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GEODESIC METRICS FOR RBF APPROXIMATION OF SOME PHYSICAL QUANTITIES MEASURED ON SPHERE

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Dedicated to late Professor Ivo Babuška

Abstract. The radial basis function (RBF) approximation is a rapidly developing field of mathematics. In the paper, we are concerned with the measurement of scalar physical quantities at nodes on sphere in the 3D Euclidean space and the spherical RBF interpolation of the data acquired. We employ a multiquadric as the radial basis function and the corresponding trend is a polynomial of degree 2 considered in Cartesian coordinates. Attention is paid to geodesic metrics that define the distance of two points on a sphere. The choice of a particular geodesic metric function is an important part of the construction of interpolation formula.

We show the existence of an interpolation formula of the type considered. The approximation formulas of this type can be useful in the interpretation of measurements of various physical quantities. We present an example concerned with the sampling of anisotropy of magnetic susceptibility having extensive applications in geosciences and demonstrate the advantages and drawbacks of the formulas chosen, in particular the strong dependence of interpolation results on condition number of the matrix of the system considered and on round-off errors in general.

Keywords: spherical interpolation; radial basis function; spherical radial basis function; geodesic metric; trend; multiquadric; magnetic susceptibility

MSC 2020: 65D12, 65D05, 65D10, 65Z05

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1. Introduction

Problems formulated on spheres have brought an increasing interest of researchers through the development of measurements in many areas of science and technology. In many cases this interest is caused by the fact that some sorts of data are measured on the Earth surface. Satellites are often the means of acquiring such data. In the paper, we consider a general $(d-1)$-dimensional sphere in the d-dimensional Euclidean space. However, the examples presented are concerned with a usual 2-dimensional sphere in the 3-dimensional space.

This paper is concerned with some ways of approximating and mapping measured physical quantities exhibiting anisotropy that can be expressed by means of a second-order tensor. A typical example we are going to employ is the measurement of magnetic susceptibility of rock having extensive applications in geosciences [13]. In the paper, we treat the data interpolation with the help of spherical radial basis functions in such a case, see [7].

Let us recall the notion of a radial basis function (RBF).

Definition 1. A function $\sigma(x,y)$, $x,y \in \mathbb{R}^d$, is called *radial* if there exists a function $\tau(r)$, $r \geqslant 0$, such that $\sigma(x,y) = \tau(r)$, where $r \in \mathbb{R}$ is the distance between x and y.

The functions appearing in the RBF interpolation formula are the basis functions chosen as radial ones and the trends, see [2]. As radial basis functions are employed, an important role in constructing the formulas is played by functions called geodesic metrics that realize the use of real functions as radial.

There are several most often used geodesic metrics, see, e.g., [1], [3], [7], [8]. The principal aim of the paper is to compare their behavior in practical computation. We confine ourselves only to real-valued functions and real data in this paper to make the exposition simpler.

In the laboratory determination of raw data, see [6], [10], [13], the rock sample rotates in magnetic field in a set of selected (tens to hundreds) directions and the data items s_i measured are of the form

$$s_i = z_i^\top K z_i + e_i, \tag{1}$$

where z_i are the unit vectors in the ith direction of measurement in Cartesian coordinates, K is a tensor and e_i are deviations from the theoretical tensor model. Multiple measurements in a single direction z_i with different results lead to a singular linear algebraic system for coefficients of the interpolation formula.

Initial data for the approximation is a set of directions and corresponding susceptibilities $(z_i, s(z_i))$. Assuming equation (1), we carry out linear regression and find an estimate of the tensor K. Then, an appropriate rotation of the coordinate system can make the tensor K diagonal,

$$K = \begin{bmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_3 \end{bmatrix}, \tag{2}$$

where the entries K_1, K_2, K_3 are *principal susceptibilities*. These Cartesian coordinates are basically used for the description of the problem in what follows. We call the graphical representation of the directional susceptibilities (1) the *lemniscate surface,* see Fig. 1. The function s corresponding to (1) is taken for the trend in our considerations that follow.

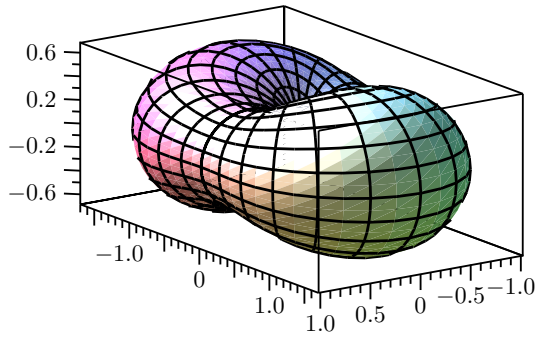


Figure 1. Lemniscate surface with $K_1 = 1.00$, $K_2 = 1.70$, $K_3 = 0.15$. Magnitude of the directional susceptibility in the ith direction z_i is given by the distance between the origin and surface measured along the vector z_i.

2. Spherical interpolation

Let $d > 0$ be the dimension of a real Euclidean space $\mathbb{R}^d$. We put

$$S^{d-1} = \left\{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d;\ \sum_{i=1}^{d} x_i^2 = 1 \right\}. \tag{3}$$

Then S^{d-1} is the $(d-1)$-*dimensional surface of a unit sphere* in the d-dimensional Euclidean space $\mathbb{R}^d$. We can consider its elements x as points on the unit sphere (3) or as vectors in $\mathbb{R}^d$ with the initial point at the center of the unit sphere and terminal point at the unit sphere. These vectors are according to (3) unit vectors.

Choose a positive integer N and a non-negative integer M, $N \geqslant M$. Given a set $X = \{X_j\}_{j=1}^N$ of mutually distinct nodes $X_j = (X_{j1}, X_{j2}, \ldots, X_{jd})$ on S^{d-1}, then a general formula for the *spherical interpolant* v has for $x \in S^{d-1}$ the form

$$v(x) = \sum_{j=1}^{N} a_j \psi(g(x, X_j)) + \sum_{k=1}^{M} b_k p_k(x), \tag{4}$$

where a_j, $j = 1, \ldots, N$, and b_k, $k = 1, \ldots, M$, are real coefficients to be found. If $M = 0$, the second sum in (4) is empty. The nonnegative function $g\colon\ S^{d-1} \times S^{d-1} \to [0, \infty)$ in (4) is the *geodesic metric* and $g(x, y) = g(y, x)$ for $x, y \in S^{d-1}$. Further, $\psi\colon\ [0, \infty) \to \mathbb{R}$ is a continuous real *spherical basis function* (SBF) or *spherical radial basis function* (SRBF), see Definition 1.

Finally, the functions $p_k\colon\ \mathbb{R}^d \to \mathbb{R}$ belong to the set $\Pi_t(\mathbb{R}^d)$ of all polynomials (called *trends*) with real coefficients and of total degree less then or equal to a nonnegative integer t. Trends are used to represent some global features of the interpolant v. Their derivatives of order $t + 1$ and higher vanish, which may influence their $\mathcal{L}$-norms. Let us formulate the interpolation problem to be solved, see [7], [9]. In the definition of interpolation problem we use the notation from [12].

Given a continuous real target function $f\colon\ S^{d-1} \to \mathbb{R}$, find the spherical interpolant (4), i.e., a continuous function $v\colon\ S^{d-1} \to \mathbb{R}$ that satisfies the *interpolation conditions*

$$v(X_i) = f_i, \quad i = 1, \ldots, N, \tag{5}$$

where $f_i = f(X_i)$ is the value measured at X_i. We use the SRBF interpolation formula (4) with a proper geodesic metric g, spherical radial basis function ψ, and trends $p_k \in \Pi_t(\mathbb{R}^d)$, $k = 1, \ldots, M$.

Let us employ the matrix notation. We substitute X_i, $i = 1, \ldots, N$, for x in formula (4) to get

$$v(X_i) = \sum_{j=1}^{N} a_j \psi(g(X_i, X_j)) + \sum_{k=1}^{M} b_k p_k(X_i), \quad i = 1, \ldots, N, \tag{6}$$

with a geodesic metric g, spherical radial basis function ψ, and trends $p_k \in \Pi_t(\mathbb{R}^d)$, $k = 1, \ldots, M$, and replace the left hand parts $v(X_i)$ of the interpolation conditions (5) with expressions (6).

Introduce an $N \times N$ symmetric matrix Ψ with the entries

$$\psi_{ij} = \psi(g(X_i, X_j)), \quad i, j = 1, \ldots, N, \tag{7}$$

and an $N \times M$ matrix P with the entries

$$p_{jk} = p_k(X_j), \quad j = 1, \ldots, N,\ k = 1, \ldots, M.$$

Moreover, we denote by $a \in \mathbb{R}^N$, $b \in \mathbb{R}^M$, and $f \in \mathbb{R}^N$ the vectors of the unknowns and the vector of the right hand parts $f(X_i)$ of the interpolation conditions (5), respectively.

Note that if $M > 0$, then we have only N interpolation conditions (5) for $N + M$ interpolation coefficients a_j and b_k in formula (4). Then it is usual to impose M additional linear constraints for the individual trends p_k (see, e.g., [7])

$$\sum_{j=1}^{N} a_j p_k(X_j) = \sum_{j=1}^{N} a_j p_{jk} = 0, \quad k = 1, \ldots, M. \tag{8}$$

Now the system of linear algebraic equations to be solved for the unknown vectors a and b consists of (5) and (8). Putting

$$U = \begin{bmatrix} \Psi & P \\ P^\top & 0 \end{bmatrix}, \tag{9}$$

which is a symmetric $(N + M) \times (N + M)$ matrix of system (9), we arrive at

$$U \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}. \tag{10}$$

We have formulated the general spherical interpolation problem. Apparently, the problem possesses the unique solution if and only if the matrix U of system (10) is nonsingular. We employ some conditions guaranteeing that U is nonsingular.

Theorem 1. *Let the $N \times N$ principal submatrix Ψ of the $(N + M) \times (N + M)$ matrix U introduced in* (9) *be symmetric positive definite and let* rank $P = M$. *Then the matrix U is nonsingular.*

P r o o f. See [4], [5], and the proof of Theorem 1 in [11] and Theorem 1 in [12]. □

R e m a r k 1. The simplest spherical interpolant v can be considered if we omit the second sum in (4), i.e., we set $M = 0$ and employ no trends. Then there is no matrix P in the formulation, $U = \Psi$ and, instead of (10), we get the $N \times N$ symmetric positive definite system

$$\Psi a = f. \tag{11}$$

Apparently, system (11) possesses the unique solution a.

R e m a r k 2. In Theorem 1, we use the hypothesis that the matrix Ψ is symmetric positive definite and rank $P = M$. However, most books and papers (see, e.g., [2], [7], [9]) employ the condition that the spherical basis function ψ is conditionally (strictly) positive definite to prove that the matrix U is nonsingular (see, e.g., [7]). The verification of any of these conditions in practical computation is about of the same difficulty as the determination of the rank of the matrix P.

A problem similar to data interpolation is data smoothing (fitting). We are not concerned with this subject in the paper.

3. Geodesic metric

The *geodesic metric* is a nonnegative real function g of two variables introduced in Section 2. As we employ a radial basis function in the spherical interpolant (4), we are supposed to define the distance between two nodes x and y on the unit sphere S^{d-1}. Consider both x and y to be unit vectors (see Section 2) in $\mathbb{R}^d$. Then the angle α of these two vectors is given as

$$\cos\alpha = x \cdot y, \tag{12}$$

where $x \cdot y$ is the inner product of two vectors. Since $\cos\alpha = \cos(2\pi - \alpha)$, we can choose for computation either the angle α or its complement to 2π, i.e. $2\pi - \alpha$.

Several particular geodesic metrics are frequently used in practice. The simplest one is the angle α itself, i.e.

$$g_0(x, y) = \frac{\alpha}{2\pi} = \frac{\cos^{-1}(x \cdot y)}{2\pi},$$

where

$$g_0 \colon S^{d-1} \times S^{d-1} \to [0, 1],$$

see, e.g., [1], [7], [8].

We introduce further two geodesic metrics based on $\cos\alpha$, $\alpha \in [0, 2\pi]$, and providing the range $[0, 1]$, i.e.

$$g \colon S^{d-1} \times S^{d-1} \to [0, 1]. \tag{13}$$

We put

$$g_1(x, y) = \sqrt{1 - \cos^2\alpha} = |\sin\alpha|, \tag{14}$$

where $\cos\alpha$ is given by (12), see [12], and

$$g_2(x, y) = \sqrt{\tfrac{1}{2}(1 - \cos\alpha)}. \tag{15}$$

R e m a r k 3. Note that the geodesic metric g_1 is periodic in α with the period π. It holds that $g_1(x, y) = 0$ for $\cos\alpha = \pm 1$, i.e. for $\alpha = 0$ and π.

Central symmetry of the data measured is expected when we apply the geodesic metric g_1. Each unit vector x is considered as a part of an axis coming through the center of the sphere and from its two possible directions no direction is prescribed. Our quantity measured (magnetic susceptibility of rock) is just of this kind, it depends only on the axis (with no direction chosen). If the angle α of two vectors x and y equals π (i.e., $y = -x$), then the values measured on the sphere at x and y should be identical since the nodes x and $-x$ of interpolation are not distinguished.

R e m a r k 4. No symmetry of data measured is supposed when we employ the geodesic metric g_2. Moreover, the geodesic metric g_2 is periodic in α with the period 2π, and it holds that $g_2(x, y) = 1$ for $\alpha = \pi$, and $g_2(x, y) = 0$ for $\alpha = 0$.

4. Interpolation of magnetic susceptibility measured

As we have mentioned in the introduction, the particular physical quantity whose sampled values are approximated by the means presented is, like in [12], magnetic susceptibility. Put $d = 3$, then S^2 is the usual two-dimensional unit sphere in the three-dimensional space. Choose a fixed positive integer N and put $M = 1$. Consider the interpolation formula (4) in the form

$$v(x) = \sum_{j=1}^{N} a_j \psi(g(x, X_j)) + bs(x), \tag{16}$$

where $x, X_j \in S^2$, i.e., P is a single column N-vector, and b and 0 are scalars in (9), (10).

The interpolation conditions (6) now read

$$v(X_i) = \sum_{j=1}^{N} a_j \psi(g(X_i, X_j)) + bs(X_i), \quad i = 1, \ldots, N. \tag{17}$$

Moreover, we add a single constraint

$$\sum_{j=1}^{N} a_j s(X_j) = 0 \tag{18}$$

corresponding to (8).

To define a SRBF formula (16) uniquely, we have to choose a geodesic metric g, spherical basis function ψ, and trend s. For $x, y \in S^2$ (both x and y are unit vectors) we employ both geodesic metrics g_1 and g_2 defined in (14) and (15).

When using the geodesic metric g_1, we consider the angle between vectors of parallel directions to be zero regardless of their orientation. Similarly, we take into account always the acute angle α of the vectors x, y.

The metric g_1 does not distinguish the vectors x and $-x$. Therefore, when g_1 is concerned, we assume that the elements X_j of the set X are mutually distinct and, moreover, that it is

$$X_j \neq \pm X_i \quad \text{for every } i, j = 1, \ldots, N. \tag{19}$$

We have chosen the *multiquadric*

$$\psi(r) = \sqrt{r^2 + c^2} \tag{20}$$

for the spherical radial basis function, where $r \in [0, 1]$ (the range of the function g) and c is a positive shape parameter that controls tension of the interpolation surface.

Finally, we take the second degree polynomial (1),

$$s(x) = K_1 x_1^2 + K_2 x_2^2 + K_3 x_3^2, \quad x = (x_1, x_2, x_3) \in S^2, \tag{21}$$

where K_1, K_2, K_3 are the positive constants from (2), for the trend.

We thus have completely described the linear algebraic system (10) and its matrix U from (9). Notice that the argument of the SRBF function ψ is from the interval $[0, 1]$ according to (13), while the argument of the trend s is from S^{d-1} like in [9].

The advantage of the formula proposed is apparent in cases when we know that the physical field measured does not principally differ from the ideal field whose values can be computed from an explicit formula, e.g. from (1). This description of an ideal field is then fitted by the trend part of the formula and the contributions obtained from the first, spherical part of the formula, are only small.

Let us verify the existence of formula (16). We claim that the matrix Ψ corresponding to (20) is symmetric positive definite.

Lemma 1 ([9], p. 20)**.** *The $N \times N$ symmetric matrix Ψ with entries*

$$\psi_{ij} = (r_{ij}^2 + c^2)^\beta,$$

where $r_{ij} = g_k(X_i, X_j)$ for $k = 1, 2$, $c > 0$, and $\beta > 0$, is symmetric positive definite.

Consider the interpolation formula (16) with the functions g, ψ, and s given by the formulae (14) or (15), (20), and (21), respectively. Choose positive constants c, K_1, K_2, K_3. For the interpolation formula (16), set up system (17) corresponding to the interpolation conditions (5) and equation (18) corresponding to the single constraint (8).

Theorem 2. *Let system* (10) *correspond to formula* (16). *If* g_1 *is the geodesic metric considered, let* (19) *hold. Let the block* P *in the block matrix* U *given by* (9) *have rank* 1. *Then the interpolation problem* (17), (18) *has the unique solution, where the coefficients* a_j, $j = 1, \ldots, N$, *and* b *solve uniquely the linear algebraic system* (10).

P r o o f. According to Lemma 1, the principal submatrix Ψ of the block matrix U of system (10) is symmetric positive definite. On the assumption that rank $P = 1$, the matrix U is nonsingular by Theorem 1 and system (10) has the unique solution a_j, $j = 1, \ldots, N$, and b. □

R e m a r k 5. P is a single column N-vector, $P^{\mathrm{T}} = (s(X_1), \ldots, s(X_N))$. The assumption of Theorem 2 that rank $P = 1$ is apparently fulfilled if at least one of the entries $s(X_k)$ is nonzero.

R e m a r k 6. The authors in [7] present, under some particular assumptions on the set X of nodes and the function ψ in the interpolation formula, a statement (Proposition 2.1) claiming that the formula $v(x)$ given by (4) minimizes a variational problem in a specific norm and satisfies the interpolation conditions (5).

5. Computational experiments

We present our numerical experience with the interpolation problem described in Section 4. According to Lemma 1, the matrix Ψ with entries (7) employing the multiquadric is symmetric positive definite and the matrix U introduced in (9) is nonsingular when the matrix P has rank $P = 1$. But the use of the lemniscate s given by (21) does not prevent difficulties in solving the linear algebraic system (10) which is for higher N ill-conditioned (for definition of the *condition number*, see, e.g., [5]).

The choice of the grid of interpolation nodes X_j is usually the first step of processing. Measurement devices usually possess their own fixed set (sets) of interpolation nodes (directions), some of them accept arbitrary sets of directions. When we use the geodesic metric g_1, the nodes X_j have to satisfy condition (19), see Remark 3.

The simplest grid, when the geodesic metric g_2 is used, is the equidistant one in the spherical coordinates ϑ and φ. For g_1, we can use the same grid but with the interpolation nodes X_j only on the north (upper) or south (lower) "hemisphere" to satisfy condition (19). System (10) can be easily solved for N of order tens, but for higher N the computation of its solution even in double precision may be hard and some particular means should be employed.

We have tried also some other grids mentioned in the literature but no one seemed exceptionally advantageous. We have also experimented with a regular icosahedron (having 20 triangular faces) inscribed in a unit sphere. The advantage could be the possibility to refine this regular triangular grid regularly, but no benefit has appeared.

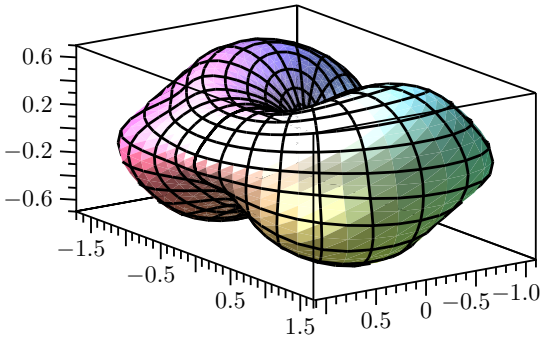


Figure 2. Lemniscate surface with $K_1 = 1.00$, $K_2 = 1.70$, $K_3 = 0.15$ like in Fig. 1. The exact sampled values were perturbed, i.e. multiplied by a random factor σ from the range $[0.85, 1.15]$. The geodesic metric g_1, symmetric grid and data, $N = 40$, $c^2 = 0.25$.

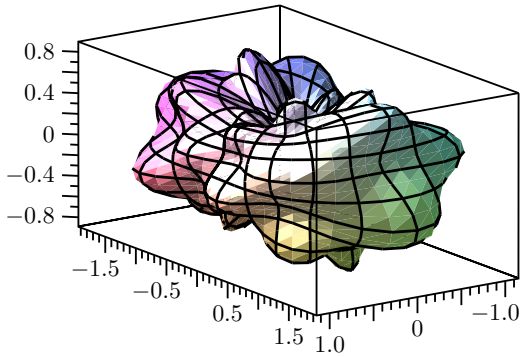


Figure 3. Lemniscate surface with $K_1 = 1.00$, $K_2 = 1.70$, $K_3 = 0.15$ like in Fig. 1. The exact sampled values were perturbed, i.e. multiplied by a random factor σ from the range $[0.999, 1.001]$. The geodesic metric g_2, $N = 74$, $c^2 = 0.25$.

In figures, we wish to show that the choice of c in the RBF cannot be arbitrary. In the metric g_1 and a wide range of perturbations, the value $c^2 = 0.25$ works well. On the other hand, in the metric g_2, it was necessary to confine range of the perturbation factor significantly to get an acceptable solution of the problem. We have also tried several other radial basis functions employed in [2], [7], [9].

6. Conclusion

We have shown sufficient conditions for the existence of a SRBF interpolant. We have considered a particular SRBF interpolation formula employing a multiquadric. The only trend function used in the formula is the second degree polynomial (21) of Section 4.

We have carried out some numerical tests with this interpolation formula whose results are summed up in Section 5. The formula performs efficiently only for a relatively small number N of interpolation nodes $X_j \in X$ given and the results exhibit week dependence on the parameter c in the case of the geodesic metric g_1, but strong dependence in the case of the geodesic metric g_2. Further research shall provide a comparison of results obtained using various other SRBFs leading to the positive definite matrix Ψ, e.g. the inverse multiquadric $1/\sqrt{r^2+c^2}$, Gaussian function $\exp(-cr^2)$ or also RBFs with compact support [2], [7], [9].

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References

[1] *B. J. Baxter, S. Hubbert*: Radial basis functions for the sphere. Recent Progress in Multivariate Approximation. International Series of Numerical Mathematics 137. Birkhäuser, Basel, 2001, pp. 33–47. doi

[2] *M. D. Buhmann*: Radial Basis Functions: Theory and Implementations. Cambridge Monographs on Applied and Computational Mathematics 12. Cambridge University Press, Cambridge, 2003. zbl MR doi

[3] *J. Duchon*: Sur l'erreur d'interpolation des fonctions de plusieurs variable par les D^m-splines. RAIRO, Anal. Numér. *12* (1978), 325–334. (In French.) zbl MR doi

[4] *M. Fiedler*: Special Matrices and Their Applications in Numerical Mathematics. Martinus Nijhoff Publishers, Dordrecht, 1986. zbl MR

[5] *G. H. Golub, C. F. Van Loan*: Matrix Computations. Johns Hopkins University Press, Baltimore, 1996. zbl MR
[6] *F. Hrouda, J. Ježek, M. Chadima*: On the origin of apparently negative minimum susceptibility of hematite single crystals calculated from low-field anisotropy of magnetic susceptibility. Geophys. J. Int. *224* (2021), 1905–1917. doi
[7] *S. Hubbert, Q. T. Lê Gia, T. M. Morton*: Spherical Radial Basis Functions: Theory and Applications. SpringerBriefs in Mathematics. Springer, Cham, 2015. zbl MR doi
[8] *J. Levesley, Z. Luo, X. Sun*: Norm estimates of interpolation matrices and their inverses associated with strictly positive definite functions. Proc. Am. Math. Soc. *127* (1999), 2127–2134. zbl MR doi
[9] *C. A. Micchelli*: Interpolation of scattered data: Distance matrices and conditionally positive definite functions. Constr. Approx. *2* (1986), 11–22. zbl MR doi
[10] *T. Nagata*: Rock Magnetism. Maruzen, Tokyo, 1961.
[11] *K. Segeth*: Some computational aspects of smooth approximation. Computing *95* (2013), S695–S708. MR doi
[12] *K. Segeth*: Spherical radial basis function approximation of some physical quantities measured. J. Comput. Appl. Math. *427* (2023), Article ID 115128, 7 pages. zbl MR doi
[13] *D. Tarling, F. Hrouda*: Magnetic Anisotropy of Rocks. Chapman and Hall, London, 1993.

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