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A new characterization of projective special unitary group $\mathbf{PSU}(5,q)$

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Abstract. Projective special unitary groups $\mathrm{PSU}(5,q)$, where

$$\frac{q^4-q^3+q^2-q+1}{(5,q+1)}$$

is a prime, is uniquely determined by its order and the size of one conjugacy class.

Keywords: element order; conjugacy class; prime graph; projective special unitary group

Classification: 20D06, 20D60

1. Introduction

Let G be a finite group. One of the important problems in finite group theory is a characterization by a specific property. Properties often involve element orders and their graphs. We say that a group G is *characterized by a property M* if every group fulfilling M is isomorphic to G.

For a natural number n let $\pi(n)$ be the set of all prime divisors of n. Let G be a finite group. Write $\pi(G)$ for $\pi(|G|)$, and denote by $N(G)$ the set of all conjugacy class sizes of the group G. The set of all orders of elements in G is denoted by $\pi_e(G)$. The largest element order of G is denoted by $k(G)$.

The *prime graph* $\Gamma(G)$ of G is constructed upon the vertex set $\pi(G)$ by joining two distinct primes p and q if and only if G has an element of order pq.

Let $t(G)$ be the number of connected components of $\pi(G)$. These components will be denoted as $\pi_1, \pi_2, \dots, \pi_{t(G)}$. If G is of an even order, then π_1 is the component containing the prime 2.

Let $m_1, m_2, \dots, m_{t(G)}$ be integers for which $|G| = m_1 \cdots m_{t(G)}$, where $\pi(m_i)$ is the vertex set of π_i. If m_i is odd, call π_i an odd order component, see [10]. Write $\mathrm{OC}(G)$ for the set $\{m_1, m_2, \dots, m_{t(G)}\}$, and $T(G)$ for the set of connected components of G.

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For a prime p denote by $s_p(G)$ the number of elements of order p. By Sylow theorems these elements form a conjugacy class if $p \in \pi(G)$ and p^2 does not divide $|G|$.

The starting point of our discussion is a conjecture of J. G. Thompson, which appears as Problem (12.38) in the Kourovka notebook [14].

Thompson's conjecture. *Let G be a group with trivial center. If M is a non-abelian simple group satisfying $N(G) = N(M)$, then $G \cong M$.*

In fact there are several results, see [1], [6], [5], [4], [3], [2], [8], [9], in which the authors prove that some groups may be characterized only by its order and the size of only one conjugacy class.

This class of groups includes all sporadic simple groups, Alt_{10}, PSL(4, 4) and PSL(2, p), p a prime, PSL(q, 2), q a prime power, ${}^2D_n(2)$, ${}^2D_{n+1}(2)$, $C_n(2)$, Sym_p, PSL(5, q) and Alt_n where $n \geq 5$, $n \in \{p, p+1, p+2\}$.

A group G is called 2-*Frobenius* if it contains a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H, respectively.

The goal of this paper is to prove that a projective special unitary groups PSU(5, q) such that $(q^4 - q^3 + q^2 - q + 1)/(5, q+1)$ is a prime number is uniquely determined by its order and by the size of one conjugacy class.

Main theorem. *Let G be a group such that $|G| = |PSU(5, q)|$. If $p = (q^4 - q^3 + q^2 - q + 1)/(5, q+1)$ is a prime, then $G \cong PSU(5, q)$ if and only if G has a conjugacy class of size $s_p(G) = |PSU(5, q)|/p$.*

2. Statements used in the proof of the main theorem

Lemma 2.1 ([11])**.** *Let G be a Frobenius group of even order with kernel K and complement H. Then*

(a) *$t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$;*
(b) *$|H|$ divides $|K| - 1$;*
(c) *K is nilpotent.*

Lemma 2.2 ([7])**.** *Let G be a 2-Frobenius group of even order. Then*

(a) *$t(G) = 2$, $\pi(H) \cup \pi(G/K) = \pi_1$ and $\pi(K/H) = \pi_2$;*
(b) *G/K and K/H are cyclic groups in which $|G/K|$ divides $|Aut\,(K/H)|$.*

Lemma 2.3 ([16])**.** *Let G be a finite group with $t(G) \geq 2$. Then one of the following statements holds:*

(a) *G is a Frobenius group;*

(b) *G is a 2-Frobenius group;*

(c) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|Out\,(K/H)|$.*

Lemma 2.4 ([17])**.** *Let q, k, l be natural numbers. Then*

(1) $(q^k - 1, q^l - 1) = q^{(k,l)} - 1$;

(2) $(q^k + 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if both } \frac{k}{(k,l)} \text{ and } \frac{l}{(k,l)} \text{ are odd,} \\ (2, q+1) & \text{otherwise;} \end{cases}$

(3) $(q^k - 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if } \frac{k}{(k,l)} \text{ is even and } \frac{l}{(k,l)} \text{ is odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$

In particular, for every $q \geq 2$ and $k \geq 1$ the inequality $(q^k - 1, q^k + 1) \leq 2$ holds.

3. Proof of the main theorem

In this section the main theorem is proved by a sequence of lemmas. The proof uses the properties of simple groups that are summarized in Tables 1–5.

The projective special unitary groups PSU$(5, q)$ is denoted by U, and the prime number $(q^4 - q^3 + q^2 - q + 1)/(5, q + 1)$ by p. By [15], PSU$(5, q)$ has only one conjugacy class of size $s_p(U) = |\mathrm{PSU}(5, q)|/p$.

Suppose that G is a group such that $|G| = |U|$ and such that G contains a conjugacy class of size $s_p(U)$. The aim is to prove $G \cong \mathrm{PSU}(5, q)$.

By the assumption on q, there exists an element α of order p in G such that $C_G(\alpha) = \langle\alpha\rangle$ and $C_G(\alpha)$ is a Sylow p-subgroup of G. By Sylow theorem, $C_G(\beta) = \langle\beta\rangle$ for any element β in G of order p. The first step is to prove that p is an isolated vertex of $\Gamma(G)$.

Lemma 3.1. *We have that p is an isolated vertex of $\Gamma(G)$.*

Proof: Assume the opposite. Then there exists $t \in \pi(G) - \{p\}$ such that $tp \in \pi_e(G)$. So $tp \geq 2p = 2(q^4 - q^3 + q^2 - q + 1)/(5, q + 1) > (q^4 + q)/(5, q + 1)$, thus $k(G) > (q^4 + q)/(5, q + 1)$. This is a contradiction since $k(G) = (q^4 + q)/(5, q + 1)$, by [13]. □

By Lemma 3.1, $t(G) \geq 2$. Hence Lemma 2.3 applies.

Lemma 3.2. *The group G is not a Frobenius group.*

Proof: Let G be a Frobenius group with kernel K and complement H. Then by Lemma 2.1, $t(G) = 2$ and $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$, and $|H|$ divides $|K| - 1$. By Lemma 3.1, p is an isolated vertex

Group	π_1	π_2
A_p, $p \neq 5, 6$, p and $p-2$ not both primes	$3 \cdot 4 \cdot (p-3)(p-2)(p-1)$	p
A_{p+1}, $p \neq 4, 5$, $p-1$ and $p+1$ not both primes	$3 \cdot 4 \cdot (p-2)(p-1)(p+1)$	p
A_{p+2}, $p \neq 3, 4$, p and $p+2$ not both prime	$3 \cdot 4 \cdot (p-1)(p+1)(p+2)$	p
$A_{p-1}(q)$, $(p, q) \neq (3, 2), (3, 4)$,	$q^{p(p-1)/2} \prod_{i=1}^{p-1}(q^i - 1)$	$\frac{q^p-1}{(q-1)(p,q-1)}$
$A_p(q)$, $q-1 \mid p+1$,	$q^{p(p+1)/2}(q^{p+1} - 1) \prod_{i=2}^{p-1}(q^i - 1)$	$\frac{q^p-1}{(q-1)}$
${}^2A_{p-1}(q)$	$q^{p(p-1)/2} \prod_{i=1}^{p-1}(q^i - (-1)^i)$	$\frac{q^p+1}{(q+1)(p,q+1)}$
${}^2A_p(q)$, $q+1 \mid p+1$, $(p, q) \neq (3, 3), (5, 2)$	$q^{p(p+1)/2}(q^{p+1} - 1) \prod_{i=2}^{p-1}(q^i - (-1)^i)$	$\frac{q^p+1}{q+1}$
${}^2A_3(2)$,	$2^6 \cdot 3^4$	5
$B_n(q)$, $n = 2^m \geq 4$, q odd	$q^{n^2}(q^n - 1) \prod_{i=1}^{n-1}(q^{2i} - 1)$	$\frac{q^n+1}{2}$
$B_p(3)$	$3^{p^2}(3^p + 1) \prod_{i=1}^{p-1}(3^{2i} - 1)$	$\frac{3^p-1}{2}$
$C_n(q)$, $n = 2^m \geq 2$	$q^{n^2}(q^n - 1) \prod_{i=1}^{n-1}(q^{2i} - 1)$	$\frac{q^n+1}{(2,q-1)}$
$C_p(q)$, $q = 2, 3$	$q^{p^2}(q^p + 1) \prod_{i=1}^{p-1}(q^{2i} - 1)$	$\frac{q^p-1}{(2,q-1)}$
$D_p(q)$, $p \geq 5$, $q = 2, 3, 5$	$q^{p(p-1)} \prod_{i=1}^{p-1}(q^{2i} - 1)$	$\frac{q^p-1}{q-1}$
$D_{p+1}(q)$, $q = 2, 3$	$\frac{1}{(2,q-1)} q^{p(p-1)} \prod_{i=1}^{p-1}(q^{2i} - 1)$	$\frac{q^p-1}{q-1}$
${}^2D_n(q)$, $n = 2^m \geq 4$	$q^{n(n-1)} \prod_{i=1}^{n-1}(q^{2i} - 1)$	$\frac{q^n+1}{(2,q+1)}$
${}^2D_n(2)$, $n = 2^m + 1 \geq 5$	$2^{n(n-1)}(2^n + 1)(2^{n-1} - 1) \prod_{i=1}^{n-2}(2^{2i} - 1)$	$2^{n-1} + 1$
${}^2D_p(3)$, $p \neq 2^m + 1$, $p \geq 5$	$3^{p(p-1)} \prod_{i=1}^{p-1}(3^{2i} - 1)$	$\frac{3^p+1}{4}$
${}^2D_n(3)$, $n \neq 2^m + 1 \neq p$, $m \geq 2$	$\frac{1}{2} 3^{n(n-1)} \prod_{i=1}^{p-1}(3^{2i} - 1)$	$\frac{3^p+1}{4}$
$G_2(q)$, $q \equiv \alpha (\mathrm{mod} 3)$, $\alpha = \pm 1$, $q > 2$	$q^6(q^3 - \alpha)(q^2 - 1)(q + \alpha)$	$q^2 - \alpha q + 1$
${}^3D_4(q)$	$q^{12}(q^6 - 1)(q^2 - 1)(q^4 + q^2 + 1)$	$q^4 - q^2 + 1$
$F_4(q)$, q odd	$q^{24}(q^8 - 1)(q^6 - 1)^2(q^4 - 1)$	$q^4 - q^2 + 1$
$E_6(q)$	$q^{36}(q^{12} - 1)(q^8 - 1)(q^6 - 1)(q^5 - 1)(q^3 - 1)(q^2 - 1)$	$\frac{q^6+q^3+1}{(3,q-1)}$
${}^2E_6(q)$, $q > 2$	$q^{36}(q^{12} - 1)(q^8 - 1)(q^6 - 1)(q^5 + 1)(q^3 + 1)(q^2 - 1)$	$\frac{q^6-q^3+1}{(3,q+1)}$
${}^2F_4(2)'$	$2^{11} \cdot 3^3 \cdot 5^2$	13

TABLE 1. The order components of simple groups with $t(G) = 2$, p is an odd prime number, see [12].

of $\Gamma(G)$. Because of that (i) $|H| = p$ and $|K| = |G|/p$ or (ii) $|H| = |G|/p$ and $|K| = p$. Since $|H|$ divides $|K| - 1$, case (ii) is impossible. So $|H| = p$ and $|K| = |G|/p$. Hence $(q^4 - q^3 + q^2 - q + 1)(5, q + 1) \mid q^{10}(q^5 + 1)(q^4 - 1)(q^3 + 1)(q^2 - 1) \times$

Group	π_1	π_2	π_3	π_4	π_5	π_6
M_{11}	$2^4\cdot 3^2$	5	11			
M_{12}	$2^6\cdot 3^3\cdot 5$	11				
M_{22}	$2^7\cdot 3^2$	5	7	11		
M_{23}	$2^7\cdot 3^2\cdot 5\cdot 7$	11	23			
M_{24}	$2^{10}\cdot 3^3\cdot 5\cdot 7$	11	23			
J_1	$2^3\cdot 3\cdot 5$	7	11	19		
J_2	$2^7\cdot 3^3\cdot 5^2$	7				
J_3	$2^7\cdot 3^5\cdot 5$	17	19			
J_4	$2^{21}\cdot 3^3\cdot 5\cdot 7\cdot 11^3$	23	29	31	37	43
HS	$2^9\cdot 3^2\cdot 5^3$	7	11			
Ru	$2^{14}\cdot 3^3\cdot 5^3\cdot 7\cdot 13$	29				
Sz	$2^{13}\cdot 3^7\cdot 5^2\cdot 7$	11	13			
He	$2^{10}\cdot 3^3\cdot 5^2\cdot 7^3$	17				
ON	$2^9\cdot 3^4\cdot 5\cdot 7^3$	11	19	31		
Mcl	$2^7\cdot 3^6\cdot 5^3\cdot 7$	11				
Ly	$2^8\cdot 3^7\cdot 5^6\cdot 7\cdot 11$	31	37	67		
Co_1	$2^{21}\cdot 3^9\cdot 5^4\cdot 7^2\cdot 11\cdot 13$	23				
Co_2	$2^{18}\cdot 3^6\cdot 5^3\cdot 7$	11	23			
Co_3	$2^{10}\cdot 3^7\cdot 5^3\cdot 7\cdot 11$	23				
F_{22}	$2^{17}\cdot 3^9\cdot 5^2\cdot 7\cdot 11$	13				
F_{23}	$2^{18}\cdot 3^{13}\cdot 5^2\cdot 7\cdot 11\cdot 13$	17	23			
F'_{24}	$2^{21}\cdot 3^{16}\cdot 5^2\cdot 7^3\cdot 11\cdot 13$	17	23	29		
$F_1=M$	$2^{46}\cdot 3^{20}\cdot 5^9\cdot 7^6\cdot 11^2\cdot 13^3\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 41$	47	59	71		
$F_2=B$	$2^{41}\cdot 3^{13}\cdot 5^6\cdot 7^2\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23$	31	47			
$F_3=Th$	$2^{15}\cdot 3^{10}\cdot 5^3\cdot 7^2\cdot 13$	19	31			
$F_5=Ha$	$2^{14}\cdot 3^6\cdot 5^6\cdot 7\cdot 13$	19				

Table 2. Components of the sporadic groups, see [12].

$(q+1)/(q^4-q^3+q^2-q+1)-1$. First if $(5,q+1)=1$, then $q^4-q^3+q^2-q+1 \mid (q^4-q^3+q^2-q+1)(q^{16}+2q^{15}-q^{13}-3q^{11}-4q^{10}+5q^6+5q^5-5q-5)+4$. Thus $p\mid 4$ which is impossible. Now assume $(5,q+1)=5$, so $(q^4-q^3+q^2-q+1)/5 \mid (q^4-q^3+q^2-q+1)(q^{16}+2q^{15}-q^{13}-3q^{11}-4q^{10}+5q^6+5q^5-5q-5)+4$, a contradiction. $\square$

Lemma 3.3. *The group G is not a 2-Frobenius group.*

Proof: Start from the opposite and assume that G is a 2-Frobenius group. This means that G has a normal series $1\trianglelefteq H\trianglelefteq K\trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H, respectively. Set $|G/K|=x$. Since p is an isolated vertex of $\Gamma(G)$, we have $|K/H|=p$. By Lemma 2.2, $|G/K|$ divides

Group	π_1	π_2	π_3	π_4
A_p, p and $p-2$ are primes	$3 \cdot 4 \cdots (p-3)(p-1)$	$p-2$	p	
$A_1(q)$, $4 \mid q+1$	$q+1$	q	$\frac{q-1}{2}$	
$A_1(q)$, $4 \mid q-1$	$q-1$	q	$\frac{q+1}{2}$	
$A_1(q)$, $2 \mid q$	q	$q+1$	$q-1$	
$A_2(2)$	8	3	7	
$A_2(4)$	2^6	5	7	11
${}^2A_5(2)$	$2^{15} \cdot 3^6 \cdot 5$	7	11	
${}^2B_2(q)$, $q = 2^{2n+1} > 2$	q^2	$q \pm \sqrt{2q} + 1$	$q-1$	
${}^2D_p(3)$, $p = 2^n + 1$, $n \geq 2$	$2 \cdot 3^{p(p-1)}(3^{p-1}-1) \cdot \prod_{i=1}^{p-2}(3^{2i}-1)$	$\frac{3^{p-1}+1}{2}$	$\frac{3^p+1}{4}$	
${}^2D_{p+1}(2)$, $p = 2^n - 1$, $n \geq 2$	$2^{p(p+1)}(2^p-1)) \cdot \prod_{i=1}^{p-1}(2^{2i}-1)$	$2^p + 1$	$2^{p+1} + 1$	
$E_7(2)$	$2^{63} \cdot 3^{11} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 31 \cdot 43$	73	127	
$F_4(q)$, $2 \mid q$, $q > 2$	$q^{24}(q^6-1)(q^4-1)^2$	$q^4 + 1$	$q^4 - q^2 + 1$	
${}^2F_4(q)$, $q = 2^{2n+1} > 2$,	$q^{12}(q^4-1)(q^3+1) \cdot (q^2+1)(q-1)$	$q^2 \pm \sqrt{2q^3} + \pm q \pm \sqrt{2q} + 1$		
$G_2(q)$, $3 \mid q$	$q^6(q^2-1)^2 q^3(q^2-1)$	$q^2 - q + 1$	$q^2 + q + 1$	
${}^2G_2(q)$, $q = 3^{2n+1}$	$q^3(q^2-1)$	$q - \sqrt{3q} + 1$	$q + \sqrt{3q} + 1$	
$E_7(3)$	$2^{23} \cdot 3^{63} \cdot 5^2 \cdot 7^3 \cdot 11^2 \cdot 13^3 \cdot 9 \cdot 7 \cdot 41 \cdot 61 \cdot 73 \cdot 547$	757	1093	
${}^2E_6(2)$	$2^{36} \cdot 3^9 \cdot 5^2 \cdot 7^2 \cdot 11$	13	17	19

Table 3. The order components of simple groups with $t(G) \geq 3$, see [12].

$|\mathrm{Aut}(K/H)|$. Thus $|G/K| \mid p-1$. We have $q^4 - q^3 + q^2 - q + 1 \mid |H|$ since $(q^4 - q^3 + q^2 - q, q^4 - q^3 + q^2 - q + 1) = 1$. Therefore $H_t \rtimes K/H$ is a Frobenius group with kernel H_t and complement K/H, where $t = q^4 - q^3 + q^2 - q + 1$. Hence $|K/H|$ divides $|H_t| - 1$. That implies $p \mid p-1$, which is a contradiction. □

Lemma 3.4. *The group G is isomorphic to the group U.*

Proof: By Lemma 3.1, p is an isolated vertex of $\Gamma(G)$. Thus $t(G) > 1$ and G satisfies one of the cases of Lemma 2.3. Lemmas 2.1 and 3.3 imply that G is neither a Frobenius group nor a 2-Frobenius group. Thus only case (c) of

Group	$E_8(q)$, $q \equiv 0,1,4 \pmod 5$
π_1	$q^{120}(q^{18}-1)(q^{14}-1)(q^{12}-1)^2(q^{10}-1)^2(q^8-1)^2(q^4+q^2+1)$
π_2	$q^8+q^7-q^5-q^4-q^3+q+1$
π_3	$q^8-q^7+q^5-q^4+q^3-q+1$
π_4	$q^8-q^6+q^4-q^2+1$
π_5	q^8-q^4+1

TABLE 4. The order components of $E_8(q)$, see [12].

Group	$E_8(q)$, $q \equiv 2,3 \pmod 5$
π_1	$q^{120}(q^{20}-1)(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^{10}-1)(q^8-1)(q^4+1)(q^4+q^2+1)$
π_2	$q^8+q^7-q^5-q^4-q^3+q+1$
π_3	$q^8-q^7+q^5-q^4+q^3-q+1$
π_4	q^8-q^4+1

TABLE 5. The order components of $E_8(q)$, see [12].

Lemma 2.3 may occur. Hence G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, and K/H is a non-abelian simple group. We know that p is an isolated vertex of $\Gamma(G)$ and that $|K/H| \mid |G|$. There must be $p \in \pi(K/H)$ since every odd component of G is also a component of K/H. According to the classification of finite simple groups the only possibilities are: alternating groups A_n, where $n > 5$; 26 sporadic finite simple groups; and simple groups of Lie type. We deal with these cases separately. In the following p always means the prime $(q^4-q^3+q^2-q+1)/(5,q+1)$.

Step 1. Let $K/H \cong \mathrm{Alt}_r$, where r and $r-2$ are prime. We have $q \geq 2$ since q is a prime power, and $p \geq 11$ since $p \in \pi(K/H)$. Therefore $r = p = 11$ and $q = 2$. Now, $|G| = |\mathrm{PSU}(5,2)| = 2^{10} \cdot 3^6 \cdot 5 \cdot 11$. This contradicts the fact that $|\mathrm{Alt}_r|$ divides $|G|$.

Step 2. Let K/H be isomorphic to one of the sporadic simple groups. Since $p \in \pi(K/H)$ the value of p must be one of 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 59, 67 and 71. For example, if $K/H \cong \mathrm{He}$, where He is the Held group, then $|\mathrm{He}| = 2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$. On the other hand, by Table 2, $\pi(\mathrm{He}) = 17$. Suppose that $p = (q^4-q^3+q^2-q+1)/(5,q+1) = 17$. If $(5,q+1) = 1$, then $q^4-q^3+q^2-q+1 = 17$. The equation $q^4-q^3+q^2-q-16 = 0$ has no solution in natural numbers. Thus $(5, q+1) = 5$ and $q^4 - q^3 + q^2 - q + 1 = 85$. However, neither $q^4 - q^3 + q^2 - q - 84 = 0$ has a solution in natural numbers. Thus $K/H \ncong \mathrm{He}$, where this equation has a solution in natural numbers.

Assume $K/H = Sz$. By Table 2, $\pi(Sz) = 11 \cdot 13$. Thus $p \in \{11, 13\}$. First, if $(5, q+1) = 1$, then $q^4 - q^3 + q^2 - q - 10 = 0$, which gives $q = 2$. Therefore $2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 = |Sz| \nmid |\mathrm{PSU}(5,2)| = 2^{10} \cdot 3^6 \cdot 5 \cdot 11$, a contradiction again. Now, if $(5, q+1) = 5$ then $(q^4 - q^3 + q^2 - q + 1)/5 = 11$, which is impossible.

A similar contradiction may be obtained for all other simple sporadic groups.

Step 3. Here, we suppose that K/H is isomorphic to a group of Lie type. First, assume $t(K/H) = 2$. Then the following isomorphisms may occur:

(I). Suppose that $K/H \cong C_{n'}(q')$, $n' = 2^{m'} \geq 2$. By Table 2, $\pi(C_{n'}(q')) = (q'^{n'} + 1)/(2, q'-1)$. Thus $p = (q'^{n'} + 1)/(2, q'-1)$. It follows that $p^5 = ((q'^{n'} + 1)/(2, q'-1))^5 < q'^{6n'}$. On the other hand, we know $|C'_n(q')| \mid |G|$. In other words, $|G|$ is divisible by $[1/(2, q'-1)]q'^{n'^2} \prod_{i=1}^{n'}(q'^{2i} - 1)$. Thus $q'^{n'^2} \leq |K/H| \leq |G| < p^5 < q'^{6n'}$. Therefore $n'^2 \leq 6n'$, which gives $n' \leq 6$. Hence $n' \in \{2, 4\}$. If $(5, q'+1) = 1$ and $(2, q'-1) = 1$, then $q^4 - q^3 + q^2 - q + 1 = q'^{n'} + 1$ and $q'^{n'} = q^4 - q^3 + q^2 - q$. If $n' = 2$ then $q'^2 = q^4 - q^3 + q^2 - q$ and $|C_2(q')| \nmid |G|$, which is a contradiction. For $n' = 4$ we get a similar contradiction. Thus $(5, q'+1) = 5$ and $(2, q'-1) = 2$. Therefore $p = (q^4 - q^3 + q^2 - q + 1)/5 = (q'^{n'} + 1)/2$. It follows that $p^5 = ((q'^{n'} + 1)/2)^5 < q'^{6n'}$. On the other hand, $(q'^{n'^2}/2) \prod_{i=1}^{n'}(q'^{2i} - 1) \leq (q^{10}/5)(q^5 + 1)(q^4 - 1)(q^3 + 1)(q^2 - 1)(q + 1)$. So $q'^{n'^2} \leq q^{10}(q^5 + 1)(q^4 - 1)(q^3 + 1)(q^2 - 1)(q + 1) < p^5 < q'^{6n'}$. It follows that $q'^{n'^2} \leq q'^{6n'}$ so $n'^2 - 6n' \leq 0$. Hence $n' \in \{0, 1, 2, 3, 4, 5, 6\}$. From that only the case $n' = 4$ is possible, because $n' = 2^{m'}$. So we deduce that $|C_4(q')|$ divides $|G|$ as $(q'^{16}/2) \prod_{i=1}^{4}(q'^{2i} - 1) \leq (q^{10}/5)(q^5+1)(q^4-1)(q^3+1)(q^2-1)(q+1) < p^5 < q'^{24}$. That gives $q'^{36} < q'^{24}$, which is impossible.

(II). Suppose that $K/H \cong D'_r(q')$, $r' \geq 5$, $q' = 2, 3, 5$. By Table 1, $\pi_1(D_{r'}(q')) = (q'^{p'} - 1)/(q' - 1)$. First assume $q' = 2$, so that $K/H \cong D_{r'}(2)$. Then $p = 2^{r'} - 1$. Therefore $p^5 = (2^{r'} - 1)^5 < 2^{5r'}$. On the other hand, $|D_{r'}(2)|$ divides $|G|$. Hence $2^{r'(r'-1)} \leq |K/H| \leq |G| < p^5 < 2^{5r'}$, and that implies $r'(r'-1) \leq 5r'$. This means $5 \leq r' \leq 7$, and thus $r' \in \{5, 7\}$. But then $|D_{r'}(2)| \nmid |G|$, which is a contradiction. Assume $q' = 3$. Thus $K/H \cong D_{r'}(3)$. Then $p = (3^{r'} - 1)/3$ and $p^5 = ((3^{r'} - 1)/3)^5 < 3^{5r'}$. On the other hand, $3^{r'(r'-1)} \leq |K/H| \leq |G| < p^7 < 3^{5r'}$. As a result $r' \leq 6$, and $r' \in \{5, 6\}$. Because of that $|D_{r'}(3)| \nmid |G|$, which is a contradiction. For $q' = 5$ we prove that $K/H \ncong D_{r'}(5)$ in a similar manner.

(III). Suppose that $K/H \cong F_4(q')$, q' odd. By Table 1, $\pi_1(F_4(q')) = q'^4 - q'^2 + 1$. Since $p = q'^4 - q'^2 + 1$, $p^7 = (q'^4 - q'^2 + 1)^7 < (q'^5)^7 = q'^{35}$. On the other

hand, $|F_4(q')|$ divides $|G|$. Thus $q'^{38} \leq |K/H| \leq |G| < p^7 < q'^{35}$, which is a contradiction.

(IV). If $K/H \cong^2 D'_n(q')$, then $n' = 2^{m'} \geq 4$. By Table 1, $\pi_1({}^2D'_n(q')) = (q'^{n'}+1)/(2, q'+1)$. Thus $p = (q'^{n'}+1)/(2, q'+1)$ and $p^6 = ((q'^{n'}+1)/(2, q'+1))^6 < q'^{7n'}$. On the other hand, $|{}^2D'_n(q')|$ divides $|G|$. Because of that $q'^{n'(n'-1)} \leq |K/H| \leq |G| < p^6 < q'^{7n'}$. Hence $n' \leq 8$ and $n' \in \{4, 8\}$. That gives $|{}^2D'_n(q')| \nmid |G|$, which is a contradiction.

(V). Suppose that $K/H \cong C_{r'}(2)$. By Table 1, $\pi(C_{r'}(2)) = 2^{r'} - 1$. Thus $p = 2^{r'} - 1$ and $p^6 = (2^{r'} - 1)^r < 2^{6r'}$. On the other hand, $|C_r(2)|$ divides $|G|$. Now, $2^{r'^2} \leq |K/H| \leq |G| < p^6 < 2^{6r'}$, and that implies $r'^2 \leq 6r'$. Thus $r' \leq 6$ and $r' \in \{2, 3, 5, 7\}$. But then $|C'_r(2)| \nmid |G|$, which is a contradiction.

(VI). Suppose that $K/H \cong A_{r'}(q')$, $q' - 1 \mid r' + 1$. By Table 1, $\pi_1(A_{r'}(q')) = (q'^{r'} - 1)/((r', q'-1)(q'-1))$. Hence $p = (q'^{r'} - 1)/((r', q'-1)(q'-1))$ and $p^7 = \big((q'^{r'} - 1)/((r', q'-1)(q'-1))\big)^7 < q'^{7r'}$. On the other hand, $|A_{r'}(q')|$ divides $|G|$. Therefore $q'^{r'(r'-1)/2} \leq |K/H| \leq |G| < p^7 < 2^{7r'}$, and so $r'^2 \leq 7r'$. Thus $r' \leq 7$ and $r' \in \{2, 3, 5, 7\}$. But then $|A_{r'}(q')| \nmid |G|$, which is a contradiction.

(VII). Suppose $K/H \cong A_{r'-1}(q')$, $(r', q') \notin \{(3, 2), (3, 4)\}$. By Table 1, $\pi_1(A_{r'}(q')) = (q'^{r'} - 1)/(q'-1)$ and $p = (q'^{r'} - 1)/(q'-1)$. We get $p^7 = ((q'^{r'} - 1)/(q'-1))^7 < q'^{7r'}$. On the other hand, $|A_{r'}(q')|$ divides $|G|$. Because of that $q'^{r'(r'+1)/2} \leq |K/H| \leq |G| < p^7 < 2^{7r'}$. Thus $r'(r'+1) \leq 14r'$, $r' \leq 13$ and $r' \in \{2, 5, 7, 11, 13\}$. But then $|A_{r'}(q')| \nmid |G|$, which is a contradiction.

$(VIII)$. Suppose that $K/H \cong E_6(q')$. By Table 1, $\pi_1(E_6(q')) = (q'^6 + q'^3 + 1)/(3, q'-1)$. Hence $p = (q'^6 + q'^3 + 1)/(3, q'-1)$ and $p^5 = ((q'^6 + q'^3 + 1)/(3, q'-1))^5 < (q'^7)^5 = q'^{35}$. On the other hand, $|E_6(q')|$ divides $|G|$. Because of that $q'^{36} \leq |K/H| \leq |G| < p^5 < q'^{35}$, which is a contradiction. A contradiction may be obtained in a similar way if $K/H \ncong^2 E_6(q')$.

Let us now turn to the cases when $t(K/H) = 3$.

(IX). Suppose that $K/H \cong L_2(q')$ and $4 \mid q'+1$. By Table 3, $\pi_1(L_2(q')) = (q'-1)/2$. Thus $p = (q'-1)/2$. If $(5, q'+1) = 1$, then $p = (q'-1)/2$. It follows that $2p + 1 = q'$. But then $|L_2(q')| \nmid |G|$, which is a contradiction. For $(5, q+1) = 5$ observe that $(q^4 - q^3 + q^2 - q + 6)/5 = (q'-1)/2$ implies $q' = (2q^4 - 2q^3 + 2q^2 - 2q + 7)/5$. Because of that $|L_2(q')| \nmid |G|$, which is a contradiction.

(X). Suppose that $K/H \cong L_2(q')$ and $4 \mid q'-1$. By Table 3, $\pi_1(L_2(q')) = (q'+1)/2$. Thus $p = (q'+1)/2$. If $(5, q'+1) = 1$, then $p = (q'+1)/2$, and $2p - 1 = q'$. That contradicts $|L_2(q')| \nmid |G|$. For $(5, q+1) = 5$ observe that

$(q^4 - q^3 + q^2 - q + 6)/5 = (q' + 1)2$ implies $q' = (2q^4 - 2q^3 + 2q^2 - 2q - 3)/5$. Because of that $|L_2(q')| \nmid |G|$, which is a contradiction.

(*XI*). Suppose that $K/H \cong L_2(q')$ and $4 \mid q'$. By Table 3, $\pi_1(L_2(q')) = q' \pm 1$. Thus $p = q' \pm 1$. If $(5, q' + 1) = 1$, then $p = q' \pm 1$ and $q' = p \pm 1$. That contradicts $|L_2(q')| \nmid |G|$. If $(5, q + 1) = 5$, then $p = (q^4 - q^3 + q^2 - q + 1)/5 = q' \pm 1$. Thus $(q^4 - q^3 + q^2 - q + 6)/5 = q'$ or $(q^4 - q^3 + q^2 - q - 4)/5 = q'$. Both cases contradict $|L_2(q')| \nmid |G|$.

(*XII*). Suppose that $K/H \cong G_2(q')$. By Table 3, $\pi_1(G_2(q')) = q'^2 \pm q' + 1$. Thus $p = q'^2 \pm q' + 1$. If $(5, q + 1) = 1$ then $q^4 - q^3 + q^2 - q = q'(q' \mp 1)$. Since $(q', q' \mp 1) = 1$, there has to be $q' = q$ and $q' = q^3 - q^2 + q - 2$. But then $|G_2(q')| \nmid |G|$, which is a contradiction. For $(5, q + 1) = 5$, we obtain $(q^4 - q^3 + q^2 - q + 1)/5 = q'^2 \pm q' + 1$. That gives $q^4 - q^3 + q^2 - q = 5q'^2 \pm 5q' + 4$, and that is impossible. Similarly we can prove that $K/H \ncong {}^3D_4(q')$.

(*XIII*). Suppose that $K/H \cong {}^2G_2(q')$, $q' = 3^{2n'+1}$. By Table 3, $\pi_1({}^2G_2(q')) = q' \pm \sqrt{3q'} + 1 = p$. If $(5, q + 1) = 1$, then $q^4 - q^3 + q^2 - q + 1 = q' \pm \sqrt{3q'} + 1$ and $q(q^3 - q^2 + q - 1) = 3^{m+1}(3^m \pm 1)$. Because of that $(3^{m+1}, 3^m + 1) = 1$, and so $q = 3^m + 1$ and $q^3 - q^2 + q - 1 = 3^{m+1}$. Therefore $3^{m+1}(3^{3m+3} - 3^{2m+2} - 3^{m+1} - 1) = 3^{m+1}(3^m \pm 1)$. That yields $3^{3m+3} - 3^{2m+2} - 3^{m+1} - 1 = 3^m \pm 1$, which has no solution in natural numbers. If $(5, q + 1) = 5$, then $(q^4 - q^3 + q^2 - q + 1)/5 = q' \pm \sqrt{3q'} + 1$. It follows that $q^4 - q^3 + q^2 - q - 4 = 5(3^{2n'+1}) \pm 5(3^{n'+1})$, which is a contradiction.

(*XIV*). Suppose that $K/H \cong {}^2D_{r'}(3)$, $r = 2^n + 1 \geq 3$. By Table 3, $\pi_1({}^2D_{r'}(3)) = (3^r + 1)/4 = p$. Therefore $p^6 = ((3^r + 1)/4) < (3^r + 1)^6 < 3^{6r}$. On the other hand, $|{}^2D_r(3)|$ divides $|G|$. This gives $3^{r'(r'+1)} \leq |K/H| \leq |G| < p^6 < 3^{6r}$, and so $r'(r' + 1) \leq 6r'$. Thus $r' \leq 5$. In other words $3 \leq 2^n + 1 \leq 5$. It follows that $r' \in \{3, 5\}$. Because of that $|{}^2D_r(3)| \nmid |G|$, which is a contradiction.

(*XV*). Suppose that $K/H \cong F_4(q')$, $2 \mid q'$. By Table 3, $\pi_1(F_4(q')) \in \{q'^4 + 1, q'^4 - q'^2 + 1\}$. Assume first that $p = q'^4 + 1$. If $(5, q + 1) = 1$, then we have a contradiction to $|F_4(q')| \nmid |G|$. If $(5, q + 1) = 5$, then $(q^4 - q^3 + q^2 - q + 1)/5 = q'^4 + 1$, and $q'^4 = (q^4 - q^3 + q^2 - q - 4)/5$. Thus $q'^{24} = ((q^4 - q^3 + q^2 - q - 4)/5)^6 \leq q^{24}$. On the other hand, we know that the p-part (the highest power of p dividing an integer) of $|K/H|$ divides the p-part of $|G|$. This means that q'^{24} must be divided q^{10}. But then $q^{24} \leq q^{10}$, which is impossible. Assume now that $p = q'^4 - q'^2 + 1$. If $(5, q + 1) = 1$, then $q^4 - q^3 + q^2 - q = q'^4 - q'^2$. As a result $q(q^3 - q^2 + q - 1) = q'^2(q'^2 - 1)$. Since $(q'^2, q'^2 - 1) = 1$ there must be $q'^2 - 1 = q$ and $q^3 - q^2 + q - 1 = q'^2$. Because of that $|F_4(q')| \nmid |G|$, which is a contradiction. For case $(5, q + 1) = 5$ proceed similarly.

(XVI). Suppose that $K/H \cong {}^2F_4(q')$, $q' = 2^{2n'+1} > 2$. By Table 3, $\pi_1({}^2F_4(q')) = q'^2 \pm \sqrt{2q'^3} + q' \pm \sqrt{2q'} + 1 = p$. If $(5, q+1) = 1$, then $q(q^3 - q^2 + q - 1) = 2^{n'+1}(2^{3n'+2} \pm 2^{2n'} + 1 + 2^{n'} \pm 1)$, which yields an obvious contradiction. If $(5, q+1) = 5$, then $(q^4 - q^3 + q^2 - q + 1)/5 = 2^{4n'+2} \pm 2^{3n'+2} + 2^{2n'+1} \pm 2^{n'+1} + 1$. It follows that $q^4 - q^3 + q^2 - q = 5(2^{4n'+2}) \pm 5(2^{3n'+2}) + 5(2^{2n'+1} \pm 5(2^{n'+1}) + 4$, which is impossible.

($XVII$). Suppose that K/H is isomorphic to ${}^2A_5(2)$ or $E_7(3)$. If $i = 2, 3$, then $\pi_i({}^2A_5(2)) = 7, 11$, by Table 3. Also $\pi_i(E_7(3)) = 757, 1093$. Thus $p \in \{7, 11, 757, 1093\}$. If $(5, q+1) = 1$, then $q^4 - q^3 + q^2 - q + 1 \in \{7, 11, 757, 1093\}$. This is satisfied by no integer $q \geq 1$. If $(5, q+1) = 5$, then $(q^4 - q^3 + q^2 - q + 1)/5 \in \{7, 11, 757, 1093\}$, and that is also impossible.

($XVIII$). Suppose that $K/H \cong {}^2B_2(q')$, where $q' = 2^{2m'+1}$ and $m' \geq 1$. By Table 3, $\pi_1({}^2B_2(q')) \in \{q' \pm \sqrt{2q'} + 1, q' - 1\}$. Assume first that $p = q' \pm \sqrt{2q'} + 1$. If $(5, q+1) = 1$, then $q(q^3 - q^2 + q - 1) = 2^{m'+1}(2^{m'} \pm 1)$. This gives $q = 2^{m'} \pm 1$ and $q^3 - q^2 + q - 1 = 2^{m'+1}$, from which a contradiction easily follows. If $(5, q+1) = 5$, then $(q^4 - q^3 + q^2 - q + 1)/5 = 2^{2m'+1} \pm 2^{m'+1} + 1$. Therefore $q^4 - q^3 + q^2 - q = 5(2^{2m'+1}) \pm 5(2^{m'+1}) + 4$. If $2^{m'} = x$, then $q^4 - q^3 + q^2 - q = 10x^2 \pm 10x + 4$, and that has no solution in positive integers. Let us now consider the case $p = q' - 1$. We thus have $q' - 1 = q(q^3 - q^2 + q - 1)$. If $(5, q+1) = 1$, then $q' = q^4 - q^3 + q^2 - q + 1$. In such a case the order of ${}^2B_2(q')$ does not divide the order of G, a contradiction. The same contradiction may be obtained if $(5, q+1) = 5$, since then $q' = (q^4 - q^3 + q^2 - q + 6)/5$.

Finally, let us assume that $t(K/H) = \{4, 5\}$.

(XIX). Suppose $K/H \cong {}^2E_6(2)$ or $A_2(4)$. By Table 3, $\pi_i({}^2E_6(2)) = 13, 17, 19$ if $i \geq 2$. Also $\pi_i(A_2(4)) = 5, 7, 11$. Thus $p \in \{5, 7, 11, 13, 19\}$. If $p = 5$, and $(5, q+1) = 1$, then $q^4 - q^3 + q^2 - q + 1 = 5$, which is a contradiction. If $(5, q+1) = 5$ and $p = 5$, then $(q^4 - q^3 + q^2 - q + 1)/5 = 5$ and $q^4 - q^3 + q^2 - q - 24 = 0$, which is impossible. For the other values of p a similar contradiction may be derived.

(XX). Suppose now that $K/H \cong E_8(q')$, where $m \geq 1$. By Tables 4 and 5 the values $\pi_i(E_8(q')$, $i \geq 2$, may be $q'^8 - q'^7 + q'^5 - q'^4 + q'^3 - q' + 1$, $q'^8 + q'^7 - q'^5 - q'^4 - q'^3 + q' + 1$, $q'^8 - q'^6 + q'^4 - q'^2 + 1$ and $q'^8 - q'^4 + 1$. If $p = q'^8 - q'^7 + q'^5 - q'^4 + q'^3 - q' + 1$, then $p < q'^9$, and so $p^8 < q'^{72}$. On the other hand, we know that $|E_8(q')|$ divides $|G|$. But $q'^{120} \leq |K/H| \leq |G| < p^8 < q'^{72}$, which is a contradiction. The other possible values of p may be brought to a contradiction in a similar manner.

What remains is the case $K/H \cong PSU_5(q')$. Since $|K/H| = |U| = |G|$ and $p \in \pi(K/H)$, there has to be $(q^4 - q^3 + q^2 - q + 1)/(5, q+1) = (q'^4 - q'^3 + q'^2 - q' + 1)/(5, q'+1)$. Thus $q = q'$. Since $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ we obtain $H = 1$. Thus $G = K \cong U$. $\square$

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