $\vdash_{(n,k)}$ are monotonic, strong adequacy is demonstrated. □

5. Concluding remarks

Despite its interest as a general result (for a countable, nonlinearly ordered family of logics), the adequate axiomatics shown here can be applied in different ways. First of all, a natural problem to be solved is the *independence* of the axiomatics presented here and it is part of a future work.

On the other hand, another of the possible uses of this axiomatics is the study of *algebraizability* of the I^nP^k-logics. It is worth commenting here that I^1P^0 is algebraizable, see [21], as in the case of I^0P^1 (this fact was already indicated).

Moreover, in [9] it was demonstrated that *all the logics of* $\mathbb{I}^n\mathbb{P}^k$ are algebraizable. So, the properties of the class of algebras associated to each I^nP^k-logic deserve to be investigated. By the way, the class of algebras associated to I^0P^1 was already studied in [14] and in [18]. In both works, the axiomatics obtained for this logic are very useful for the study of the so-called *class of* P^1*-algebras*. This is because there is a connection between the axiomatics of an algebraizable logic and its equivalent algebraic semantics, cf. [1]. As a generalization of this fact, the axiomatics shown here would allow to study the different classes of (say) I^nP^k*-algebras* in a more efficient way.

In addition, note this fact about the complexity of the formulas: given a fixed logic I^nP^k, every formula $\varphi \in L(C)$ with complexity $\mathrm{Comp}(\varphi) \geq \max\{n, k\}$ behaves "in a classical way" (this fact is related to the inclusion of Ax_5 and Ax_6 in the axiomatics presented in this paper). This would suggest to define a special kind of logics: the family $\mathbb{SC}$ of "stationary classically logics". Obviously, $\mathbb{I}^n\mathbb{P}^k$ would be a particular subclass of $\mathbb{SC}$. The study of the latter class deserves special attention.

Finally, it should be noted that Kalmár's technique for the proof of completeness is studied by its own right, independently from the logic to be treated, see [11], [12], [15], [17] for instance, specially by its constructive character. So, the results and modifications here presented can contribute to this abstract theoretic study in a future research.

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