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Hichem Khelifi, Youssef El Hadfi

Abstract. This paper introduces two novel nonlinear anisotropic Picone identities with variable exponents that expand upon the traditional identity used for the ordinary Laplace equation. Additionally, the research explores potential applications of these findings in anisotropic Sobolev spaces featuring variable exponents.

Keywords: anisotropic Picone identity; variable exponent

Classification: 35A23, 35A02, 35J75

1. Introduction

Let Ω be a bounded open domain in $\mathbb{R}^N$, $N \geq 3$, with Lipschitz continuous boundary $\partial\Omega$. Let $p_i\colon \overline{\Omega} \to \mathbb{R}$, for $i = 1, \dots, N$, be continuous functions representing the variable exponents. In recent years, there has been significant interest in studying anisotropic operator with variable exponents

$$\sum_{i=1}^{N} D_i(|D_i u|^{p_i(x)-2} D_i u), \qquad \text{where } D_i u = \frac{\partial u}{\partial x_i}, \tag{1.1}$$

they participate in various areas of applied sciences. In certain instances, they provide realistic models for studying natural occurrences in electrorheological fluids (refer to citations in [1]). Another significant application is related to image processing [5].

Note that in the classical case $p_i(x) = p$, equation (1.1) represents the p-Laplacian operator. However, in the constant case $p_i(x) = p_i$, equation (1.1) becomes the anisotropic Laplacian operator given by $\sum_{i=1}^{N} D_i(|D_i u|^{p_i-2} D_i u)$, where $p_i > 1$. The p_i-Laplacian operator exhibits homogeneity for every $i = 1, \dots, N$. On the other hand, equation (1.1) represents a nonhomogeneous operator, as it incorporates different exponents p_i for each coordinate direction. The $p_i(x)$-Laplacian does not necessarily have a more complex nonlinearity compared to

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other types of Laplacians. The complexity of the nonlinearity depends on the specific form of the function $p_i(x)$, since there is not equivalence relation between the norm $\|u\|_{p_i(x)}$ and $p_i(x)$ modular $\varrho(u) = \int_\Omega |u|^{p_i(x)}\,\mathrm{d}x + \int_\Omega |D_i u|^{p_i(x)}\,\mathrm{d}x$ on the anisotropic variable exponent Sobolev spaces $W^{1,p_i(\cdot)}(\Omega)$, see [4], [10], [17], [18], and the references therein. For the isotropic case (i.e., $p_i(x) = p(x)$), we define the $p(x)$-Laplacian operator as $\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, where $p(x) \in C(\overline{\Omega})$ and $p(x) > 1$ almost everywhere in Ω.

In [20], the following homogeneous linear differential system of the second order is under study by M. Picone,

$$(a_1(x)u')' + a_2(x)u = 0,$$
$$(b_1(x)v')' + b_2(x)v = 0,$$

and it has been proven that for differentiable functions u and v, both nonzero, the following identity holds.

$$\left(\frac{u}{v}(a_1u'v - b_1uv')\right)' = (b_2 - a_2)u^2 + (a_1 - b_1)u'^2 + b_1\left(u' - \frac{v'u}{v}\right)^2. \tag{1.2}$$

In [21], (1.2) was extended to the Laplace operator. Specifically, for differentiable functions $u \geq 0$ and $v > 0$, we have

$$|\nabla u|^2 + \frac{u^2}{v^2}|\nabla v|^2 - 2\frac{u}{v}\nabla u \cdot \nabla v = |\nabla u|^2 - \nabla\left(\frac{u^2}{v}\right) \cdot \nabla v \geq 0. \tag{1.3}$$

The extension of equation (1.3) to the p-Laplacian operator with $1 < p < \infty$ was achieved by W. Allegretto and Y. Huang in their work cited as reference [3]. N. Yoshida in [22] proved the Picone identity with variable exponent

$$\begin{aligned}\frac{|\nabla u|^{p(x)}}{p(x)} &- \left(\frac{u}{v}\right)^{p(x)-1}|\nabla v|^{p(x)-2}\nabla v \cdot \nabla u + \frac{p(x)-1}{p(x)}\left(\frac{u}{v}\right)^{p(x)}|\nabla v|^{p(x)} \\ &= \frac{|\nabla u|^{p(x)}}{p(x)} - |\nabla v|^{p(x)-2}\nabla v \cdot \nabla\left(\frac{u^{p(x)}}{p(x)v^{p(x)-1}}\right) \geq 0.\end{aligned} \tag{1.4}$$

Other authors have established different variable exponent Picone identities than (1.4). Those alternative Picone identities can be found in various sources, including [2], [8]. Additionally, T. Feng and X. Cui introduced in [7] a linear Picone identity for the anisotropic Laplace operator. Furthermore, T. Feng and K. Zhang extended in [7] the nonlinear Picone identity of T. Feng and X. Cui beyond the Laplace operator. This extension presents a generalization of their findings to Euclidean space and can be found in [9].

In this paper, our aim is to derive two nonlinear anisotropic Picone identities with variable exponents and explore their applications. The following are the main results presented in this study.

Theorem 1.1. *Let $v > 0$ and $u > 0$ be two differentiable functions in the set $\Omega \subset \mathbb{R}^N$, $N \geq 3$, and denote*

$$\begin{aligned} L(u,v) = {} & \sum_{i=1}^{N} |D_i u|^{p_i(x)} - \sum_{i=1}^{N} \frac{u^{p_i(x)} \ln\left(\frac{u}{v}\right)}{v^{p_i(x)-1}} |D_i v|^{p_i(x)-2} D_i v \cdot D_i p_i(x) \\ & - \sum_{i=1}^{N} p_i(x) \frac{u^{p_i(x)-1}}{v^{p_i(x)-1}} |D_i v|^{p_i(x)-2} D_i v \cdot D_i u \\ & + (p_i(x)-1)\Big(\frac{u}{v}\Big)^{p_i(x)} |D_i v|^{p_i(x)}, \\ R(u,v) = {} & \sum_{i=1}^{N} |D_i u|^{p_i(x)} - \sum_{i=1}^{N} |D_i v|^{p_i(x)-2} D_i v \cdot D_i\Big(\frac{u^{p_i(x)}}{v^{p_i(x)-1}}\Big). \end{aligned}$$

Then

$$L(u,v) = R(u,v). \tag{1.5}$$

Moreover, if $D_i v(x) \cdot D_i p_i(x) = 0$ for every $i = 1, \ldots, N$ and every $x \in \Omega$, then we have $L(u,v) \geq 0$. Furthermore, $L(u,v) = 0$ almost everywhere in Ω if and only if $D_i(u/v) = 0$ almost everywhere in Ω for any $i = 1, \ldots, N$.

Theorem 1.2. *Let $v > 0$ and $u > 0$ be two differentiable functions in the set $\Omega \subset \mathbb{R}^N$, $N \geq 3$, and denote*

$$\begin{aligned} L(u,v) = {} & \sum_{i=1}^{N} |D_i u|^{p_i(x)} - \sum_{i=1}^{N} \frac{u^{p_i(x)} \ln u}{f(v)} |D_i v|^{p_i(x)-2} D_i v \cdot D_i p_i(x) \\ & - \sum_{i=1}^{N} p_i(x) \frac{u^{p_i(x)-1}}{f(v)} |D_i v|^{p_i(x)-2} D_i v \cdot D_i u + \sum_{i=1}^{N} \frac{u^{p_i(x)} f'(v)}{[f(v)]^2} |D_i v|^{p_i(x)}, \\ R(u,v) = {} & \sum_{i=1}^{N} |D_i u|^{p_i(x)} - \sum_{i=1}^{N} D_i\Big(\frac{u^{p_i(x)}}{f(v)}\Big) \cdot |D_i v|^{p_i(x)-2} D_i v, \end{aligned}$$

where $f(v) > 0$ and $f'(v) \geq (p_i(x)-1)[f(v)]^{(p_i(x)-2)/(p_i(x)-1)}$ for all $p_i(x) > 1$ and for all $i = 1, \ldots, N$. Then

$$L(u,v) = R(u,v). \tag{1.6}$$

Moreover, if $D_i v(x) \cdot D_i p_i(x) = 0$ for every $i = 1, \ldots, N$ and every $x \in \Omega$, we have $L(u,v) \geq 0$. Furthermore, $L(u,v) = 0$ almost everywhere in Ω if and only if $D_i(u/v) = 0$ almost everywhere in Ω for any $i = 1, \ldots, N$.

2. Preliminary results

This section provides a brief overview of anisotropic spaces with variable exponents. For a more detailed discussion on Lebesgue–Sobolev spaces with variable exponents, we recommend to consult [6], [13], [14], [23] and [16], [15], [12], [11], as well as related references. Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$. We denote

$$p^+ = \max_{x\in\overline{\Omega}} p(x), \qquad p^- = \min_{x\in\overline{\Omega}} p(x),$$

and

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega})\colon\ p^- > 1\}.$$

Let $p(\cdot) \in C_+(\overline{\Omega})$. We define the space

$$L^{p(\cdot)}(\Omega) = \left\{u\colon\ \Omega \to \mathbb{R}^N \text{ measurable:}\ \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x < \infty\right\},$$

and

$$\|u\|_{p(\cdot)} := \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0\colon\ \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)}\mathrm{d}x \leq 1\right\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxemburg norm. We define also the Banach space

$$W_0^{1,p(\cdot)}(\Omega) = \{f \in L^{p(\cdot)}(\Omega)\colon\ |\nabla f| \in L^{p(\cdot)}(\Omega) \text{ and } f = 0 \text{ on } \partial\Omega\}$$

endowed with the norm $\|f\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla f\|_{p(\cdot)}$. The space $W_0^{1,p(\cdot)}(\Omega)$ is separable and reflexive provided that with $1 < p^- \leq p^+ = \max_{x\in\Omega} p(x) < \infty$.

Let $p_i\colon\ \overline{\Omega} \to (1,\infty)$ be continuous functions. We introduce the anisotropic variable exponent Sobolev space

$$W^{1,p_i(\cdot)}(\Omega) := \{u \in L^{p_i}(\Omega)\cdot\ D_i u \in L^{p_i}(\Omega)\},$$

and

$$W_0^{1,p_i(\cdot)}(\Omega) := \{u \in W_0^{1,1}(\Omega)\cdot\ D_i u \in L^{p_i}(\Omega)\},$$

which are Banach spaces under the norm

$$\|u\|_{W_0^{1,p_i(\cdot)}(\Omega)} = \|u\|_{L^{p_i(\cdot)}(\Omega)} + \|D_i u\|_{L^{1,p_i(\cdot)}(\Omega)}, \qquad i = 1,\dots,N.$$

Proposition 2.1. *Let $a \geq 0$, $b \geq 0$ and let $p_i, q_i\colon\ \overline{\Omega} \to (1,\infty)$ be continuous functions for every $i = 1,\dots,N$, with $1/p_i(x) + 1/q_i(x) = 1$. Then*

$$ab \leq \frac{a^{p_i(x)}}{p_i(x)} + \frac{b^{q_i(x)}}{q_i(x)}, \qquad \forall i = 1,\dots,N. \tag{2.1}$$

Moreover, the equality holds if and only if $a^{p_i(x)} = b^{q_i(x)}$ for every $i = 1,\dots,N$.

3. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1: We see with a direct computation that

$$\begin{aligned} R(u,v) = & \sum_{i=1}^{N} |D_i u|^{p_i(x)} \\ & - \sum_{i=1}^{N} \frac{v^{p_i(x)-1} D_i(u^{p_i(x)}) - u^{p_i(x)} D_i(v^{p_i(x)-1})}{(v^{p_i(x)-1})^2} |D_i v|^{p_i(x)-2} D_i v \\ = & \sum_{i=1}^{N} |D_i u|^{p_i(x)} - \sum_{i=1}^{N} \frac{D_i(u^{p_i(x)})}{v^{p_i(x)-1}} |D_i v|^{p_i(x)-2} D_i v \\ & + \sum_{i=1}^{N} \frac{u^{p_i(x)} D_i(v^{p_i(x)-1})}{(v^{p_i(x)-1})^2} |D_i v|^{p_i(x)-2} D_i v \\ = & \sum_{i=1}^{N} |D_i u|^{p_i(x)} \\ & - \sum_{i=1}^{N} \frac{u^{p_i(x)} \ln u D_i p_i(x) + p_i(x) u^{p_i(x)-1} D_i u}{v^{p_i(x)-1}} |D_i v|^{p_i(x)-2} D_i v \\ & + \sum_{i=1}^{N} \frac{u^{p_i(x)}(v^{p_i(x)-1} \ln v D_i p_i(x) + (p_i(x)-1) v^{p_i(x)-2} D_i v)}{(v^{p_i(x)-1})^2} \\ & \times |D_i v|^{p_i(x)-2} D_i v \\ = & \sum_{i=1}^{N} |D_i u|^{p_i(x)} - \sum_{i=1}^{N} \frac{u^{p_i(x)} \ln u}{v^{p_i(x)-1}} |D_i v|^{p_i(x)-2} D_i v \cdot D_i p_i(x) \\ & - \sum_{i=1}^{N} p_i(x) \frac{u^{p_i(x)-1}}{v^{p_i(x)-1}} |D_i v|^{p_i(x)-2} D_i v \cdot D_i u = L(u,v). \end{aligned}$$

Now we prove that $L(u,v)$ is nonnegative. We rewrite $L(u,v)$ as

$$\begin{aligned} L(u,v) = & \sum_{i=1}^{N} p_i(x) \Big[\frac{1}{p_i(x)} |D_i u|^{p_i(x)} + \frac{p_i(x)-1}{p_i(x)} \Big(\Big(\frac{u}{v} |D_i v| \Big)^{p_i(x)-1} \Big)^{p_i(x)/(p_i(x)-1)} \Big] \\ & - \sum_{i=1}^{N} p_i(x) \frac{u^{p_i(x)-1}}{v^{p_i(x)-1}} |D_i v|^{p_i(x)-1} |D_i u| \\ & + \sum_{i=1}^{N} p_i(x) \frac{u^{p_i(x)-1}}{v^{p_i(x)-1}} |D_i v|^{p_i(x)-2} [|D_i v||D_i u| - D_i v D_i u] \end{aligned}$$

$$
\begin{aligned}
&-\sum_{i=1}^{N} \frac{u^{p_i(x)} \ln \frac{u}{v}}{v^{p_i(x)-1}} |D_i v|^{p_i(x)-2} D_i v \cdot D_i p_i(x) \\
&= \mathcal{I}_{i,1} + \mathcal{I}_{i,2} + \mathcal{I}_{i,3}, \qquad (3.1)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{I}_{i,1} =& \sum_{i=1}^{N} p_i(x) \Big[\frac{1}{p_i(x)} |D_i u|^{p_i(x)} + \frac{p_i(x)-1}{p_i(x)} \Big(\Big(\frac{u}{v} |D_i v| \Big)^{p_i(x)-1} \Big)^{p_i(x)/(p_i(x)-1)} \Big] \\
& - \sum_{i=1}^{N} p_i(x) \frac{u^{p_i(x)-1}}{v^{p_i(x)-1}} |D_i v|^{p_i(x)-1} |D_i u|, \\
\mathcal{I}_{i,2} =& \sum_{i=1}^{N} p_i(x) \frac{u^{p_i(x)-1}}{v^{p_i(x)-1}} |D_i v|^{p_i(x)-2} \big[|D_i v||D_i u| - D_i v D_i u \big], \\
\mathcal{I}_{i,3} =& -\sum_{i=1}^{N} \frac{u^{p_i(x)} \ln \frac{u}{v}}{v^{p_i(x)-1}} |D_i v|^{p_i(x)-2} D_i v \cdot D_i p_i(x).
\end{aligned}
$$

By Young's inequality, see Proposition 2.1, inequality (1.6), (here $a = |D_i u|$, $b = (u|D_i v|/v)^{p_i(x)-1}$ and $q_i(x) = p_i(x)/(p_i(x)-1)$), we obtain

$$
\begin{aligned}
\mathcal{I}_{i,1} \geq& \sum_{i=1}^{N} p_i(x) |D_i u| \Big(\frac{u}{v} |D_i v| \Big)^{p_i(x)-1} - \sum_{i=1}^{N} p_i(x) \frac{u^{p_i(x)-1}}{v^{p_i(x)-1}} |D_i v|^{p_i(x)-1} |D_i u| \\
\geq& \ 0. \qquad (3.2)
\end{aligned}
$$

Using the fact that $|D_i v||D_i u| - D_i v D_i u \geq 0$, we have

$$
\mathcal{I}_{i,2} \geq 0. \qquad (3.3)
$$

From (3.2)–(3.3) and the fact that $\mathcal{I}_{i,3} = 0$ (since $D_i v \cdot D_i p_i(x) = 0$), it follows that $L(u,v) \geq 0$.

If $D_i(u/v) = 0$ a.e. in Ω for every $i = 1, \ldots, N$, then $u = cv$, and thus, $L(u,v) = 0$. Now we show that $L(u,v) = 0$ implies $D_i(u/v) = 0$ for every $i = 1, \ldots, N$. In fact, if $L(u,v)(x_0) = 0$ for all $x_0 \in \Omega$, then we consider the two cases $u(x_0) \neq 0$ and $u(x_0) = 0$, respectively.

(1) If $u(x_0) \neq 0$, then $\mathcal{I}_{i,1}(x_0) = 0$, $\mathcal{I}_{i,2}(x_0) = 0$ and $\mathcal{I}_{i,3}(x_0) = 0$. One shows by $\mathcal{I}_{i,1}(x_0) = 0$ that

$$
|D_i u| = \frac{u}{v} |D_i v|, \qquad \forall i = 1, \ldots, N. \qquad (3.4)
$$

Using $\mathcal{I}_{i,2}(x_0) = 0$, we get

$$
D_i u = c D_i v, \qquad \forall i = 1, \ldots, N, \qquad (3.5)
$$

where c is a positive constant. Putting (3.8) into (3.7) yields $u = cv$, namely $D_i(u/v) = 0$ for every $i = 1, \dots, N$.

(2) If $u(x_0) = 0$, denote $S = \{x \in \Omega\colon\ u(x) = 0\}$ and then for all $i = 1, \dots, N$, $D_i u = 0$ a.e. in S. Thus

$$D_i\Big(\frac{u}{v}\Big) = \frac{vD_iu - uD_iv}{v^2} = 0.$$

This finishes the proof of Theorem 1.1. □

Proof of Theorem 1.2: Expanding $R(u,v)$ by direct calculation one easily sees that $L(u,v) = R(u,v)$. Now the proof that $L(u,v)$ is nonnegative, is similar to proof of Theorem 1.1. We rewrite $L(u,v)$ as

$$L(u,v) = \mathcal{J}_{i,1} + \mathcal{J}_{i,2} + \mathcal{J}_{i,3} + \mathcal{J}_{i,4}, \tag{3.6}$$

where

$$\begin{aligned}
\mathcal{J}_{i,1} = {} & \sum_{i=1}^{N} p_i(x)\Big[\frac{1}{p_i(x)}|D_iu|^{p_i(x)} \\
& + \frac{p_i(x)-1}{p_i(x)}\Big(\frac{u^{p_i(x)-1}}{f(v)}|D_iv|^{p_i(x)-1}\Big)^{p_i(x)/(p_i(x)-1)}\Big] \\
& - \sum_{i=1}^{N} p_i(x)\frac{u^{p_i(x)-1}}{f(v)}|D_iv|^{p_i(x)-1}|D_iu|, \\
\mathcal{J}_{i,2} = {} & \sum_{i=1}^{N} p_i(x)\frac{u^{p_i(x)-1}}{f(v)}|D_iv|^{p_i(x)-2}[\,|D_iv||D_iu| - D_ivD_iu], \\
\mathcal{J}_{i,3} = {} & \sum_{i=1}^{N} -(p_i(x)-1)\Big(\frac{u^{p_i(x)-1}}{f(v)}|D_iv|^{p_i(x)-1}\Big)^{p_i(x)/(p_i(x)-1)} \\
& + \frac{u^{p_i(x)}f'(v)}{[f(v)]^2}|D_iv|^{p_i(x)}, \\
\mathcal{J}_{i,4} = {} & -\sum_{i=1}^{N} \frac{u^{p_i(x)}\ln u}{f(v)}|D_iv|^{p_i(x)-2}D_iv\cdot D_ip_i(x).
\end{aligned}$$

By Young's inequality, see Proposition 2.1, inequality (1.6), (here $a = |D_iu|$, $b = (u^{p_i(x)-1}/f(v))|D_iv|^{p_i(x)-1}$ and $q_i(x) = p_i(x)/(p_i(x)-1)$), and using the fact that $|D_iv||D_iu| - D_ivD_iu \geq 0$, we obtain

$$\mathcal{J}_{i,1} \geq 0 \qquad \text{and} \qquad \mathcal{J}_{i,2} \geq 0. \tag{3.7}$$

Since $f(v)>0$ and $f'(v)\geq (p_i(x)-1)[f(v)]^{(p_i(x)-2)/(p_i(x)-1)}$, we have

$$
\begin{aligned}
\mathcal{J}_{i,3} \geq\ & \sum_{i=1}^{N}(p_i(x)-1)\Big[-\Big(\frac{u^{p_i(x)-1}}{f(v)}|D_iv|^{p_i(x)-1}\Big)^{p_i(x)/(p_i(x)-1)} \\
& + \frac{u^{p_i(x)}[f(v)]^{(p_i(x)-2)/(p_i(x)-1)}}{[f(v)]^2}|D_iv|^{p_i(x)}\Big] = 0.
\end{aligned}
\tag{3.8}
$$

From (3.6)–(3.8) and the fact that $\mathcal{J}_{i,4}=0$ (since $D_iv\cdot D_ip_i(x)=0$), it follows that $L(u,v)\geq 0$. On the other hand, similar arguments as those used in the proof of Theorem 1.1 show that $L(u,v)=0$ a.e. in Ω if and only if $D_i(u/v)=0$ a.e. in Ω for any $i=1,\dots,N$. This completes the proof of Theorem 1.2. □

4. Some applications and examples

4.1 Applications. We give three examples of applications of Theorems 1.1–1.2. First we will show a sturmian comparison principle to the anisotropic elliptic equation with variable exponent, by Theorem 1.1.

Proposition 4.1. *Let $\lambda>0$ and $k(x)$ be a nonnegative weight. Suppose that a function $v\in C^2(\Omega)$ and $v>0$, satisfying*

$$
-\sum_{i=1}^{N}\frac{u^{p_i(x)}}{v^{p_i(x)-1}}D_i(|D_iv|^{p_i(x)}D_iv)\geq \sum_{i=1}^{N}\lambda k(x)u^{p_i(x)} \qquad \text{in } \Omega. \tag{4.1}
$$

Then for any $u\in C_0^1(\Omega)$ with $u\geq 0$, there holds

$$
\sum_{i=1}^{N}\int_\Omega |D_iu|^{p_i(x)}\,\mathrm{d}x \geq \lambda\sum_{i=1}^{N}\int_\Omega k(x)u^{p_i(x)}\,\mathrm{d}x. \tag{4.2}
$$

Proof: By Theorem 1.1 and (4.1), we obtain

$$
\begin{aligned}
0 &\leq \int_\Omega L(u,v)\,\mathrm{d}x = \int_\Omega R(u,v)\,\mathrm{d}x \\
&= \sum_{i=1}^{N}\int_\Omega |D_iu|^{p_i(x)}\,\mathrm{d}x - \sum_{i=1}^{N}\int_\Omega |D_iv|^{p_i(x)-2}D_iv\cdot D_i\Big(\frac{u^{p_i(x)}}{v^{p_i(x)-1}}\Big)\,\mathrm{d}x \\
&= \sum_{i=1}^{N}\int_\Omega |D_iu|^{p_i(x)}\,\mathrm{d}x + \sum_{i=1}^{N}\int_\Omega \frac{u^{p_i(x)}}{v^{p_i(x)-1}}D_i(|D_iv|^{p_i(x)-2}D_iv)\,\mathrm{d}x \\
&\leq \sum_{i=1}^{N}\int_\Omega |D_iu|^{p_i(x)}\,\mathrm{d}x - \lambda\sum_{i=1}^{N}\int_\Omega k(x)u^{p_i(x)}\,\mathrm{d}x,
\end{aligned}
$$

which gives (4.2). □

Let Ω be an open and bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a Lipschitz boundary $\partial\Omega$. Consider a nonnegative function f that belongs to $L^m(\Omega)$, where $m \geq 1$. We assume that the exponent variable $\gamma(\cdot)\colon \overline{\Omega} \to (0,\infty)$ is a continuous and smooth function, and that $p_i(\cdot)\colon \overline{\Omega} \to (1,\infty)$, $i = 1,\dots,N$, are continuous functions that satisfy the following conditions:

$$p_1(x) \leq p_2(x) \leq \cdots \leq p_N(x), \qquad \overline{p}(x) < N, \ \forall x \in \overline{\Omega},$$

where

$$\frac{1}{\overline{p}(x)} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i(x)}, \qquad \forall x \in \overline{\Omega}.$$

Let us now present the following proposition to provide an example of using Picon's inequality to demonstrate the uniqueness of the solution.

Proposition 4.2. *Let $u_1 \geq 0$, $u_2 \geq 0$ be two functions in $W_0^{1,p_i(\cdot)}(\Omega)$ which are an energy solution of the problem*

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} D_i(|D_i u|^{p_i(x)-2}D_i u) = \frac{f}{u^{\gamma(x)}} & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

Then $u_1 = u_2$.

Proof: The existence of the solution is proven by the author in [19]. Let u_1 and u_2 be two solutions of problem (4.3), suppose that $u_2 \geq u_1$. Using the anisotropic Picone identities in Theorem 1.1 to u_1 and u_2, it follows that

$$\begin{aligned} 0 \leq R(u,v) &= L(u,v)\\ &= \sum_{i=1}^{N}|D_i u|^{p_i(x)} - \sum_{i=1}^{N}|D_i v|^{p_i(x)-2}D_i v\cdot D_i\Big(\frac{u^{p_i(x)}}{v^{p_i(x)-1}}\Big). \end{aligned}$$

Then

$$\begin{aligned} \sum_{i=1}^{N}\int_\Omega |D_i\cdot u_2|^{p_i(x)}\,\mathrm{d}x &\geq \sum_{i=1}^{N}\int_\Omega D_i(|D_i\cdot u_1|^{p_i(x)-2}D_i\cdot u_1)\frac{u_2^{p_i(x)}}{u_1^{p_i(x)-1}}\\ &= \int_\Omega \frac{f}{u_1^{\gamma(x)}}\frac{u_2^{p_i(x)}}{u_1^{p_i(x)-1}} = \int_\Omega \frac{f u_2^{p_i(x)}}{u_1^{\gamma(x)+p_i(x)-1}}. \end{aligned} \tag{4.4}$$

On the other hand, we get

$$\sum_{i=1}^{N}\int_\Omega |D_i u_2|^{p_i(x)}\,\mathrm{d}x = \int_\Omega f u_2^{-\gamma(x)+1}. \tag{4.5}$$

Combining (4.4) and (4.5), we have

$$\int_\Omega f u_2^{-\gamma(x)+1} \geq \int_\Omega f u_2^{p_i(x)} u_1^{1-\gamma(x)-p_i(x)}.$$

Case 1. If $\gamma(x) = 1$ then

$$\int_\Omega f\Big[1 - \frac{u_2^{p_i(x)}}{u_1^{p_i(x)}}\Big]\,\mathrm{d}x \geq 0. \tag{4.6}$$

Since $u_2 \geq u_1$, we have $1 - u_2^{p_i(x)}/u_1^{p_i(x)} \leq 0$. By (4.6), we obtain

$$\int_\Omega f\Big[1 - \frac{u_2^{p_i(x)}}{u_1^{p_i(x)}}\Big]\,\mathrm{d}x = 0.$$

Thanks to Theorem 1.1, we get

$$-\sum_{i=1}^{N}\int_\Omega |D_i u_2|^{p_i(x)}\,\mathrm{d}x = \sum_{i=1}^{N}\int_\Omega D_i(|D_i u_1|^{p_i(x)-2} D_i u_1)\frac{u_2^{p_i(x)}}{u_1^{p_i(x)-1}},$$

which implies that

$$R(u_1,u_2) = L(u_1,u_2) = 0 \qquad \text{a.e. in } \Omega.$$

Then

$$D_i \cdot \Big(\frac{u_1}{u_2}\Big) = 0, \qquad \text{a.e. in } \Omega,\ \forall i = 1,\dots,N,$$

we obtain $u_1 = cu_2$ for some $c > 0$. Taking into consideration that u_1 and u_2 solve problem (4.3) it follows that $c_1 = 1$, and then $u_1 = u_2$.

Case 2. If $\gamma(x) \neq 1$ we have

$$\int_\Omega f u_2^{-\gamma(x)+1} \geq \int_\Omega f u_2^{p_i(x)} u_1^{1-\gamma(x)-p_i(x)},$$

which is equivalent to

$$\int_\Omega f\Big[1 - \frac{u_2^{p_i(x)}}{u_1^{p_i(x)}}\Big]\,\mathrm{d}x \geq 0. \tag{4.7}$$

In order to get (since $u_1 \leq u_2$)

$$\begin{aligned} 1 - \frac{u_1^{p_i(x)+\gamma(x)-1}}{u_2^{p_i(x)+\gamma(x)-1}} \leq 0 \;&\Leftrightarrow\; u_2^{p_i(x)+\gamma(x)-1} \leq u_2^{p_i(x)+\gamma(x)-1} \\ &\Leftrightarrow\; p_i(x) + \gamma(x) - 1 \geq 0 \end{aligned}$$

it suffices that $\gamma(x) \geq 1 - p_i(x)$, and that is true since $\gamma(x) \geq 0$ and $p_i(x) \geq 1$. This completes the proof. □

We finish with a theorem of Liouville for an anisotropic elliptic system with variable exponents. Let $v > 0$, $f(v) > 0$ and $f'(v) \geq (p_i(x)-1)[f(v)]^{(p_i(x)-2)/(p_i(x)-1)}$ for $p_i(x) \geq 1$ and for all $i = 1, \ldots, N$.

Proposition 4.3. *Let $g(u, v)$ be an integrable function in Ω. Assume that $(u, v) \in W_0^{1,p_i(\cdot)}(\Omega) \times W_0^{1,p_i(\cdot)}(\Omega)$ is a pair of solution to an anisotropic elliptic system*

$$\begin{cases} \sum_{i=1}^{N} |D_i u|^{p_i(x)} = g(u,v) & \text{in } \Omega, \\ \sum_{i=1}^{N} D_i\Big(\frac{u^{p_i(x)}}{f(v)}\Big)|D_i v|^{p_i(x)-2} D_i v = g(u,v) & \text{in } \Omega, \\ u > 0, \ v > 0 & \text{in } \Omega, \\ u = 0, \ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.8}$$

Then $u = cv$ a.e. in Ω for some constant c.

PROOF: By Theorem 1.2 and (4.8), we obtain

$$\begin{aligned} \int_\Omega L(u,v)\,\mathrm{d}x &= \int_\Omega R(u,v)\,\mathrm{d}x \\ &= \sum_{i=1}^{N} |D_i u|^{p_i(x)} - \sum_{i=1}^{N} D_i\Big(\frac{u^{p_i(x)}}{f(v)}\Big)|D_i v|^{p_i(x)-2} D_i v \\ &= g(u,v) - g(u,v) = 0. \end{aligned}$$

Thus, $u = cv$ a.e. in Ω for some constant c. □

4.2 Examples.

Example 1. Let Ω be a set contained in $\mathbb{R}^N$, and Ω_i be subsets of Ω such that $\Omega = \bigcup_{i=1}^N \Omega_i$ with $\Omega_i \cap \Omega_j = \emptyset$ for all $i \neq j$. Consider a function v defined on $\mathbb{R}^N$ with values in $\mathbb{R}$, given by

$$v(x) = \sum_{i=1}^{N} \chi_{\Omega_i}(x) x_i.$$

The characteristic function on the set Ω_i is defined as follows

$$\chi_{\Omega_i}(x) = \begin{cases} 1 & \text{if } x \in \Omega_i, \\ 0 & \text{otherwise.} \end{cases}$$

Define $p_i(x)$ as

$$p_i(x) = \begin{cases} 0 & \text{if } x \in \Omega_i, \\ x_i & \text{otherwise.} \end{cases}$$

We can easily verify that

$$D_i v(x) D_i p_i(x) = 0, \qquad \forall i = 1, \ldots, N.$$

For example, when $N = 3$, the function v is given by

$$v(x) = \begin{cases} x_1 & \text{if } x \in \Omega_1, \\ x_2 & \text{if } x \in \Omega_2, \\ x_3 & \text{if } x \in \Omega_3, \end{cases}$$

and

$$p_1(x) = \begin{cases} 0, & x \in \Omega_1 \\ x_1, & x \in \Omega_2 \\ x_1, & x \in \Omega_3 \end{cases}, \quad p_2(x) = \begin{cases} x_2, & x \in \Omega_1 \\ 0, & x \in \Omega_2 \\ x_2, & x \in \Omega_3 \end{cases}, \quad p_3(x) = \begin{cases} x_3 & x \in \Omega_1 \\ x_3, & x \in \Omega_2 \\ 0, & x \in \Omega_3 \end{cases}.$$

Then we obtain

$$D_1 v(x) \cdot D_1 p_1(x) = \begin{cases} 1 \cdot 0 = 0 & \text{if } x \in \Omega_1, \\ 0 \cdot 1 = 0 & \text{if } x \in \Omega_2, \\ 0 \cdot 1 = 0 & \text{if } x \in \Omega_3, \end{cases}$$

$$D_2 v(x) \cdot D_2 p_2(x) = \begin{cases} 0 \cdot 1 = 0 & \text{if } x \in \Omega_1, \\ 1 \cdot 0 = 0 & \text{if } x \in \Omega_2, \\ 0 \cdot 1 = 0 & \text{if } x \in \Omega_3, \end{cases}$$

and

$$D_3 v(x) \cdot D_3 p_3(x) = \begin{cases} 0 \cdot 1 = 0 & \text{if } x \in \Omega_1, \\ 0 \cdot 1 = 0 & \text{if } x \in \Omega_2, \\ 1 \cdot 0 = 0 & \text{if } x \in \Omega_3. \end{cases}$$

Then, $D_i v(x) D_i p_i(x) = 0$ for all $i = 1, \ldots, N$ and all $x \in \Omega$.

Example 2. Let v be a function defined on $\mathbb{R}^N$ with values in $\mathbb{R}$, given by

$$v(x) = \sum_{\substack{j=1 \\ j \neq i}}^{N} x_j,$$

and

$$p_i(x) = x_i^2 + 1, \qquad \forall i = 1, \ldots, N.$$

We can easily verify that

$$D_i v(x) = \begin{cases} 1, & j \neq i, \\ 0, & j = i. \end{cases}$$

On the other hand, we have

$$D_i p_i(x) = 2x_i + 1, \qquad \forall i = 1, \dots, N,$$

then,

$$D_i v(x) D_i p_i(x) = 0, \qquad \forall i = 1, \dots, N.$$

Let us consider the trivial cases, for $N = 3$, we obtain

$$\begin{aligned} v(x) &= x_2 + x_3 \ \text{ and } \ p_1(x) = x_1^2 + 1 \qquad \text{for } i = 1, \\ v(x) &= x_1 + x_3 \ \text{ and } \ p_2(x) = x_2^2 + 1 \qquad \text{for } i = 2, \\ v(x) &= x_1 + x_2 \ \text{ and } \ p_3(x) = x_3^2 + 1 \qquad \text{for } i = 3. \end{aligned}$$

Hence, $D_i v(x) D_i p_i(x) = 0$ for all $i = 1, 2, 3$ and for all $x \in \Omega$.

Example 3. Let $v\colon \mathbb{R}^N \to \mathbb{R}$ be a function defined by

$$v(x) = \prod_{\substack{j=1 \\ j \neq i}}^{N} x_j, \quad p_i(x) = \mathrm{e}^{x_i} + 1, \qquad \forall i = 1, \dots, N.$$

We can readily confirm that

$$D_i v(x) = \begin{cases} \displaystyle\prod_{\substack{l=1 \\ l \neq i}}^{N} x_l, & j \neq i, \\ 0, & j = i. \end{cases}$$

On the other hand, we observe that

$$D_i p_i(x) = \mathrm{e}^{x_i}, \qquad \forall i = 1, \dots, N,$$

then,

$$D_i v(x) D_i p_i(x) = 0, \qquad \forall i = 1, \dots, N, \ \forall x \in \Omega.$$

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