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PROPERTIES OF QUANTUM LOGIC MAPS AS FUZZY RELATIONS ON A SET OF ALL SYMMETRIC AND IDEMPOTENT BINARY MATRICES

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A quantum logic is one of possible mathematical models for non-compatible random events. In this work we solve a problem proposed at the conference FSTA 2006. Namely, it is proved that s-maps are symmetric fuzzy relations on a set of all symmetric and idempotent binary matrices. Consequently s-maps are not antisymmetric fuzzy relations. This paper also explores other properties of s-maps, j-maps and d-maps. Specifically, it is proved that s-maps are neither reflexive, nor irreflexive, and nor transitive, j-maps have the same properties as s-maps and d-maps are reflexive and not transitive.

Keywords: quantum logic, s-map, fuzzy relations

Classification: 03E72, 03G12

1. INTRODUCTION

In 2006, at the 8th Conference on Fuzzy Set Theory and Applications, a number of problems related to the classification of strict t-norms, Lipschitz t-norms, interval semi-groups, copulas, semicopulas and quasicopulas, fuzzy implications, mean values, fuzzy relations, MV-algebra, and effect algebras were raised [3]. This work offers a solution for the problem of s-maps as fuzzy relations, which was introduced in this conference. Namely, it is proved that s-maps are symmetric fuzzy relations on a set of all symmetric and idempotent binary matrices. This paper also explores properties of s-maps, j-maps and d-maps.

The paper is structured in the following way: in the Section 2 the initial problem is given; we study the properties of s-maps in Section 3; we propose the solution for the initial problem in Section 4; in Section 5 and 6, we examine j-maps and d-maps on a set of all symmetric and idempotent binary matrices.

2. THE GIVEN PROBLEM

Let $\mathcal{A}$ be a set of all symmetric matrices, such that each element of a matrix is either 0 or 1 and for each $A \in \mathcal{A}$ we have $A \cdot A = A$. Let I be the identity matrix and let Θ be the zero matrix.

Definition 2.1. (Nánásiová and Pulmannová [5]) A fuzzy relation $s : \mathcal{A} \times \mathcal{A} \rightarrow [0, 1]$ is called an s-map, if:

1. $s(I, I) = 1$;
2. if $A, B \in \mathcal{A}$ such that $A \cdot B = B \cdot A = \Theta$, then $s(A, B) = 0$;
3. if $A, B \in \mathcal{A}$ such that $A \cdot B = B \cdot A = \Theta$, then for all $C \in \mathcal{A}$:
 - $s(A + B, C) = s(A, C) + s(B, C)$,
 - $s(C, A + B) = s(C, A) + s(C, B)$.

Let us formulate the main theorem that is ought to be proven:

Theorem 2.2. (Mesiar and Klement [3]) If the fuzzy relation $s : \mathcal{A} \times \mathcal{A} \rightarrow [0, 1]$ is an s-map, then it is symmetric, i. e., $s(A, B) = s(B, A)$ for all $A, B \in \mathcal{A}$.

3. PROPERTIES OF S-MAPS

From the given conditions, we will formulate the properties of the s-map, which will help us to solve the symmetry problem and understand the given fuzzy relation.

Let us prove that the s-map of any matrix from set $\mathcal{A}$ and the identity matrix is symmetric:

Lemma 3.1. (Al-Adilee and Nánásiová [1]) If the fuzzy relation s is an s-map, then for all $A \in \mathcal{A} : s(A, I) = s(I, A)$.

Proof. For a given matrix A , put $B = I - A$. Then $A + B = I$, $A \cdot B = B \cdot A = \Theta$ and using of 2. of Definition 2.1, we compute $s(A, I) = s(A, A) = s(I, A)$. $\square$

Definition 3.2. A set of matrices $\mathcal{B} \subset \mathcal{A}$ is called a null set if for all $A, B \in \mathcal{B}$ and $A \neq B : s(A, B) = s(B, A) = 0$, where s is an s-map.

Let us prove that s-maps have the distributive property over addition:

Lemma 3.3. If the fuzzy relation s is an s-map, $H_1, H_2, \dots, H_i \in \mathcal{B}$ ($i \in \mathbb{N} \setminus \{1\}$), then for all $A \in \mathcal{A}$:

$$s(H_1 + H_2 + \dots + H_i, A) = s(H_1, A) + s(H_2, A) + \dots + s(H_i, A),$$

$$s(A, H_1 + H_2 + \dots + H_i) = s(A, H_1) + s(A, H_2) + \dots + s(A, H_i).$$

Proof. The proof follows from 2. of Definition 2.1 by induction. $\square$

Next we will look at the properties of s-maps to examine them more closely. We will start by proving that s-maps are not reflexive or irreflexive fuzzy relations.

Remark 3.4. If a fuzzy relation s is an s-map, then it is neither reflexive nor irreflexive.

Namely, $s(\Theta, \Theta) = 0$ and $s(I, I) = 1$ which contradicts the definitions of reflexivity and irreflexivity [2], respectively.

Now let us prove that s-maps are not transitive.

Theorem 3.5. If a fuzzy relation s is an s-map, then it is not transitive.

Proof. Suppose we are given an s-map of s matrices of order n . We find two matrices $A, B \in \mathcal{A}$ such that $A \neq \Theta$ and $B \neq \Theta$, and $A \cdot B = B \cdot A = \Theta$. We assume that the s-map is transitive. Then by the definition of transitivity [2] it follows that: $s(A, B) \wedge s(B, C) \leq s(A, C)$ for all $A, B, C \in \mathcal{A}$. Therefore:

$$s(A, I) \wedge s(I, B) \leq s(A, B).$$

But since $s(A, B) = s(B, A) = 0$, it follows that:

$$s(A, I) \wedge s(I, B) \leq 0$$

Hence, one of the values $s(A, I)$ or $s(I, B)$ is equal to 0.

Assume that $s(A, I) = 0$ (An analogous proof if one chooses that $s(I, B) = 0$). But then from Lemma 3.1 it follows that:

$$s(A, I) = s(A, A) = 0.$$

But in that case, the only matrix for which $s(A, A) = 0$ is satisfied is the zero matrix, i.e. $A = \Theta$. But that contradicts our initial assumption. $\square$

4. SYMMETRY OF S-MAPS

It is easy to see that the only matrices which belong to set $\mathcal{A}$ are diagonal. To make it more convenient to write the matrices, let's denote them respectively by tuples of order n , which consist of the elements of the main diagonal of matrices from the set $\mathcal{A}$. In general:

$$\begin{pmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{pmatrix} \Leftrightarrow (a_{11}, a_{22}, \dots, a_{nn}) \Leftrightarrow (a_1, a_2, \dots, a_n).$$

Now we will define the terms base matrices, basis set and explore their properties.

Definition 4.1. A matrix $C \in \mathcal{A}$ is called a base matrix if only one of its elements is 1.

The set of all possible basis matrices $\Delta \subset \mathcal{A}$ will be called a basis set:

$$\Delta = \{C_1 = (1, 0, \dots, 0), C_2 = (0, 1, \dots, 0), \dots, C_n = (0, 0, \dots, 1)\}.$$

It is easy to see that every matrix A from set $\mathcal{A}$ can be expressed as the sum of base matrices. In fact, if $A = (a_1, a_2, \dots, a_n)$, then $A = a_1 \cdot C_1 + a_2 \cdot C_2 + \dots + a_n \cdot C_n$.

Lemma 4.2. The basis set $\Delta \subset \mathcal{A}$ is a null set.

Proof. We choose arbitrary matrices $A, B \in \Delta$, $A \neq B$. Then, for an arbitrary index i , if $a_i = 1$ then $b_i = 0$ and vice versa, $b_i = 1$ implies $a_i = 0$. Therefore $A \cdot B = B \cdot A = \Theta$ and this gives Δ is a null set. $\square$

Now we have defined and proved everything necessary for the solution of the problem. Let us prove that from the given conditions, it follows that the s-map is always symmetric on a set of symmetric binary matrices.

Theorem 4.3. If the fuzzy relation s is an s-map, then $s(A, B) = s(B, A)$ for all $A, B \in \mathcal{A}$.

Proof. We consider the s-map $s : \mathcal{A} \times \mathcal{A} \rightarrow [0, 1]$ and choose matrices $A, B \in \mathcal{A}$, which are of order n . We divide the matrix A into the sum of the basis matrices:

$$A = a_1 \cdot C_1 + a_2 \cdot C_2 + \dots + a_n \cdot C_n = H_1 + H_2 + \dots + H_j = \sum_{t=1}^j H_t \quad (1 \leq j \leq n),$$

where H_t are the basis matrices for which $a_i = 1$ ($i = 1, 2, \dots, n$).

$$s(A, B) = s\left(\sum_{t=1}^j H_t, B\right).$$

We use Lemma 3.1:

$$s\left(\sum_{t=1}^j H_t, B\right) = \sum_{t=1}^j s(H_t, B).$$

Next, we divide matrix B into the sum of base matrices. As with the matrix A , let's denote by G_k ($1 \leq k \leq n$) those base matrices for which $b_i = 1$ ($i = 1, 2, \dots, n$).

$$B = b_1 \cdot C_1 + b_2 \cdot C_2 + \dots + b_n \cdot C_n = G_1 + G_2 + \dots + G_k = \sum_{m=1}^k G_m.$$

Again using Lemma 3.1 we get:

$$\sum_{t=1}^j s(H_t, B) = \sum_{t=1}^j s\left(H_t, \sum_{m=1}^k G_m\right) = \sum_{t=1}^j \sum_{m=1}^k s(H_t, G_m).$$

Consider an arbitrary s-map $s(H_i, G_r)$, where $1 \leq i \leq j$ and $1 \leq r \leq k$.

- if $H_i = G_r$, then $s(H_i, G_r) = s(G_r, H_i) = s(H_i, H_i) = s(G_r, G_r)$;
- if $H_i \neq G_r$, then $s(H_i, G_r) = s(G_r, H_i) = 0$.

Hence:

$$\begin{aligned} s(A, B) &= \sum_{t=1}^j \sum_{m=1}^k s(H_t, G_m) = \sum_{t=1}^j \sum_{m=1}^k s(G_m, H_t) = \\ \sum_{m=1}^k \sum_{t=1}^j s(G_m, H_t) &= \sum_{m=1}^k s(G_m, \sum_{t=1}^j H_t) = s\left(\sum_{m=1}^k G_m, \sum_{t=1}^j H_t\right) = s(B, A). \end{aligned}$$

□

Consequences: If a fuzzy relation s is an s -map, then it is not antisymmetric.

Example 4.4. For matrices of order n , the fuzzy relation $s(A, B) = \frac{tr(A \cdot B)}{n}$ is an s -map (see Table 1). Note that in this and the following examples $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

$s(A, B)$	Θ	A	B	I
Θ	0	0	0	0
A	0	$\frac{1}{2}$	0	$\frac{1}{2}$
B	0	0	$\frac{1}{2}$	$\frac{1}{2}$
I	0	$\frac{1}{2}$	$\frac{1}{2}$	1

Tab. 1. S-map values for 2×2 matrices.

5. PROPERTIES OF J-MAPS

Let us start with the definition of a j -map.

Definition 5.1. (Al-Adilee and Nánásiová [1]) A fuzzy relation $q : \mathcal{A} \times \mathcal{A} \rightarrow [0, 1]$ is called a j -map, if:

1. $q(\Theta, \Theta) = 0$ and $q(I, I) = 1$;
2. if $A, B \in \mathcal{A}$ such that $A \cdot B = B \cdot A = \Theta$, then $q(A, B) = q(A, A) + q(B, B)$;
3. if $A, B \in \mathcal{A}$ such that $A \cdot B = B \cdot A = \Theta$, then for all $C \in \mathcal{A}$:
 - $q(A + B, C) = q(A, C) + q(B, C) - q(C, C)$,
 - $q(C, A + B) = q(C, A) + q(C, B) - q(C, C)$.

It is easy to see from the second property of j -maps, that j -maps are symmetric for all matrices in the null set.

Lemma 5.2. (Al-Adilee and Nánásiová [1]) If $A, B \in \mathcal{A}$ such that $A \cdot B = B \cdot A = \Theta$, then $q(A + B, A + B) = q(A, B)$.

Proof. The proof follows from 3. of Definition 5.1. □

It is easy to see, that j-maps are neither reflexive nor irreflexive nor transitive. The proofs are very similar to the ones in chapter "Properties of s-maps". Now we are going to prove that j-maps are symmetric for all $A, B \in \mathcal{A}$.

Theorem 5.3. If the fuzzy relation q is a j-map, then $q(A, B) = q(B, A)$ for all $A, B \in \mathcal{A}$.

Proof. We consider the j-map $q : \mathcal{A} \times \mathcal{A} \rightarrow [0, 1]$. We can then write matrices A and B which are of order n as sum of their respective base matrices:

$$A = \sum_{t=1}^j H_t \quad (1 \leq j \leq n) \quad \text{and} \quad B = \sum_{k=1}^m G_k \quad (1 \leq m \leq n).$$

Hence from 3. of Definition 5.1:

$$\begin{aligned} q(A, B) &= q\left(\sum_{t=1}^j H_t, \sum_{k=1}^m G_k\right) = \sum_{t=1}^j q\left(H_t, \sum_{k=1}^m G_k\right) - (j-1) \cdot q\left(\sum_{k=1}^m G_k, \sum_{k=1}^m G_k\right) \\ &= \sum_{t=1}^j \sum_{k=1}^m q(H_t, G_k) - (m-1) \cdot \sum_{t=1}^j q(H_t, H_t) - (j-1) \cdot q\left(\sum_{k=1}^m G_k, \sum_{k=1}^m G_k\right). \end{aligned}$$

Then we can apply Lemma 5.2 and 3. of Definition 5.1 to get:

$$q\left(\sum_{k=1}^m G_k, \sum_{k=1}^m G_k\right) = \sum_{k=1}^m q(G_k, G_k).$$

We can acknowledge that $q(H_t, G_k) = q(G_k, H_t)$ for all base matrices, where $t = 1, 2, \dots, j$ and $k = 1, 2, \dots, m$. Hence we can conclude that:

$$\begin{aligned} q(A, B) &= \sum_{t=1}^j \sum_{k=1}^m q(H_t, G_k) - (m-1) \cdot \sum_{t=1}^j q(H_t, H_t) - (j-1) \cdot \sum_{k=1}^m q(G_k, G_k) \\ &= \sum_{t=1}^j \sum_{k=1}^m q(G_k, H_t) - (m-1) \cdot \sum_{t=1}^j q(H_t, H_t) - (j-1) \cdot \sum_{k=1}^m q(G_k, G_k) = q(B, A). \end{aligned}$$

□

$q(A, B)$	Θ	A	B	I
Θ	0	$\frac{1}{2}$	$\frac{1}{2}$	1
A	$\frac{1}{2}$	$\frac{1}{2}$	1	1
B	$\frac{1}{2}$	1	$\frac{1}{2}$	1
I	1	1	1	1

Tab. 2. J-map values for 2×2 matrices.

Example 5.4. For matrices of order n , the fuzzy relation $q(A, B) = \frac{\sum_{i=1}^n \max(a_i, b_i)}{n}$ is a j-map (see Table 2).

6. PROPERTIES OF D-MAPS

Definition 6.1. (Al-Adilee and Nánásiová [1]) A fuzzy relation $d : \mathcal{A} \times \mathcal{A} \rightarrow [0, 1]$ is called a d-map, if:

1. $d(\Theta, I) = d(I, \Theta) = 1$;
2. $d(A, A) = 0$ for all $A \in \mathcal{A}$
3. if $A, B \in \mathcal{A}$ such that $A \cdot B = B \cdot A = \Theta$, then $d(A, B) = d(A, \Theta) + d(\Theta, B)$;
4. if $A, B \in \mathcal{A}$ such that $A \cdot B = B \cdot A = \Theta$, then for all $C \in \mathcal{A}$:
 - $d(A + B, C) = d(A, C) + d(B, C) - d(\Theta, C)$,
 - $d(C, A + B) = d(C, A) + d(C, B) - d(C, \Theta)$.

From 2. of Definition 6.1, we get that d-maps are reflexive. As with all other quantum logic maps, d-maps are not transitive.

From the following examples, we can notice that the symmetry of d-maps does not follow from the given properties.

Example 6.2.

$d(A, B)$	Θ	A	B	I
Θ	0	$\frac{1}{3}$	$\frac{2}{3}$	1
A	$\frac{2}{3}$	0	1	$\frac{1}{3}$
B	$\frac{1}{3}$	1	0	$\frac{2}{3}$
I	1	$\frac{2}{3}$	$\frac{1}{3}$	0

Tab. 3. D-map values for 2×2 matrices.

Example 6.3. (Nánásiová and Valášková [4])

$$d(A, B) = s(A, A) + s(B, B) - 2s(A, B),$$

where $s: \mathcal{A} \times \mathcal{A} \rightarrow [0, 1]$ is an s-map.

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REFERENCES

- [1] M. A. Al-Adilee and O. Nánásiová: Copula and s-map on a quantum logic. *Inform. Sci.* **24** (2009), 4199–4207. DOI:10.1016/j.ins.2009.08.011
- [2] A. Lostaks: L-sets and L-valued structures. Nr. 2. University of Latvia, 2003.
- [3] R. Mesiar and E. P. Klement: Open problems posed at the eighth international conference on fuzzy set theory and applications. *Kybernetika* **42** (2006), 2, 225–235.
- [4] O. Nánásiová and Á. Valášková: Maps on a quantum logic. *Soft Computing* **14** (2010), 1047–1052. DOI:10.1007/s00500-009-0483-4
- [5] O. Nánásiová and S. Pulmannová: S-map and tracial states. *Inform. Sci.* **5** (2009), 515–520. DOI:10.1016/B978-1-4160-5899-1.50030-6

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