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L-limited-like properties on Banach spaces

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Abstract. We study weakly precompact sets and operators. We show that an operator is weakly precompact if and only if its adjoint is pseudo weakly compact. We study Banach spaces with the p-L-limited* and the p-(SR*) properties and characterize these classes of Banach spaces in terms of p-L-limited* and p-Right* subsets. The p-L-limited* property is studied in some spaces of operators.

Keywords: p-Right* set; Right* set; DP p-convergent operator; weakly precompact operator; limited p-convergent operator

Classification: 46B20, 46B25, 46B28

1. Introduction

The concepts of sequentially Right (SR) property and L-limited property on Banach spaces were introduced in [30], [32]. Right sets and L-limited sets in dual Banach spaces were used to give characterizations of Banach spaces with the (SR) property and the L-limited property, see [26], [32]. The p-Right subsets and p-L-limited subsets of dual Banach spaces, and Banach spaces with the p-sequentially Right property and the p-L-limited property were studied in [21].

Right* sets and Banach spaces with the sequentially Right* (SR*) property were studied in [9], [18]. The p-Right* sets and Banach spaces with the p-sequentially Right* (p-(SR*)) property were introduced in [1]. The concepts of p-L-limited* (or R_p^*) sets and Banach spaces with the p-L-limited* (or SR_p^*) property were introduced in [10]. In this paper we study p-L-limited* and p-Right* sets, and Banach spaces with the p-Right* and p-L-limited* properties.

We obtain some characterizations of Banach spaces with the property that their p-L-limited* and p-Right* subsets are weakly precompact, relatively weakly compact, and relative norm compact. We show that an operator $T\colon Y \to X$ is weakly precompact if and only if T^* is pseudo weakly compact if and only if T^* is DP p-convergent, $2 \le p < \infty$. We show that a subset of a Banach space X is weakly precompact if and only if it is a Right* set if and only if it is a p-Right* set, $2 \le p < \infty$. We also study whether some spaces of operators have the p-L-limited* property.

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2. Definitions and notation

Throughout this paper, X and Y will denote Banach spaces. The unit ball of X will be denoted by B_X, the identity map on X will be denoted by i_X, and X^* will denote the continuous linear dual of X. The space X embeds in Y (in symbols $X \hookrightarrow Y$) if X is isomorphic to a closed subspace of Y. An operator $T\colon X \to Y$ will be a continuous and linear function. The space of all operators from X to Y will be denoted by $L(X,Y)$.

A subset S of a Banach space X is said to be *weakly precompact* (or *weakly conditionally compact*) provided that every sequence from S has a weakly Cauchy subsequence. An operator $T\colon X \to Y$ is called *weakly precompact* if $T(B_X)$ is weakly precompact. A Banach space X is called *weakly sequentially complete* if every weakly Cauchy sequence in X is weakly convergent. A Banach space X has the *Grothendieck property* if w^*-convergent sequences in X^* are weakly convergent.

An operator $T\colon X \to Y$ is called *completely continuous* (or *Dunford–Pettis*) if T maps weakly convergent sequences to norm convergent sequences.

A Banach space X has the *Dunford–Pettis property* (DPP) if for any Banach space Y, every weakly compact operator $T\colon X \to Y$ is completely continuous. If X is a $C(K)$-space or an L_1-space, then X has the DPP. The reader can check [11] for results related to the DPP.

A subset A of a Banach space X is called a *Dunford–Pettis* (or DP) (*limited*, respectively) subset of X if each weakly null (w^*-null, respectively) sequence (x_n^*) in X^* tends to 0 uniformly on A; i.e.

$$\sup_{x\in A} |x_n^*(x)| \to 0.$$

An operator $T\colon X \to Y$ is called *unconditionally convergent* if it takes weakly unconditionally convergent (wuc) series in X to unconditionally convergent series in Y.

A bounded subset A of X^* (or of X) is called a V-subset of X^* (a V^*-subset of X, respectively) provided that

$$\sup_{x^*\in A} |x^*(x_n)| \to 0 \qquad \left(\sup_{x\in A} |x_n^*(x)| \to 0, \quad \text{respectively}\right)$$

for each wuc series $\sum x_n$ in X $\left(\sum x_n^* \text{ in } X^*, \text{ respectively}\right)$.

A Banach space X has *property* (V) if every V-subset of X^* is relatively weakly compact, see [29]. Reflexive Banach spaces and $C(K)$ spaces have property (V), see [29, Theorem 1, Proposition 7].

A Banach space X has *property* (V^*) if every V^*-subset of X is relatively weakly compact, see [29]. Reflexive Banach spaces and L_1-spaces have property (V^*), see [29].

For $1 \leq p < \infty$, p^* denotes the conjugate of p. If $p = 1$, c_0 plays the role of l_{p^*}. The unit vector basis of l_p will be denoted by (e_n).

Let $1 \leq p < \infty$. A sequence (x_n) in X is called *weakly p-summable* if $(\langle x^*, x_n\rangle) \in l_p$ for each $x^* \in X^*$, see [12, page 32]. Let $l_p^w(X)$ denote the set of all weakly p-summable sequences in X. Let $c_0^w(X)$ be the space of weakly null sequences in X. If $p = \infty$, then we consider $c_0^w(X)$ instead of $l_\infty^w(X)$.

If $p < q$, then $l_p^w(X) \subseteq l_q^w(X)$. The weakly 1-summable sequences are precisely the wuc series and the weakly ∞-summable sequences are precisely weakly null sequences.

Let $1 \leq p \leq \infty$. An operator $T\colon X \to Y$ is called *p-convergent* if T maps weakly p-summable sequences into norm null sequences, see [7]. The set of all p-convergent operators is denoted by $C_p(X, Y)$.

Let $1 \leq p \leq \infty$. A sequence (x_n) in X is called *weakly p-convergent* to $x \in X$ if the sequence $(x_n - x)$ is weakly p-summable, see [7]. A bounded subset K of X is *relatively weakly p-compact* if every sequence in K has a weakly p-convergent subsequence with limit in X.

An operator $T\colon X \to Y$ is *weakly p-compact* if $T(B_X)$ is relatively weakly p-compact, see [7]. The set of weakly p-compact operators $T\colon X \to Y$ will be denoted by $W_p(X, Y)$.

A Banach space $X \in W_p$ (or $X \in C_p$) if $i_X \in W_p(X, X)$ ($i_X \in C_p(X, X)$, respectively), see [7].

3. The p-L-limited* and p-SR* properties

In this section we study Banach spaces with the L-limited*, p-L-limited*, Right*, and p-Right* properties, $1 \leq p < \infty$.

We also study the p-L-limited* property in the space $K_{w^*}(X^*, Y)$ of w^*-w compact operators from X^* to Y.

For the definition of Right topology on a Banach space X, see [30]. A sequence (x_n) in a Banach space X is Right null if and only if it is DP weakly null, see [19, Proposition 1].

An operator $T\colon X \to Y$ is *pseudo weakly compact* (pwc) or *Dunford–Pettis completely continuous* (DPcc) (or *limited completely continuous* (lcc)) if it takes DP (limited, respectively) weakly null sequences in X into norm null sequences in Y.

A subset K of X^* is called a *Right set*, see [26], (or *L-limited set*, see [32]) if each Right null (limited weakly null, respectively) sequence (x_n) in X tends to 0 uniformly on K.

A Banach space X is *sequentially Right* (SR) (or X *has property* (SR)), see [30], if for any Banach space Y, every pseudo weakly compact operator $T\colon X \to Y$ is weakly compact.

A Banach space X has the *L-limited property*, see [32], if every L-limited subset of X^* is relatively weakly compact.

Let $1 \le p < \infty$. A subset A of a dual space X^* is called a *p-Right set* (or *p-L-limited set*, see [21], or *L_p-limited set*, see [10]) if for every DP (limited, respectively) weakly p-summable sequence (x_n) in X, $\sup_{x^* \in A} |x^*(x_n)| \to 0$.

Let $1 \le p < \infty$. A Banach space X has the p-(SR) (or *p-L-limited*) *property* if every p-Right (p-L-limited, respectively) subset of X^* is relatively weakly compact, see [21].

The case $p = \infty$ can be included in the previous definitions, if we consider the ∞-Right (or ∞-L-limited) subsets of X^* to be precisely the Right (L-limited, respectively) sets and the ∞-(SR) (∞-L-limited, respectively) property to be the (SR) (L-limited, respectively) property.

Let $1 \le p < \infty$. An operator $T\colon X \to Y$ is called *DP p-convergent*, see [21], (or *limited p-convergent*, see [16]), if it takes DP (limited, respectively) weakly p-summable sequences to norm null sequences.

A subset K of X is called a *Right* set* (or R^*-set), see [9], (*L-limited** set, respectively) if each DP (limited, respectively) weakly null sequence (x_n^*) in X^* tends to 0 uniformly on K.

A Banach space X has the (SR^*), see [9], (or *L-limited**) *property* if every Right* (L-limited*, respectively) subset of X is relatively weakly compact.

Let $1 \le p \le \infty$. A subset A of a Banach space X is called a *p-Right* set* (or p-R^*-set), see [1], (*p-L-limited* set* or R_p^* *set*, respectively, see [10]) if for every DP (limited, respectively) weakly p-summable sequence (x_n^*) in X^* holds $\sup_{x \in A} |x_n^*(x)| \to 0$.

Let $1 \le p \le \infty$. A Banach space X has *property* p-(SR*), see [1], (or *p-L-limited** or SR_p^*, see [10]) if every p-Right* (*p-L-limited**, respectively) subset of X is relatively weakly compact.

We prefer the R_p^* sets to be called p-L-limited* sets, so that there is no confusion between these sets and the p-R^* sets. Similarly, we prefer the SR_p^* property to be called the p-L-limited* property so that there is no confusion between this property and the p-(SR*) property.

The ∞-L-limited* (or ∞-Right*) sets are precisely the L-limited* (Right*, respectively) sets, and the ∞-L-limited* (∞-Right*, respectively) property is precisely the L-limited* (Right*, respectively) property.

If $p < q$, then a q-L-limited* set is a p-L-limited* set, since $l_p^w(X^*) \subseteq l_q^w(X^*)$. If X has the p-L-limited* property, then it has the q-L-limited* property for $p < q$.

Every p-L-limited* (or p-Right*) set is bounded. Indeed, if A is an unbounded p-L-limited* (p-Right*, respectively) subset of X, then there are sequences (x_n) in A and (x_n^*) in B_{X^*} such that $|x_n^*(x_n)| > n^2$ for all n. Let $u_n^* = x_n^*/n^2$. Then $\sum \|u_n^*\|^p = \sum \|x_n^*\|^p/n^{2p} < \infty$. Hence (u_n^*) is a limited (DP, respectively) weakly p-summable sequence in X^* and $|u_n^*(x_n)| > 1$ for all n. This is a contradiction.

Let $1 \leq p \leq \infty$. A bounded subset A of X is called a p-(V^*) set, see [28], (or *weakly-p-Dunford–Pettis* set, see [20]) if $\sup_{x\in A} |x_n^*(x)| \to 0$ for every weakly p-summable (weakly null for $p = \infty$) sequence (x_n^*) in X^*.

A Banach space X has property p-(V^*), see [28], (or RDP_p^* [20]) if every p-(V^*) subset of X is relatively weakly compact.

The 1-(V^*) subsets of X are precisely the V^*-sets and the ∞-(V^*) subsets of X are precisely the DP sets. Property 1-(V^*) is precisely property (V^*) and property ∞-(V^*) is precisely the RDP* property, see [3] for this definition.

If $\sum x_n$ is wuc in X, then (x_n) is DP. Indeed, let $S\colon c_0 \to X$ be an operator such that $S(e_n) = x_n$ for all n, see [12, Proposition 2.2, page 36]. Since c_0 has the DPP and (e_n) is weakly null, (e_n) is DP, see [11, Theorem 1]. Then (x_n) is DP. Therefore the 1-Right subsets of X^* are precisely the V-sets and the 1-Right* subsets of X are the V^*-sets. Thus property 1-(SR) is property (V) and property 1-(SR*) is property (V^*).

It follows from definitions that for $1 \leq p \leq \infty$, every p-(V^*) set in X is a p-Right* set and every p-Right* set is a p-L-limited* set (since any limited weakly p-summable sequence is also DP weakly p-summable).

If X has property (V^*), then X has property (SR*). If X has property (SR*), then X has property RDP*, see [9], (see [9] for this definition). If X has property (V), then it has property (SR), see [26].

Proposition 3.1. *Let $1 \leq p \leq \infty$ and let X be a Banach space.*

- (i) *If K is a weakly precompact subset of X, then K is a p-Right* set and a p-L-limited* set.*
- (ii) *If X has property p-L-limited*, then X has property p-(SR*).*
- (iii) *If X has property p-(SR*), then X has property (SR*). Consequently, X is weakly sequentially complete.*
- (iv) *If X has property (V), then X has property p-(SR). If X has property (V^*), then X has property p-(SR*).*

(v) *If X has property p-(SR^*), then X has property p-(V^*).*

(vi) *If X has property p-(SR^*) and X^* has the Grothendieck property, then X has property p-L-limited*.*

(vii) *If X^* has property (V), then X has property p-L-limited*.*

Proof: (i) Let K be a weakly precompact subset of X. Then K is a Right* set, see [18, Theorem 17], and hence K is an L-limited* set. Further, K is a p-Right* set, and thus a p-L-limited* set.

(ii) Any p-Right* subset of X is a p-L-limited* set, thus relatively weakly compact. Hence X has property p-(SR*).

(iii) Any Right* subset of X is a p-Right* set, thus relatively weakly compact. Therefore X has property (SR*) and X is weakly sequentially complete, see [18, Corollary 18].

(iv) If X has property (V), then any p-Right subset of X^* is a 1-Right set (V-set), and thus relatively weakly compact. If X has property (V^*), any p-Right* subset of X is a 1-Right* set (V^*-set), and thus relatively weakly compact.

(vi) Since X^* has the Grothendieck property, any DP set in X^* is limited. Then every DP weakly p-summable sequence in X^* is limited weakly p-summable. Therefore any p-L-limited* set in X is a p-Right* set, and thus relatively weakly compact.

(vii) Suppose X^* has property (V). Then X has property (V^*), see [29], and X has property p-(SR*) by (iv). Since X^* has property (V), X^* has the Grothendieck property, see [11, page 40], [24, Corollary 32 (ii)]. Then X has property p-L-limited* by (vi). □

Example 1. (i) Since l_1 has the Schur property, B_{c_0} is a p-(V^*) (hence a p-Right* and a p-limited*) set, and it is not relatively weakly compact. Then c_0 does not have properties p-(V^*), p-(SR*) and p-L-limited*.

(ii) Let $X = l_1$ or $X = L_1$. Since X^* has property (V), see [29], X has property p-L-limited* by Proposition 3.1 (vii), $1 \le p \le \infty$.

An operator $T\colon X \to Y$ is called *limited*, see [5], (*weakly limited*, respectively), if $T(B_X)$ is limited (DP, respectively).

We note that $T\colon X \to Y$ is limited (weakly limited, respectively) if and only if T^* is w^*-norm sequentially continuous (completely continuous, respectively).

If $p = \infty$, the limited p-convergent (DP p-convergent, respectively) operators are precisely the limited completely continuous (pseudo weakly compact operators, respectively).

Theorem 3.2. *Let $1 \le p \le \infty$. Let $T\colon Y \to X$ be an operator. The following statements are equivalent:*

(i) *$T(B_Y)$ is a p-L-limited* (p-Right*, respectively) set.*

(ii) *$T^*: X^* \to Y^*$ is limited p-convergent (DP p-convergent, respectively).*

If $1 < p < \infty$, then we have one more equivalent condition:

(iii) *If $S: l_{p^*} \to X^*$ is a limited (weakly limited, respectively) operator, then $T^*S: l_{p^*} \to Y^*$ is compact.*

Proof: (i) $\Rightarrow$ (ii) We only prove the result for p-L-limited* sets. The other case is similar. Suppose $T(B_Y)$ is a p-L-limited* set. Then for every limited weakly p-summable sequence (limited weakly null if $p = \infty$) (x_n^*) in X^*,

$$\|T^*(x_n^*)\| = \sup_{y \in B_Y} |\langle T^*(x_n^*), y\rangle| = \sup_{y \in B_Y} |\langle T(y), x_n^*\rangle| \to 0.$$

Therefore $T^*: X^* \to Y^*$ is limited p-convergent (limited completely continuous if $p = \infty$).

Tracing back, we can prove (ii) $\Rightarrow$ (i).

(ii) $\Leftrightarrow$ (iii) If $1 < p < \infty$, the result follows from [21, Theorem 3.24]. □

A Banach space X has the *Gelfand–Phillips* (GP) *property* (or is a *Gelfand–Phillips space*) if every limited subset of X is relatively compact.

Separable spaces and Schur spaces have the GP property, see [5].

Let $1 \leq p < \infty$. A Banach space X has the *p-Gelfand–Phillips* (p-GP) *property* (or is a *p-Gelfand-Phillips space*) if every limited weakly p-summable sequence in X is norm null, see [16].

The case $p = \infty$ can be included in the previous definition, if we consider the ∞-GP property to be the GP property. If X has the GP property, then X has the p-GP property for any $1 \leq p < \infty$.

A Banach space X has the *Dunford–Pettis relatively compact property* (DPrcP) if every DP subset of X is relatively compact, see [14].

Schur spaces have the DPrcP. The space X does not contain l_1 if and only if X^* has the DPrcP, see [14].

Let $1 \leq p < \infty$. A Banach space X has the *p-Dunford–Pettis relatively compact property* (p-DPrcP) if every DP weakly p-summable sequence in X is norm null, see [21].

If X has the DPrcP, then X has the p-DPrcP for any $1 \leq p < \infty$.

We note that every reflexive space has the p-L-limited* (or the p-(SR*)) property, since every p-L-limited* (p-Right*, respectively) subset of X is bounded (as seen before), and thus relatively weakly compact.

Corollary 3.3.

(i) ([10, Theorem 3.4], [1, Lemma 3.4 (iv)]) *Let* $1 \leq p \leq \infty$. *The Banach space* Y^* *has the p-Gelfand–Phillips (or the p-DPrcP) property if and only if* B_Y *is a p-L-limited* (p-Right*, respectively) set in* Y.

(ii) *If* X^* *has the p-GP property (or the p-DPrcP), then* X *has the p-L-limited* (the p-(SR*), respectively) property if and only if* X *is reflexive.*

(iii) *If* X *is a nonreflexive Banach space and* X *has the p-(SR*) (the p-L-limited*, respectively) property, then* X *contains a copy of* l_1.

Proof: (i) By Theorem 3.2 applied to the identity map i_Y on Y, B_Y is a p-L-limited* set if and only if $i_Y^* = i_{Y^*}$ is limited p-convergent if and only if Y^* has the p-GP property.

(ii) Suppose X^* has the p-GP property and X has the p-L-limited* property. By (i), B_X is a p-L-limited* subset of X, thus relatively weakly compact.

(iii) Suppose X is a nonreflexive Banach space with the p-(SR*) property. If X does not contain a copy of l_1, then B_X is weakly precompact by Rosenthal's l_1 theorem. By Proposition 3.1, X is weakly sequentially complete. Then B_X is relatively weakly compact. This contradiction concludes the proof. □

Let (e_n^*) be the unit basis of l_1. The following result contains [10, Theorem 3.9].

Theorem 3.4. *Let* X *be a Banach space and let* $1 \leq p \leq \infty$. *The following statements in each of the following collections are equivalent:*

1. (i) *If* $T\colon Y \to X$ *is an operator so that* $T^*\colon X^* \to Y^*$ *is limited p-convergent, then* T *is weakly precompact (weakly compact; compact, respectively).*
   (ii) *Same as* (i) *with* $Y = l_1$.
   (iii) *Every p-L-limited* subset of* X *is weakly precompact (relatively weakly compact; relatively compact, respectively).*
2. (i) *If* $T\colon Y \to X$ *is an operator so that* $T^*\colon X^* \to Y^*$ *is DP p-convergent, then* T *is weakly precompact (weakly compact; compact, respectively).*
   (ii) *Same as* (i) *with* $Y = l_1$.
   (iii) *Every p-Right* subset of* X *is weakly precompact (relatively weakly compact; relatively compact, respectively).*

Proof: We will show 1. (i) ⇒ (ii) ⇒ (iii) ⇒ (i) in the weakly precompact case, if $1 \leq p < \infty$. The arguments for the remaining implications and for $p = \infty$ follow the same pattern.

1. (i) ⇒ (ii) is obvious.

1. (ii) ⇒ (iii) Suppose that A is a p-L-limited* subset of X, and let (x_n) be a sequence in A. We recall that A is bounded (as seen before). Define $T\colon l_1 \to X$

by $T(b) = \sum b_i x_i$. Note that $T^*\colon X^* \to l_\infty$, $T^*(x^*) = (x^*(x_i))$. Let (x_n^*) be a limited weakly p-summable sequence in X^*. Since A is a p-L-limited* set, it follows that

$$\|T^*(x_n^*)\| = \sup_i |x_n^*(x_i)| \to 0, \qquad \text{as } \ n \to \infty,$$

and $T^*\colon X^* \to l_\infty$ is limited p-convergent. Then T is weakly precompact by hypothesis. Thus $(x_n) = (T(e_n^*))$ has a weakly Cauchy subsequence.

1. (iii) $\Rightarrow$ (i) Let $T\colon Y \to X$ be an operator so that $T^*\colon X^* \to Y^*$ is limited p-convergent. Then $T(B_Y)$ is a p-L-limited* subset of X (by Theorem 3.2), and thus weakly precompact by assumptions. □

Corollary 3.5. *Let X be a Banach space. The following statements in each of the following collections are equivalent:*

1. (i) *If $T\colon Y \to X$ is an operator so that $T^*\colon X^* \to Y^*$ is limited completely continuous, then T is weakly precompact (weakly compact; compact, respectively).*
   (ii) *Same as* (i) *with $Y = l_1$.*
   (iii) *Every L-limited* subset of X is weakly precompact (relatively weakly compact; relatively compact, respectively).*
2. ([18, Theorem 14])
   (i) *If $T\colon Y \to X$ is an operator so that $T^*\colon X^* \to Y^*$ is pseudo weakly compact, then T is weakly precompact (weakly compact; compact, respectively).*
   (ii) *Same as* (i) *with $Y = l_1$.*
   (iii) *Every Right* subset of X is weakly precompact (relatively weakly compact; relatively compact, respectively).*

Corollary 3.6. *Let $1 \le p \le \infty$, X be a Banach space, and Y be a closed linear subspace of X. If X has the p-L-limited* (p-(SR*), respectively) property, then Y has the p-L-limited* (p-(SR*), respectively) property.*

Proof: Let $T\colon E \to Y$ be an operator such that $T^*\colon Y^* \to E^*$ is limited p-convergent. Let $S\colon Y \to X$ be the natural embedding of Y into X. Since $T^*S^*\colon X^* \to E^*$ is limited p-convergent, $ST\colon E \to X$ is weakly compact by Theorem 3.4. If (a_n) is a sequence in B_E, then $(ST(a_n)) = (T(a_n))$ has a weakly convergent subsequence in X, and thus in Y. Therefore T is weakly compact, and Y has the p-L-limited* property by Theorem 3.4. □

Theorem 3.7.

(i) *Let $2 \le p < \infty$. If $T\colon Y \to X$ is an operator such that $JT\colon Y \to l_p$ is compact for all 2-summing operators $J\colon X \to l_p$, then T is weakly precompact.*

(ii) *Let* $2 \le p < \infty$. *If* $T\colon Y \to X$ *is an operator such that* $T^*\colon X^* \to Y^*$ *is DP* p*-convergent, then* T *is weakly precompact.*

(iii) *If* $T\colon Y \to X$ *is an operator such that* $T^*\colon X^* \to Y^*$ *is pseudo weakly compact, then* T *is weakly precompact.*

Proof: (i) The proof is the same as the proof of [20, Theorem 16 (i)].

(ii) Suppose $T\colon Y \to X$ is an operator such that $T^*\colon X^* \to Y^*$ is DP p-convergent, and let $J\colon X \to l_p$ be a 2-summing operator. Then $J^{**}\colon X^{**} \to l_p$ is 2-summing [12, Proposition 2.19, page 50]. Therefore J^{**} is weakly compact and completely continuous [12, Theorem 2.17, page 50]. Since J^{**} is completely continuous, $J^*(B_{l_{p^*}})$ is a DP set in X^*, and $T^*J^*\colon l_{p^*} \to Y^*$ is compact by Theorem 3.2. Then $JT\colon Y \to l_p$ is compact. Apply (i).

(iii) Let $2 \le p < \infty$ and let $T\colon Y \to X$ be an operator such that $T^*\colon X^* \to Y^*$ is pseudo weakly compact. Then T^* is DP p-convergent, and thus T is weakly precompact by (ii). □

In [20, Theorem 16 (ii)] it is shown that if $2 < p < \infty$ and $T\colon Y \to X$ is an operator such that $T^*\colon X^* \to Y^*$ is p-convergent, then T is weakly precompact. We note that if T^* is p-convergent, then T^* is DP p-convergent. Thus Theorem 3.7 (ii) improves [20, Theorem 16 (ii)].

Corollary 3.8.

(i) *Let* $2 \le p < \infty$ *and let* $T\colon Y \to X$ *be an operator. Then* T *is weakly precompact if and only if* T^* *is pseudo weakly compact if and only if* T^* *is DP* p*-convergent.*

(ii) *Let* $2 \le p < \infty$. *Let* X *be a Banach space and let* A *be a subset of* X. *Then* A *is weakly precompact if and only if* A *is* p*-Right** *set if and only if* A *is a Right** *set.*

(iii) *Let* $2 \le p < \infty$. *The Banach space* X *is weakly sequentially complete if and only if* X *has property* p-(SR^*) *if and only if* X *has property* (SR^*).

(iv) *Let* $2 \le p < \infty$. *If* X *has the Schur property, then* p*-Right** *and Right** *subsets of* X *are relatively compact.*

Proof: (i) If T^* is DP p-convergent (pseudo weakly compact, respectively), then T is weakly precompact by Theorem 3.7. If T is weakly precompact, then T^* is pseudo weakly compact by [18, Corollary 7], and thus DP p-convergent.

(ii) Let $T\colon Y \to X$ be an operator such that $T^*\colon X^* \to Y^*$ is DP p-convergent (pseudo weakly compact, respectively). Then T is weakly precompact by Theorem 3.7. It follows that every p-Right* (Right*, respectively) subset of X is weakly precompact by Theorem 3.4 (Corollary 3.5, respectively) part 2.

If A is weakly precompact, then A is a p-Right* (Right*, respectively) set by Proposition 3.1 ([18, Theorem 17], respectively).

(iii) Suppose X is weakly sequentially complete. If A is a p-Right* (Right*, respectively) subset of X, then A is weakly precompact, and thus relatively weakly compact. If X has property p-(SR*) ((SR*), respectively), then X is weakly sequentially complete by Proposition 3.1 ([18, Corollary 18], respectively). □

Since any DP subset of X is a Right* set, we obtain the well known result about the weak precompactness of DP sets (by Corollary 3.8).

Corollary 3.9.

(i) ([31, page 377], [2]) *Every Dunford–Pettis subset of X is weakly precompact.*

(ii) ([20, Corollary 17]) *Let $2 \leq p < \infty$. Every p-(V^*) subset of X is weakly precompact.*

A subset A of X is *p-limited*, $1 \leq p < \infty$, see [27], if for every weak*(weak) p-summable sequence (x_n^*) in X^*, there exists $(\alpha_n) \in l_p$ such that $|x_n^*(x)| \leq \alpha_n$ for all $x \in A$ and $n \in \mathbb{N}$.

A sequence (x_n^*) in X^* is weak* p-summable sequence if and only if it is weakly p-summable, see [16, page 3].

Let $1 \leq p < \infty$. A subset A of X^* is a *p-L-set* in X^*, see [22], if for every weakly p-summable sequence (x_n) in X, there exists $(\alpha_n) \in l_p$ such that $|x^*(x_n)| \leq \alpha_n$ for all $x^* \in A$ and $n \in \mathbb{N}$.

A subset A of X^* is a p-L-set in X^* if and only if it is a p-limited set in X^*, see [22, Corollary 3]. We recall that an operator $T\colon X \to Y$ is completely continuous if and only if T maps weakly Cauchy sequences to norm convergent sequences.

Theorem 3.10. *An operator $T\colon Y \to X$ is weakly precompact if and only if $T^*(A)$ is relatively compact whenever A is a 2-limited set in X^*.*

Proof: Suppose that $T\colon Y \to X$ is weakly precompact, and let A be a 2-limited set in X^*. Then A is a 2-L-set in X^*. Let (x_i^*) be a sequence in A and let $J\colon X \to l_\infty$ be defined by $J(x) = (x_i^*(x))$, $x \in X$. Note that $J^*(e_n^*) = x_n^*$ for each $n \in \mathbb{N}$. If (x_n) is weakly 2-summable in X, then

$$(\|J(x_n)\|)_n = \Big(\sup_i |x_i^*(x_n)|\Big)_n \in l_2,$$

since A is a 2-L-set. Therefore J is 2-summing, and thus weakly compact and completely continuous [12, Theorem 2.17, page 50]. Since $T(B_Y)$ is weakly precompact and J is completely continuous, $JT(B_Y)$ is relatively compact, and so JT is compact. Hence $(T^*J^*(e_n^*)) = (T^*(x_n^*))$ is relatively compact.

Conversely, suppose that $T^*(A)$ is relatively compact for any 2-limited set A in X^*. Let $J\colon X \to l_p$ be a 2-summing operator, $2 \le p < \infty$. Then $J^*(B_{l_{p^*}})$ is a 2-limited set in X^*, see [22, Theorem 2]. Therefore $T^*J^*(B_{l_{p^*}})$ is relatively compact and T^*J^* is compact. Thus JT is compact and T is weakly precompact by Theorem 3.7. □

Proposition 3.11. *Let $1 \le p \le \infty$ and let X be a Banach space.*

(i) *If X has the p-(SR) property, then X^* has the p-(SR*) property.*
(ii) *If X^* has the p-(SR) property, then X has the p-(SR*) property.*
(iii) *If X has the p-L-limited property, then X^* has the p-L-limited* property.*
(iv) *If X^* has the p-L-limited property, then X has the p-L-limited* property.*

PROOF: (i) Suppose X has the p-(SR) property. If (x_n) is a DP weakly p-summable sequence in X, then it is a DP weakly p-summable sequence in X^{**}. Hence every p-Right* subset of X^* is a p-Right set, thus relatively weakly compact.

(ii) Suppose X^* has the p-(SR) property. Note that every p-Right* subset of X is a p-Right subset of X^{**}, thus relatively weakly compact.

(iii) Suppose X has the p-L-limited property. If (x_n) is a limited weakly p-summable sequence in X, then it is a limited weakly p-summable sequence in X^{**}. Hence every p-L-limited* subset of X^* is a p-L-limited set, thus relatively weakly compact.

(iv) Suppose X^* has the p-L-limited property. Every p-L-limited* subset of X is a p-L-limited subset of X^{**}, thus relatively weakly compact. □

Example 2. (i) This example shows that there is a Banach space that has property p-(V^*) and does not have property p-(SR*), $1 < p < \infty$. Let J_p be the James p-space, $1 < p < \infty$. Then J_p has property p-(V^*), see [28, Theorem 2.14]. The spaces J_p and J_p^* are separable non-reflexive spaces, and J_p contains no copy of l_1. The space J_p does not have properties p-(SR*) and p-L-limited* by Corollary 3.3 (iii) (or by Proposition 3.1, since it is not weakly sequentially complete).

(ii) This example shows that there is a Banach space X that has property p-(SR*) and does not have property p-L-limited*, $2 \le p \le \infty$. Let X be the first Bourgain–Delbaen space, see [4]. The space X is a separable $\mathcal{L}_\infty$-space which has the Schur property, and X^* is weakly sequentially complete. If $2 \le p \le \infty$, every p-Right* set in X is relatively compact (by Corollary 3.8). Further, X^* is isomorphic to $C[0,1]^*$, and thus X^* has the GP property (since L_1-spaces have the GP property, see [13]). Then X^* has the p-GP property and B_X is a p-L-limited* set (by Corollary 3.3 (i)) which is not weakly precompact.

(iii) If X is the first Bourgain–Delbaen space, then X contains a p-(V^*) set which is not weakly precompact for $1 < p < 2$. Let $T\colon l_{p^*} \to X^*$ be an operator. Since X^{**} is isomorphic to a $C(K)$ space, $T^*\colon X^{**} \to l_p$ is compact [33, page 100]. Hence T is compact and $X^* \in C_p$, if $1 < p < 2$, by [7, Proposition 1.5]. Then B_X is a p-(V^*) set [20, Corollary 25], and thus a p-Right* set, and B_X is not weakly precompact.

(iv) There is a Banach space X such that X^* has the p-L-limited* (p-(SR*), respectively) property but X does not have the p-L-limited (p-(SR), respectively) property. Let X be the first Bourgain–Delbaen space. Since X^* is isomorphic to an L_1-space, X^* has property p-L-limited* (by Example 1 (ii)), and thus property p-(SR*) for all $1 \le p \le \infty$. The identity operator on X is completely continuous, hence limited p-convergent (DP p-convergent, respectively) and not weakly compact. Hence X does not have the p-L-limited (p-(SR), respectively) property for all $1 \le p \le \infty$, see [21], [32].

A Banach space X has the DP*-*property* (DP*P) if all weakly compact sets in X are limited, see [6].

If X has the DP*P, then it has the DPP. If X is a Schur space or if X has the DPP and the Grothendieck property, then X has the DP*P, see [6], [24].

Let $1 \le p \le \infty$. A Banach space X has the *Dunford–Pettis property of order* p (DPP$_p$, respectively), if every weakly compact operator $T\colon X \to Y$ is p-convergent for any Banach space Y, see [7].

Let $1 \le p \le \infty$. A Banach space X has the DP*-*property of order* p (DP$^*P_p$) if all weakly p-compact sets in X are limited, see [15].

The DPP$_\infty$ is precisely the DPP, and every Banach space has the DPP$_1$. If X has the DPP, then X has the DPP$_p$ for all $1 \le p < \infty$. The DP*P$_\infty$ is precisely the DP*P and every Banach space has the DP*P$_1$. If X has the DP*P, then X has the DP*P$_p$ for all $1 \le p < \infty$. If X has the DP*P$_p$, then X has the DPP$_p$.

Proposition 3.12.

(i) *Let $1 < p \le \infty$. If X^* has the DPP$_p$, then a subset of X is p-Right* set if and only if it is a p-(V^*) set.*

(ii) *Let $1 < p \le \infty$. If X^* has the DP$^*P_p$, then a subset of X is p-L-limited* set if and only if it is a p-(V^*) set.*

(iii) *Let $1 < p \le \infty$. Suppose X^* has the DP$^*P_p$ (DPP$_p$, respectively). Then X has property p-(V^*) if and only if X has property p-L-limited* (p-(SR*), respectively).*

(iv) *Let $2 \leq p \leq \infty$. Suppose X^* has the DP*P$_p$ and A is a subset of X. Then A is a p-L-limited* set if and only if A is a p-(V^*) set if and only if A is weakly precompact.*

(v) *Let $2 \leq p \leq \infty$. Suppose X^* has the DPP$_p$ and A is a subset of X. Then A is a p-Right* set if and only if A is a p-(V^*) set if and only if A is weakly precompact.*

Proof: (i) Let $1 < p < \infty$. Suppose X^* has the DPP$_p$. Then every weakly p-summable sequence in X^* is DP, see [7, Proposition 3.2]. Hence every p-Right* subset of X is a p-(V^*) set.

Let $p = \infty$. Suppose X^* has the DPP. Then every weakly null sequence in X^* is DP, see [11, Theorem 1]. Hence every Right* subset of X is a DP set.

Conversely, any p-(V^*) set is a p-Right* set.

(ii) Let $1 < p < \infty$. Since X^* has the DP*P$_p$, every weakly p-summable sequence in X^* is limited, see [16, Theorem 2.4]. Hence every p-L-limited* subset of X is a p-(V^*) set. Let $p = \infty$. Since X^* has the DP*P, every weakly null sequence in X^* is limited, see [6], [24]. Then every L-limited* subset of X is a DP set. Conversely, any p-(V^*) set is a p-L-limited* set.

(iii) Suppose X^* has the DP*P$_p$ (DPP$_p$, respectively) and X has property p-(V^*). Every p-L-limited* (p-Right*, respectively) subset of X is a p-(V^*) set, and thus relatively weakly compact. The converse follows from Proposition 3.1.

(iv) Let $2 \leq p \leq \infty$. Every p-L-limited* subset of X is a p-(V^*) set (by (ii)), and thus weakly precompact by Corollary 3.9. If A is weakly precompact, then A is a p-L-limited* set and a p-(V^*) set by Proposition 3.1. ☐

The following result gives a characterization of p-L-limited* (p-Right*, respectively) sets and contains [10, Theorem 3.12].

Proposition 3.13 ([25]). *Let $1 < p < \infty$. Suppose that A is a bounded subset of a Banach space X. Then the following assertions are equivalent:*

(i) *A is a p-L-limited* (p-Right*, respectively) set.*

(ii) *$T(A)$ is relatively compact whenever Y is a Banach space and $T\colon X \to Y$ is an operator such that T^* is limited (weakly limited, respectively) weakly p-compact.*

(iii) *$T(A)$ is relatively compact whenever $Y^* \in W_p$ and $T\colon X \to Y$ is an operator such that T^* is limited (weakly limited, respectively).*

(iv) *$T(A)$ is relatively compact whenever $T\colon X \to l_p$ is an operator such that T^* is limited (weakly limited, respectively).*

(v) *If (x_n^*) is a limited (DP, respectively) weakly p-summable sequence in X^* and (x_n) is a sequence in A, then $\lim x_n^*(x_n) = 0$.*

If A is a bounded subset of X and $l_\infty(A) = B(A)$ is the Banach space of all bounded real-valued functions defined on A (endowed with sup norm), then let $E\colon X^* \to l_\infty(A)$ be defined by $Ex^*(x) = x^*(x)$, $x^* \in X^*$, $x \in A$. The operator E is called the evaluation map of A.

If A is a bounded subset of X^*, let $E\colon X \to l_\infty(A) = B(A)$ be the evaluation map defined by $Ex(x^*) = x^*(x)$, $x \in X$, $x^* \in A$.

Motivated by [3, Theorem 3.1], we give characterizations of p-L-limited sets and p-L-limited* sets using evaluation maps.

Proposition 3.14. *Let $1 \le p \le \infty$.*

(i) *A bounded subset A of X^* is a p-L-limited set if and only if the evaluation map $E\colon X \to B(A)$ defined by $E(x)(x^*) = x^*(x)$, $x \in X$, $x^* \in A$, is limited p-convergent.*

(ii) *A bounded subset A of X is a p-L-limited* set if and only if the evaluation map $E\colon X^* \to B(A)$ defined by $E(x^*)(x) = x^*(x)$, $x^* \in X^*$, $x \in A$, is limited p-convergent.*

(iii) *A subset A of X is a p-L-limited* set if and only if there is a Banach space Y and an operator $T\colon Y \to X$ such that T and T^* are limited p-convergent and $A \subset T(B_Y)$.*

Proof: (i) Let A be a bounded subset of X^*, and let $E\colon X \to B(A)$ be the evaluation map. Then A is a p-L-limited set in X^* if and only if

$$\|E(x_n)\|_\infty = \sup_{x^* \in A} |x^*(x_n)| \to 0$$

for any limited weakly p-summable (limited weakly null for $p = \infty$) sequence (x_n) in X, if and only if E is limited p-convergent (limited completely continuous for $p = \infty$).

(ii) is similar to (i).

(iii) Let A be a p-L-limited* subset of X and let $\overline{\text{aco}}(A)$ be the closed absolutely convex hull of A. Note that $\overline{\text{aco}}(A)$ is also a p-L-limited* set. Let $Y = l_1(A)$, and define $T\colon Y \to X$ by $T(f) = \sum_{x \in A} f(x)x$, $f \in Y$. It is clear that T is a bounded linear operator, and $A \subset T(B_Y) \subset \overline{\text{aco}}(A)$. Since $Y = l_1(A)$ has the Schur property, the operator T is completely continuous, and thus limited p-convergent. Moreover, T^* is the evaluation map $E\colon X^* \to B(A)$, and thus T^* is limited p-convergent by (ii).

Conversely, suppose that $A \subset T(B_Y)$, where $T\colon Y \to X$ is an operator so that T and T^* are limited p-convergent. Then $T(B_Y)$, and thus A, is a p-L-limited* subset of X by Theorem 3.2. □

For each $1 \le p \le \infty$, every p-L-limited* subset of X^* is a p-L-limited set, but the converse is not true. Since c_0 is separable, it has the GP property, and thus the p-GP property. Then B_{l_1} is a p-L-limited set, see [21, Corollary 3.9], [32, Theorem 2.3]. The sequence (e_n) is weakly null, and thus limited in l_∞, since l_∞ has the DP*P, see [24]. Since (e_n) is weakly p-summable, l_∞ does not have the p-GP property, and B_{l_1} is not a p-L-limited* set by Corollary 3.3 (i).

Theorem 3.15. *Let $1 \leq p \leq \infty$ and let X be a Banach space. Then the following are equivalent:*

(i) *Every p-L-limited subset of X^* is a p-L-limited* set.*

(ii) *T^{**} is limited p-convergent whenever Y is a Banach space and $T\colon X \to Y$ is a limited p-convergent operator.*

(iii) *Same as* (ii) *with $Y = l_\infty$.*

Proof: (i) ⇒ (ii) Suppose that every p-L-limited subset of X^* is a p-L-limited* set and let $T\colon X \to Y$ be a limited p-convergent operator. Then $T^*(B_{Y^*})$ is a p-L-limited set, see [21, Theorem 3.4], and thus a p-L-limited* set. Therefore T^{**} is limited p-convergent by Theorem 3.2. (ii) ⇒ (iii) is obvious.

(iii) ⇒ (i) Let A be a p-L-limited subset of X^* and let (x_n^*) be a sequence in A. Define $T\colon X \to l_\infty$ by $T(x) = (x_i^*(x))$. Let (x_n) be a limited weakly p-summable sequence in X. Since A is a p-L-limited set,

$$\lim_n \| T(x_n)\| = \lim_n \sup_i |x_i^*(x_n)| = 0.$$

Therefore T is limited p-convergent. Since T^{**} is limited p-convergent (by (iii)), $T^*(B_{l_\infty^*})$ is a p-L-limited* set by Theorem 3.2. Hence $(T^*(e_n^*)) = (x_n^*)$, and thus A is a p-L-limited* set. □

Corollary 3.16. *Let X be a Banach space. Then the following are equivalent:*

(i) *Every L-limited subset of X^* is an L-limited* set.*

(ii) *T^{**} is limited completely continuous whenever Y is a Banach space and $T\colon X \to Y$ is limited completely continuous.*

(iii) *Same as* (ii) *with $Y = l_\infty$.*

Let $1 \leq p \leq \infty$. A bounded subset A of X^* is called a p-(V) set, see [28], (or *weakly-p-L-set*, see [20]) if for all weakly p-summable (weakly null for $p = \infty$) sequences (x_n) in X, $\sup_{x^* \in A} |x^*(x_n)| \to 0$.

A Banach space X has property p-(V), see [28], (or RDP$_p$, see [20]) if every p-(V) subset of X^* is relatively weakly compact.

Let K be a compact Hausdorff space. We recall that $C(K)$ has the Grothendieck property if and only if it contains no complemented copy of c_0, see [8].

Proposition 3.17.

(i) *Let $1 < p < \infty$. Suppose X has the DP*P$_p$. Then X has property p-(V) if and only if it has the p-L-limited property.*

(ii) *Let $1 < p \leq \infty$. Let K be a compact Hausdorff space. If $C(K)$ has the Grothendieck property, then it has the p-L-limited property.*

Proof: (i) Suppose X has the DP*P$_p$ and property p-(V). By [21, Corollary 3.19], every p-L-limited set in X^* is a p-(V) set, and thus relatively weakly compact. The converse follows since every p-(V) set in X^* is a p-L-limited set.

(ii) If $C(K)$ has the Grothendieck property, then it has the DP*P, see [24, Corollary 5], thus the DP*P$_p$. Hence $C(K)$ has the p-L-limited property by (i) if $1 < p < \infty$ and by [32, Corollary 2.13] for $p = \infty$. □

Corollary 3.18. *Let $1 < p < \infty$ and let X be a Banach space.*

(i) *If every p-Right (Right, respectively) subset of X^* is weakly precompact, then T^{**} is DP p-convergent (pseudo weakly compact, respectively) whenever $T\colon X \to Y$ is DP p-convergent (pseudo weakly compact, respectively).*

(ii) *If every p-L-limited (L-limited, respectively) subset of X^* is weakly precompact, then T^{**} is limited p-convergent (limited completely continuous, respectively) whenever $T\colon X \to Y$ is limited p-convergent (limited completely continuous, respectively).*

(iii) *Let K be a compact Hausdorff space. If $C(K)$ has the Grothendieck property, then T^{**} is limited p-convergent (limited completely continuous, respectively) whenever $T\colon C(K) \to Y$ has the same property.*

Proof: (i) Let A be a p-Right (Right, respectively) subset of X^*. Since A is weakly precompact, it is a p-Right* (Right*, respectively) set by Proposition 3.1. Then T^{**} is DP p-convergent (pseudo weakly compact, respectively) if and only if T has the same property, see [1, Theorem 3.6] ([1, Corollary 3.7], respectively).

(ii) If every p-L-limited (or L-limited) subset of X^* is weakly precompact, then every p-L-limited (L-limited, respectively) is a p-L-limited* set by Proposition 3.1. Apply Theorem 3.15 (Corollary 3.16, respectively).

(iii) By Proposition 3.17, $C(K)$ has the p-L-limited property. Apply (ii). □

Since $X = C(K)$ has property (V), see [29], it has property p-(SR) by Proposition 3.1, and thus it satisfies the hypothesis of Corollary 3.18 (i).

The $w^* - w$ continuous (compact, respectively) operators from X^* to Y will be denoted by $L_{w^*}(X^*, Y)$ ($K_{w^*}(X^*, Y)$, respectively).

If H is a subset of $L_{w^*}(X^*, Y)$, $x^* \in X^*$ and $y^* \in Y^*$, let $H(x^*) = \{T(x^*)\colon T \in H\}$ and $H^*(y^*) = \{T^*(y^*)\colon T \in H\}$.

Theorem 3.19.

1. *Let $1 \leq p \leq \infty$. If X has the p-wL-limited* property and Y has the p-L-limited* property (or if X has the p-L-limited* property and Y has*

the p-wL-limited property), then* $K_{w^*}(X^*, Y)$ *has the p-wL-limited* property.*

2. *Let* $1 \le p \le \infty$. *Suppose* $L_{w^*}(X^*, Y) = K_{w^*}(X^*, Y)$. *If* X *and* Y *have the p-L-limited* property, then* $K_{w^*}(X^*, Y)$ *has the p-L-limited* property.*

Proof: 1. Suppose that X has the p-wL-limited* property and Y has the p-L-limited* property. Let H be a p-L-limited* set in $K_{w^*}(X^*, Y)$. If $x^* \in X^*$ and $y^* \in Y^*$, let $\phi_{x^*}: K_{w^*}(X^*, Y) \to Y$ and $\psi_{y^*}: K_{w^*}(X^*, Y) \to X$ be defined by $\phi_{x^*}(T) = T(x^*)$, $\psi_{y^*}(T) = T^*(y^*)$, $T \in K_{w^*}(X^*, Y)$. Continuous linear images of p-L-limited* sets are p-L-limited* sets. For each $x^* \in X^*$, $H(x^*) = \phi_{x^*}(H)$ is a p-L-limited* set in Y, thus relatively weakly compact. Similarly $H^*(y^*)$ is a p-L-limited* set in X, thus weakly precompact for each $y^* \in Y^*$. Then H is is weakly precompact by [17, Theorem 3.2].

2. Suppose X and Y have the p-L-limited* property. Let H be a p-L-limited* set in $K_{w^*}(X^*, Y)$. By the previous proof, $H(x^*)$ and $H^*(y^*)$ are p-L-limited* sets, hence relatively weakly compact for each $x^* \in X^*$ and $y^* \in Y^*$. Then H is relatively weakly compact by [23, Theorem 4.8]. □

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