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# The positive cone of a Banach lattice. Coincidence of topologies and metrizability

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*Abstract.* Let $X$ be a Banach lattice, and denote by $X_+$ its positive cone. The weak topology on $X_+$ is metrizable if and only if it coincides with the strong topology if and only if $X$ is Banach-lattice isomorphic to $l^1(\Gamma)$ for a set $\Gamma$. The weak* topology on $X_+^*$ is metrizable if and only if $X$ is Banach-lattice isomorphic to a $C(K)$-space, where $K$ is a metrizable compact space.

*Keywords:* normed lattice; Banach lattice; positive cone; AM-space; AL-space; Banach lattice $C(K)$; Banach lattice $l^1(\Gamma)$; strong topology; weak topology; weak* topology; coincidence of topologies; metrizability; nonatomic measure

*Classification:* 46B42, 46E05, 54E35

## 1. Introduction

Let $X$ be a Banach space, with its strong and weak topologies denoted by $s$ and $w$, respectively. It is well known that $w$ is metrizable if and only if $X$ is finite-dimensional. Moreover, an analogous result holds for the weak* topology $w^*$ of $X^*$.

The situation is different when we consider the positive cone $X_+$ of a (real) Banach lattice $X$. Indeed, it was shown in [9] that if $X = l^1(\Gamma)$, where $\Gamma$ is an arbitrary (nonempty) set, then $(X_+, w)$ is not only metrizable, but we have $s = w$ on $X_+$ (see also Lemma 3 in Section 3). In this paper we show the converse: if $(X_+, w)$ is metrizable, then $X$ is Banach-lattice isomorphic to $l^1(\Gamma)$ for some $\Gamma$ (see Theorem 2 in Section 4). One ingredient of the proof is the theorem that the positive part of the unit sphere in $L^1(\mu)$, where $\mu$ is a nonatomic probability measure, equipped with $w$, is not metrizable (see Theorem 1 in Section 4, a result due to J. Spurný).

We also show that $(X_+^*, w^*)$, where $X$ is a Banach lattice, is metrizable (if and) only if $X$ is Banach-lattice isomorphic to a $C(K)$-space, where $K$ is a metrizable compact space (see Theorem 3 in Section 5). Moreover, we establish simple

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necessary and sufficient conditions that $s = w^*$, respectively, $w = w^*$ on $X^*_+$ (see Proposition 5 in Section 5).

In the proofs we apply, not always explicitly, many standard results from both general functional analysis and the theory of Banach lattices. Among those results are classical representation theorems for *AL*- and *AM*-spaces and the well-known duality property between those spaces (see, e.g., [1, Section 12]).

The notation and terminology we use are mostly standard. Nevertheless, some relevant explanations are given in Section 2. Auxiliary results are gathered together in Section 3. The main results are presented in Sections 4 and 5, which are independent as far as the proofs are concerned.

Finally, we note that related topological properties of order intervals in $C(K)$- and $C_b(K)$-spaces, where $K$ is a completely regular space, are investigated in [10].

The author is much indebted to Jiří Spurný for his kind permission to incorporate his unpublished result (see Theorem 1 in Section 4; our proof is a modification of the original one). The result was obtained in answer to a question of the author and will be also applied in another paper of his (in preparation).

## 2. Preliminaries

All linear spaces under consideration are supposed to be over the field $\mathbb{R}$ of real numbers.

Let $X$ be a normed space and let $X^*$ stand for its dual space. The closed unit balls of $X$ and $X^*$ are denoted by $B$ and $B^*$, respectively. The strong and weak topologies of $X$ are denoted by $s$ and $w$, respectively, while the weak$^*$ topology of $X^*$ is denoted by $w^*$. Given $Z \subset X^*$, we denote by $\sigma(X, Z)$ the weakest topology on $X$ making the elements of $Z$ continuous.

Let now $X$ be a linear lattice (= Riesz space, in the terminology of [1]). The order of $X$ is denoted by the symbol $\leqslant$ and its positive cone by $X_+$. The symbol $|x|$ stands for the absolute value of $x \in X$. An element $e$ of $X_+$ is called an *order unit* of $X$ if, for every $x \in X$, there exists $t \in \mathbb{R}$ with $|x| \leqslant te$.

Let next $X$ be a normed lattice. Then $X_+$ and $X^*_+$ are $w$-closed and $w^*$-closed, respectively. We set

$$B_+ = B \cap X_+ \qquad \text{and} \qquad B^*_+ = B^* \cap X^*_+.$$

It follows that $B_+$ is $w$-closed. Moreover, the Banach–Alaoglu theorem (see, e.g., [11, Theorem 3.15]), implies that $B^*_+$ is $w^*$-compact. We shall use the latter result in establishing Propositions 4 and 5 in Section 5.

Let $(T, \tau)$ be a topological space. For $S \subset T$ we write $(S, \tau)$ meaning that $S$ is equipped with the corresponding relative topology. If $\sigma$ is another topology on $T$, we write

$$\tau = \sigma \qquad \text{on } S$$

meaning that $\tau$ and $\sigma$ restricted to $S$ coincide.

## 3. Auxiliary results

The following lemma will be used in establishing Propositions 1 and 2 in this section and Proposition 4 in Section 5.

**Lemma 1.** *Let $X$ be a normed lattice and let $\tau$ be a linear topology on $X$ such that $X_+$ and $B$ are $\tau$-closed. The following two conditions are equivalent:*

(i) $(B_+, \tau)$ *is compact* [*and metrizable*]*;*
(ii) $(B, \tau)$ *is compact* [*and metrizable*].

Clearly, (ii) implies (i). Since $B \subset B_+ - B_+$, the converse implication holds, by standard results (see [12, Propositions 7.1.5 and 7.6.3]; see also [7, Theorem 4.4.15] for a generalization of the latter proposition).

The next two propositions will be used in establishing Proposition 5 in Section 5.

**Proposition 1.** *For a normed lattice $X$ the following three conditions are equivalent:*

(i) $(B_+, s)$ *is compact;*
(ii) $(B, s)$ *is compact;*
(iii) $X$ *is finite-dimensional.*

The equivalence of (i) and (ii) is a special case of Lemma 1. The equivalence of (ii) and (iii) holds for an arbitrary normed space $X$ (cf. [11, Theorem 1.22]).

**Proposition 2.** *For a normed lattice $X$ the following three conditions are equivalent:*

(i) $(B_+, w)$ *is compact* [*and metrizable*]*;*
(ii) $(B, w)$ *is compact* [*and metrizable*]*;*
(iii) $X$ *is reflexive* [*and separable*].

The equivalence of (i) and (ii) is a special case of Lemma 1. The equivalence of (ii) and (iii) holds for an arbitrary normed space $X$ (cf. [5, Theorems V.4.7 and V.5.1]).

Proposition 2 suggests the following problem, which seems to be open.

*Characterize Banach lattices $X$ such that $(B_+, w)$ is noncompact but metrizable.*

For an in-depth study of (separable) Banach spaces $X$ such that $(B,w)$ is completely metrizable, see the influential paper [6] by G. A. Edgar and R. F. Wheeler.

The next lemma will be used in the proof of the proposition which follows it.

**Lemma 2.** *Let $X$ be a Banach lattice and let $x_1, x_2, \ldots \in X$ be such that for every $x \in X$ we have $x \leqslant x_n$ for some $n$. Then there is an $m$ such that $x_m$ is an order unit of $X$.*

Proof: Given $x \in X$ and $r > 0$, denote by $B(x,r)$ the closed ball in $X$ with centre $x$ and radius $r$.

Set

$$E_n = \{x \in X\colon\ |x| \leqslant x_n\}, \qquad n = 1, 2, \ldots$$

The $E_n$ are closed and, by assumption, we have

$$X = \bigcup_{n=1}^{\infty} E_n.$$

The Baire category theorem implies that, for some $x_0 \in X$, $r_0 > 0$, and $m$, we have $B(x_0, r_0) \subset E_m$. It follows that

$$B(0, 2r_0) = B(x_0, r_0) - B(x_0, r_0) \subset 2E_m,$$

and so $B(0, r_0) \subset E_m$. Thus, $x_m$ is an order unit of $X$. □

The following proposition is the main tool in the proofs of Theorem 2 in Section 4 and of Theorem 3 in Section 5.

**Proposition 3.** *Let $X$ be a normed lattice and let $Z$ be a closed linear sublattice of $X^*$. If $(X_+, \sigma(X,Z))$ is metrizable, then $Z$ is Banach-lattice isomorphic to an AM-space with unit.*

Proof: Let $(V_n)$ be a base of neighbourhoods of $0$ in $(X_+, \sigma(X,Z))$. We may assume that

$$V_n = \{x \in X_+\colon\ z_n(x) < 1\}, \qquad \text{where } z_n \in Z_+,\ n = 1, 2, \ldots$$

Given $z \in Z_+$, we then have

$$V_n \subset \{x \in X_+\colon\ z(x) < 1\}$$

for some $n$. Thus, $z_n(x) < 1$ implies $z(x) < 1$ whenever $x \in X_+$. By homogeneity, we get $z \leqslant z_n$. Lemma 2 now implies that $Z$ has an order unit. This yields the assertion, by a classical result (see [1, Theorem 12.20] and the discussion following it). □

**Remark 1.** It follows from Proposition 3 that $(l^1_+, w^*)$, where $l^1$ is identified with $c_0^*$, is not metrizable, see Theorem 3 in Section 5 for a more general result. This can be also established by a simple direct argument.

Indeed, let $(V_n)$ be a sequence of neighbourhoods of 0 in $(l^1_+, w^*)$. We claim that there is a neighbourhood $V$ of 0 in $(l^1_+, w^*)$ with $V_n \setminus V \neq \emptyset$ for all $n$. We may assume that

$$V_n = \{z \in l^1_+ :\ z(x_n) < 1\}, \qquad n = 1, 2, \dots,$$

where $x_n \in (c_0)_+$ and $x_1 \leqslant x_2 \leqslant \dots$ Let $i_1 < i_2 < \dots$ be such that $x_n(i_n) \to 0$. Take $x \in \mathbb{R}^{\mathbb{N}}$ with

$$x_n(i_n) < x(i_n) < x_n(i_n) + \frac{1}{n}, \qquad n = 1, 2, \dots, \text{ and } x(i) = 0 \text{ otherwise.}$$

Clearly, $x \in (c_0)_+$. Set

$$V = \{z \in l^1_+ :\ z(x) < 1\}, \quad t_n = \frac{2}{x_n(i_n) + x(i_n)}, \qquad n = 1, 2, \dots$$

We then have $t_n x_n(i_n) < 1 < t_n x(i_n)$, and so

$$t_n e_{i_n} \in V_n \setminus V, \qquad n = 1, 2, \dots,$$

where $(e_n)$ is the standard basis of $l^1$. This proves our claim. Thus, $(l^1_+, w^*)$ is not metrizable.

The next lemma will be used in the proof of Theorem 2 in Section 4.

**Lemma 3** (= [9, Lemma 1])**.** *For every set $\Gamma$ we have $s = w$ on $l^1_+(\Gamma)$.*

In the proof of Theorem 1 in Section 4 we shall need the following simple result.

**Lemma 4.** *Let $(\Omega, \mathfrak{M}, \mu)$ be a probability measure space with $\mu$ nonatomic. Given $\mathfrak{M}$-simple functions $g_1, \dots, g_n$ on $\Omega$ and $\delta \in [0,1]$, there exists $M \in \mathfrak{M}$ such that*

$$\mu(M) = \delta \quad \text{and} \quad \int_M g_i \, \mathrm{d}\mu = \delta \int_\Omega g_i \, \mathrm{d}\mu, \qquad i = 1, \dots, n.$$

Proof: By linearity, it is enough to establish the assertion in the case where $g_i = 1_{M_i}$, $i = 1, \dots, n$, where $\{M_1, \dots, M_n\}$ is an $\mathfrak{M}$-partition of $\Omega$. For every $i$ choose $\widetilde{M_i} \in \mathfrak{M}$ such that

$$\widetilde{M_i} \subset M_i \quad \text{and} \quad \mu(\widetilde{M_i}) = \delta\mu(M_i), \qquad i = 1, \dots, n.$$

Then $M := \widetilde{M_1} \cup \dots \cup \widetilde{M_n}$ is as desired. □

**Remark 2.** In fact, Lemma 4 holds for arbitrary $g_1, \ldots, g_n \in L^1(\mu)$. This is a direct consequence of the Lyapunov convexity theorem (see, e.g., [11, Theorem 5.5]) applied to the vector measure $\varphi\colon \mathfrak{M} \to \mathbb{R}^{n+1}$ defined by the formula

$$\varphi(M) = \left(\mu(M), \int_M g_1 \,\mathrm{d}\mu, \ldots, \int_M g_n \,\mathrm{d}\mu\right), \qquad M \in \mathfrak{M}.$$

## 4. The weak topology of $X_+$

We start by a nonmetrizability result. It will be applied in the proof of Theorem 2 below, but it is also of interest in itself.

**Theorem 1** (J. Spurný, personal communication of March 15, 2011)**.** *Let $(\Omega, \mathfrak{M}, \mu)$ be a probability measure space with $\mu$ nonatomic, and set*

$$S_+ = \left\{ f \in L^1_+(\mu)\colon \int_\Omega f \,\mathrm{d}\mu = 1 \right\}.$$

*Then $(S_+, w)$ is not metrizable.*

Proof: We shall show that there is no countable base of neighbourhoods of $1_\Omega$ in $(S_+, w)$. Suppose, to get a contradiction, that $(V_n)$ is such a base. We may assume that for all $n$

$$V_n = \left\{ f \in S_+\colon \left| \int_\Omega f g_i \,\mathrm{d}\mu - \int_\Omega g_i \,\mathrm{d}\mu \right| < \varepsilon_n, \ \ i = 1, \ldots, n \right\},$$

where $g_1, g_2, \ldots$ are $\mathfrak{M}$-simple functions on $\Omega$ and $\varepsilon_1 > \varepsilon_2 > \ldots > 0$. For every $n$, choose $M_n \in \mathfrak{M}$ according to Lemma 4 with $\delta = 1/n$. We then have $n 1_{M_n} \in V_n$. Since $V_1 \supset V_2 \supset \ldots$, it follows that $n 1_{M_n} \to 1_\Omega$ weakly. Set $\mu_n(M) = n\mu(M \cap M_n)$ for all $M \in \mathfrak{M}$ and $n = 1, 2, \ldots$ Then $\mu_n$ are measures on $\mathfrak{M}$ which are absolutely continuous with respect to $\mu$ and $\mu_n(M) \to \mu(M)$ for all $M \in \mathfrak{M}$. Moreover, $\mu_n(M_n) = 1$ for all $n$. Taking into account that $\mu(M_n) \to 0$, we get a contradiction with the Vitali–Hahn–Saks theorem (see, e.g., [5, Theorem III.7.2]). □

It follows from Theorem 1 that, under its assumptions, the strong and weak topologies do not coincide on $S_+$, which is in contrast with Lemma 3. In the case where $\Omega = [0, 1]$ and $\mu$ is Lebesgue measure on the Lebesgue $\sigma$-algebra over $\Omega$, this can also be seen, by a simple modification of the sequence of Rademacher functions.

**Theorem 2.** *For a Banach lattice $X$ the following three conditions are equivalent:*

(i) $s = w$ *on* $X_+$;
(ii) $(X_+, w)$ *is metrizable;*
(iii) $X$ *is Banach-lattice isomorphic to* $l^1(\Gamma)$ *for a set* $\Gamma$.

PROOF: Clearly, (i) implies (ii). By Lemma 3, (iii) implies (i).

Suppose (ii) holds. By Proposition 3, $X^*$ is Banach-lattice isomorphic to an $AM$-space. Therefore, $X^{**}$ is Banach-lattice isomorphic to an $AL$-space (see [1, Theorem 12.22]) and so is $X$ (see [12, Theorem 3.9.8]). It follows that $X$ is Banach-lattice isomorphic to $L^1(\mu)$ for some positive measure $\mu$ on a $\sigma$-algebra $\mathfrak{M}$ of sets (see [1, Theorem 12.26]). In view of Theorem 1, for every $M \in \mathfrak{M}$ with $0 < \mu(M) < \infty$, there is a $\mu$-atom $N$ with $N \subset M$. This yields (iii). □

**Remark 3.** It follows from Theorem 2, (ii) ⇒ (iii), that $((c_0)_+, w)$ is not metrizable. This can be also established by a much simpler and more direct argument, which is a modification of the solution to Problem 21 in [8]. Indeed, suppose, to get a contradiction, that there exists a base $(V_n)$ of neighbourhoods of $0$ in $((c_0)_+, w)$. We may assume that

$$V_n = \{x \in (c_0)_+ \colon\ x_n^*(x) < 1\},$$

where $x_n^* \in l^1_+$, $n = 1, 2, \ldots$, and $x_1^* \leqslant x_2^* \leqslant \ldots$ Denote by $(e_n)$ the standard basis of $c_0$. Choose $k_1 < k_2 < \ldots$ so that $x_n^*(e_{k_n}) < 1/n$ for all $n$. The sequence $(ne_{k_n})$ is, clearly, unbounded, but it converges weakly to $0$, a contradiction.

## 5. The weak* topology of $X_+^*$

The next result will be used in the proof of Theorem 3 below.

**Proposition 4.** *For a normed lattice $X$ the following three conditions are equivalent:*

(i) $(B_+^*, w^*)$ *is metrizable;*
(ii) $(B^*, w^*)$ *is metrizable;*
(iii) $X$ *is separable.*

The equivalence of (i) and (ii) is a special case of Lemma 1, since $(B^*, w^*)$ is compact, by a standard result. The equivalence of (ii) and (iii) holds for an arbitrary normed space $X$ (see [1, Theorem 10.7]).

**Theorem 3.** *For a Banach lattice $X$ the following two conditions are equivalent:*

(i) $(X_+^*, w^*)$ *is metrizable;*
(ii) $X$ *is Banach-lattice isomorphic to a $C(K)$-space, where $K$ is a metrizable compact space.*

The implication (ii) ⇒ (i) is known; see [13, Theorem 3.1] or [3, Theorem 31.5, (d) ⇒ (b)] for a more general result. Nevertheless, a simple proof is given below for completeness.

Proof of Theorem 3: Recall that the Banach lattice $C(K)$, where $K$ is a compact space, is separable if and only if $K$ is metrizable (see [12, Proposition 7.6.2]).

(i) ⇒ (ii): In view of Proposition 3, (i) implies that $X$ is Banach-lattice isomorphic to an *AM*-space with unit. A classical result (see [1, Theorem 12.28]) now shows that $X$ is Banach-lattice isomorphic to a $C(K)$-space, where $K$ is compact. By Proposition 4, (i) implies (iii), $X$ is, moreover, separable. Therefore, $K$ is metrizable, and so (ii) holds.

(ii) ⇒ (i): Let $K$ be a metrizable compact space. Let $\{f_n\colon n \in \mathbb{N}\}$ be a dense subset of $C(K)$ with $f_1 = 1_K$. Set

$$d(\mu,\nu) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\int_K f_n \,\mathrm{d}(\mu-\nu)|}{1+|\int_K f_n \,\mathrm{d}(\mu-\nu)|} \qquad \text{for } \mu,\nu \in C(K)^*_+.$$

The metric $d$ defines a topology $\tau$ on $C(K)^*_+$ that is, clearly, weaker than $w^*$. Since $(B^*_+, w^*)$ is compact, $w^* = \tau$ on bounded subsets of $C(K)^*_+$. Since $\tau$-convergent nets are eventually bounded, $w^* = \tau$ on $C(K)^*_+$ , and so (i) holds. □

For completeness we note the following simple result.

**Proposition 5.** *Let $X$ be a normed lattice.*

(a) *$s = w^*$ on $X^*_+$ if and only if $X$ is finite-dimensional.*
(b) *$w = w^*$ on $X^*_+$ if and only if $X$ is reflexive.*

The "if" part of (a) follows from standard results, while the "if" part of (b) is obvious. As noted before, $(B^*_+, w^*)$ is compact. Therefore, the "only if" parts of (a) and (b) follow from Propositions 1 and 2, respectively, and standard results.

## Added in proof

1. Theorems 1 and 2 are already contained in [4], see Proposition 2 and Theorem 3* thereof. Our proofs are, however, different from those of S. J. Dilworth.
2. Lemma 3 also appears in [2, Lemma 4.1].

## References


[1] Aliprantis C. D., Burkinshaw O., *Positive Operators*, Pure and Applied Mathematics, 119, Academic Press, Orlando, 1985.

[2] Barroso C. S., Kalenda O. F. K., Lin P.-K., *On the approximate fixed point property in abstract spaces*, Math. Z. **271** (2012), no. 3–4, 1271–1285.

[3] Bauer H., *Measure and Integration Theory*, De Gruyter Studies in Mathematics, 26, Walter de Gruyter, Berlin, 2001.

[4] Dilworth S. J., *A weak topology characterization of* $l_1(m)$, Geometry of Banach Spaces, Strobl 1989, London Math. Soc. Lecture Note Ser., 158, Cambridge University Press, Cambridge, 1990, pages 89–94.
[5] Dunford N., Schwartz J. T., *Linear Operators. I. General Theory*, Pure and Applied Mathematics, 7, Interscience Publishers, New York, 1958.
[6] Edgar G. A., Wheeler R. F., *Topological properties of Banach spaces*, Pacific J. Math. **115** (1984), no. 2, 317–350.
[7] Engelking R., *General Topology*, Monografie Matematyczne, 60, PWN—Polish Scientific Publishers, Warszawa, 1977.
[8] Halmos P. R., *A Hilbert Space Problem Book*, D. van Nostrand Co., Princeton, 1967.
[9] Lipecki Z., *On compactness and extreme points of some sets of quasi-measures and measures. IV*, Manuscripta Math. **123** (2007), no. 2, 133–146.
[10] Lipecki Z., *Order intervals in* $C(K)$*. Compactness, coincidence of topologies, metrizability*, Comment. Math. Univ. Carolin. **63** (2022), no. 3, 295–306.
[11] Rudin W., *Functional Analysis*, McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book Co., New York, 1973.
[12] Semadeni Z., *Banach Spaces of Continuous Functions. Vol. I*, Monografie Matematyczne, 55, PWN—Polish Scientific Publishers, Warszawa, 1971.
[13] Varadarajan V. S., *Weak convergence of measures on separable metric spaces*, Sankhyā **19** (1958), 15–22.

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