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Inequalities involving norm and numerical radius of Hilbert space operators

Nasrollah Goudarzi, Zahra Heydarbeygi

Abstract. This paper presents several numerical radii and norm inequalities for Hilbert space operators. These inequalities improve some earlier related inequalities. For an operator A, we prove that

$$\omega^2(A) \le \Big\| \frac{A^*A + AA^*}{2} - \frac{1}{2R}\big((1-t)A^*A + tAA^* - ((1-t)(A^*A)^{1/2} + (AA^*)^{1/2})^2\big)\Big\|$$

where $R = \max\{t, 1-t\}$ and $0 \le t \le 1$.

Keywords: bounded linear operator; numerical radius; operator norm; inequality

Classification: 47A12, 47A30, 47A63

1. Introduction

Let $(\mathscr{H}; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. The numerical range of an operator A is the subset of the complex numbers $\mathbb{C}$ given by:

$$W(A) = \{\langle Ax, x\rangle \colon\ x \in \mathscr{H},\ \|x\| = 1\}.$$

The numerical radius of an operator A on $\mathscr{H}$ is shown by:

$$\omega(A) = \sup\{|\langle Ax, x\rangle| \colon\ x \in \mathscr{H},\ \|x\| = 1\}.$$

It is well-known that $\omega(\cdot)$ is a norm on the Banach algebra $\mathscr{B}(\mathscr{H})$ of all bounded linear operators $A\colon \mathscr{H} \to \mathscr{H}$. This norm is equivalent to the operator norm. The following more precise result holds:

$$\frac{1}{2}\|A\| \le \omega(A) \le \|A\|. \tag{1.1}$$

F. Kittaneh has established in [6] that if $A \in \mathscr{B}(\mathscr{H})$,

$$\omega^2(A) \le \frac{1}{2}\big\||A|^2 + |A^*|^2\big\| \tag{1.2}$$

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where $|A| = (A^*A)^{1/2}$. The inequality (1.2) is stronger than the second inequality in (1.1). This can be noticed by using the fact that

$$\frac{1}{2}\||A|^2 + |A^*|^2\| \le \frac{1}{2}\||A|^2\| + \frac{1}{2}\||A^*|^2\| = \|A\|^2.$$

Interestingly, inequality (1.2) has been refined in [8] and [13] by using the operator Hermite–Hadamard inequality. Moreover, in [3], M. El-Haddad and F. Kittaneh generalize (1.2) in the following form

$$\omega^{2r}(A) \le \left\|(1-t)|A|^{2r} + t|A^*|^{2r}\right\|, \qquad 0 \le t \le 1, \quad r \ge 1. \tag{1.3}$$

For some recent and interesting results regarding inequalities for the numerical radius, see [7], [9], and [10]. Section 2 establishes significant improvements in the second inequality in (1.1). Our method enables us to refine inequalities (1.2) and (1.3).

2. Results

The following result gives a multiplicative refinement of the second inequality in (1.1).

Theorem 2.1. *Let $A \in \mathscr{B}(\mathscr{H})$ and $A = U|A|$ be the polar decomposition of A, then*

$$\omega(A) \le \Big(\frac{\omega(U)+1}{2}\Big)\|A\|.$$

Proof: If A is a positive contraction, then by [14, Corollary 3.2], we have for any $x, y \in \mathscr{H}$,

$$|\langle Ax, y\rangle| \le \frac{1}{2}(|\langle x, y\rangle| + \|x\|\|y\|). \tag{2.1}$$

By replacing $A/\|A\|$ in (2.1), we get

$$|\langle Ax, y\rangle| \le \frac{\|A\|}{2}(|\langle x, y\rangle| + \|x\|\|y\|) \tag{2.2}$$

for positive operator A. Now, if $A = U|A|$ is the polar decomposition of A, then

$$\begin{aligned} |\langle Ax, x\rangle| &= |\langle U|A|x, x\rangle| = |\langle |A|x, U^*x\rangle| \\ &\le \frac{\|\,|A|\,\|}{2}(|\langle x, U^*x\rangle| + \|x\|\|U^*x\|) \qquad \text{(by (2.2))} \\ &= \frac{\|A\|}{2}(|\langle Ux, x\rangle| + \|x\|\|U^*x\|) \\ &\le \frac{\|A\|}{2}(|\langle Ux, x\rangle| + \|U^*\|) \end{aligned}$$

$$\leq \frac{\|A\|}{2}(|\langle Ux, x\rangle| + 1) \qquad (\text{since } \|U\| = \|U^*\| \leq 1)$$
$$\leq \frac{\|A\|}{2}(\omega(U) + 1),$$

where we assume $x \in \mathscr{H}$ is a unit vector. Therefore,

$$\omega(A) \leq \Big(\frac{\omega(U)+1}{2}\Big)\|A\|.$$

This verifies the desired result. □

Remark 2.1. Notice that $\omega(U) \leq \|U\| \leq 1$, implies $(\omega(U)+1)/2 \leq 1$. Therefore,

$$\omega(A) \leq \Big(\frac{\omega(U)+1}{2}\Big)\|A\| \leq \|A\|.$$

Remark 2.2. It should be mentioned here that if A is a normal and invertible operator, then U is a unitary operator, and consequently $\omega(U) = \|U\| = 1$.

The following lemmas will aid in proving the next result.

Lemma 2.1 ([5])**.** *Let $A \in \mathscr{B}(\mathscr{H})$ and let $x, y \in \mathscr{H}$ be any vectors. If $0 \leq t \leq 1$,*

$$|\langle Ax, y\rangle|^2 \leq \big\langle |A|^{2(1-t)}x, x\big\rangle \big\langle |A^*|^{2t}y, y\big\rangle.$$

Lemma 2.2. *Let $A \in \mathscr{B}(\mathscr{H})$ be a self-adjoint operator, and let $x \in \mathscr{H}$ be a unit vector. Then,*

$$\langle Ax, x\rangle^2 \leq \big\langle A^2x, x\big\rangle.$$

Proof: Utilizing the Cauchy–Schwarz inequality, we have

$$\langle Ax, x\rangle^2 \leq \|Ax\|^2 = \langle Ax, Ax\rangle = \big\langle A^2x, x\big\rangle.$$

□

We derive the following new bound employing Lemmas 2.1 and 2.2.

Theorem 2.2. *Let $A \in \mathscr{B}(\mathscr{H})$ and let $0 \leq t \leq 1$. Then*

$$\omega^4(A) \leq \left\|(1-t)|A|^4 + t|A^*|^4 - 2r\left(\frac{|A|^4 + |A^*|^4}{2} - \left(\frac{|A|^2 + |A^*|^2}{2}\right)^2\right)\right\|, \tag{2.3}$$

where $r = \min\{t, 1-t\}$.

Proof: If $0 \le t \le 1/2$, then

$$\begin{aligned}\big((1-t)|A|^2+t|A^*|^2\big)^2 &= \left((1-2t)|A|^2+2t\left(\frac{|A|^2+|A^*|^2}{2}\right)\right)^2\\ &\le (1-2t)|A|^4+2t\left(\frac{|A|^2+|A^*|^2}{2}\right)^2\\ &\qquad \text{(since } f(t)=t^2 \text{ is operator convex; see Theorem 1.5.8 in [1])}\\ &= (1-t)|A|^4+t|A^*|^4-2r\left(\frac{|A|^4+|A^*|^4}{2}-\left(\frac{|A|^2+|A^*|^2}{2}\right)^2\right).\end{aligned}$$

A similar discussion holds for $1/2 \le t \le 1$. So we obtain

$$\big((1-t)|A|^2+t|A^*|^2\big)^2 \le (1-t)|A|^4+t|A^*|^4-2r\left(\frac{|A|^4+|A^*|^4}{2}-\left(\frac{|A|^2+|A^*|^2}{2}\right)^2\right).$$

On the other hand,

$$\begin{aligned}|\langle Ax,x\rangle|^4 &\le \big(\langle |A|^{2(1-t)}x,x\rangle\langle |A^*|^{2t}x,x\rangle\big)^2 \qquad \text{(by Lemma 2.1)}\\ &\le \big(\langle |A|^2x,x\rangle^{1-t}\langle |A^*|^2x,x\rangle^t\big)^2 \qquad \text{(by [3, Lemma 3])}\\ &\le \big((1-t)\langle |A|^2x,x\rangle+t\langle |A^*|^2x,x\rangle\big)^2\\ &\qquad \text{(by the weighted arithmetic-geometric mean inequality)}\\ &= \big\langle \big((1-t)|A|^2+t|A^*|^2\big)x,x\big\rangle^2\\ &\le \big\langle \big((1-t)|A|^2+t|A^*|^2\big)^2x,x\big\rangle \qquad \text{(by Lemma 2.2)}\end{aligned}$$

for any unit vector $x \in \mathscr{H}$. Hence,

$$\begin{aligned}|\langle Ax,x\rangle|^4 &\le \Bigg\langle \Bigg((1-t)|A|^4+t|A^*|^4\\ &\qquad -2r\left(\frac{|A|^4+|A^*|^4}{2}-\left(\frac{|A|^2+|A^*|^2}{2}\right)^2\right)\Bigg)x,x\Bigg\rangle \qquad (2.4)\\ &\le \left\|(1-t)|A|^4+t|A^*|^4-2r\left(\frac{|A|^4+|A^*|^4}{2}-\left(\frac{|A|^2+|A^*|^2}{2}\right)^2\right)\right\|.\end{aligned}$$

Therefore, from (2.4) it follows that

$$\omega^4(A) \le \left\|(1-t)|A|^4+t|A^*|^4-2r\left(\frac{|A|^4+|A^*|^4}{2}-\left(\frac{|A|^2+|A^*|^2}{2}\right)^2\right)\right\|.$$

□

The following remark shows that the inequality (2.3) is sharper than the inequality (1.3).

Remark 2.3. Since $f(t)=t^2$ is operator convex on $(0,\infty)$, we have

$$\frac{|A|^4+|A^*|^4}{2}-\left(\frac{|A|^2+|A^*|^2}{2}\right)^2\geq 0,$$

therefore, we have

$$\begin{aligned}\Big\|(1-t)|A|^4+t|A^*|^4-2r\Big(\frac{|A|^4+|A^*|^4}{2}-\Big(\frac{|A|^2+|A^*|^2}{2}\Big)^2\Big)\Big\|\\ \leq \big\|(1-t)|A|^4+t|A^*|^4\big\|.\end{aligned}$$

We conclude this section with the following result. The reader may compare it to [15, Theorem 2.3].

Theorem 2.3. *Let* $A\in\mathscr{B}(\mathscr{H})$ *and let* $0\leq t\leq 1$*. Then*

$$\begin{aligned}\omega^2(A)\leq \Big\|&\frac{|A|^2+|A^*|^2}{2}\\ &-\frac{1}{2R}\big((1-t)|A|^2+t|A^*|^2-((1-t)|A|+t|A^*|)^2\big)\Big\|,\end{aligned}\tag{2.5}$$

where $R=\max\{t,1-t\}$.

Proof: By Lemma 3.12 in [11],

$$(1-t)|A|^2+t|A^*|^2\leq((1-t)|A|+t|A^*|)^2+2R\Big(\frac{|A|^2+|A^*|^2}{2}-\Big(\frac{|A|+|A^*|}{2}\Big)^2\Big),$$

where $0\leq t\leq 1$ and $R=\max\{t,1-t\}$. This inequality gives

$$\begin{aligned}\Big(\frac{|A|+|A^*|}{2}\Big)^2\leq\ &\frac{|A|^2+|A^*|^2}{2}\\ &-\frac{1}{2R}\Big((1-t)|A|^2+t|A^*|^2-\big((1-t)|A|+t|A^*|\big)^2\Big).\end{aligned}\tag{2.6}$$

On the other hand,

$$\begin{aligned}|\langle Ax,x\rangle|^2&\leq\langle|A|x,x\rangle\langle|A^*|x,x\rangle\qquad\text{(by Lemma 2.1)}\\ &\leq\Big(\frac{\langle|A|x,x\rangle+\langle|A^*|x,x\rangle}{2}\Big)^2\\ &\quad\text{(by the arithmetic-geometric mean inequality)}\\ &=\Big\langle\Big(\frac{|A|+|A^*|}{2}\Big)x,x\Big\rangle^2\leq\Big\langle\Big(\frac{|A|+|A^*|}{2}\Big)^2x,x\Big\rangle\qquad\text{(by Lemma 2.2).}\end{aligned}$$

Utilizing (2.6) we get

$$
\begin{aligned}
|\langle Ax,x\rangle|^2 &\leq \Big\langle \Big(\frac{|A|^2+|A^*|^2}{2} \\
&\qquad - \frac{1}{2R}\big((1-t)|A|^2+t|A^*|^2-((1-t)|A|+t|A^*|)^2\big)\Big)x,x\Big\rangle \\
&\leq \Big\|\frac{|A|^2+|A^*|^2}{2} \\
&\qquad - \frac{1}{2R}\big((1-t)|A|^2+t|A^*t|^2-((1-t)|A|+t|A^*|)^2\big)\Big\|.
\end{aligned} \tag{2.7}
$$

Taking the supremum in (2.7) over $x \in \mathscr{H}$, $\|x\| = 1$ we deduce the desired result. □

The following remark shows that the inequality (2.5) is sharper than the inequality (1.2).

Remark 2.4. Since

$$(1-t)|A|^2+t|A^*|^2-((1-t)|A|+t|A^*|)^2 \geq 0,$$

we have

$$\Big\|\frac{|A|^2+|A^*|^2}{2}-\frac{1}{2R}\big((1-t)|A|^2+t|A^*|^2-((1-t)|A|+t|A^*|)^2\big)\Big\| \leq \frac{1}{2}\|\,|A|^2+|A^*|^2\|.$$

It is well-known that for any $A, B \in \mathscr{B}(\mathscr{H})$

$$\omega\left(\begin{bmatrix} O & A \\ B & O\end{bmatrix}\right) \leq \frac{1}{2}(\|A\|+\|B\|). \tag{2.8}$$

This follows from the following fact, see [4, (4.6)],

$$\omega\left(\begin{bmatrix} O & A \\ B & O\end{bmatrix}\right) = \frac{1}{2}\sup_{\theta\in\mathbb{R}}\|A+\mathrm{e}^{\mathrm{i}\theta}B^*\|.$$

To obtain the following result, which contains a refinement of (2.8), we mimic some ideas from [12, Corollary 2.1].

Theorem 2.4. *Let $A, B \in \mathscr{B}(\mathscr{H})$. Then for any $t \in \mathbb{R}$,*

$$\omega\left(\begin{bmatrix} O & A \\ B & O\end{bmatrix}\right) \leq \frac{1}{2}\left(\omega\left(\begin{bmatrix} O & (1+t)A \\ (1-t)B & O\end{bmatrix}\right)+\omega\left(\begin{bmatrix} O & (1-t)A \\ (1+t)B & O\end{bmatrix}\right)\right).$$

Proof: It has been shown in [2, Corollary 2.4] that

$$\|A+B\| \leq \Big\|tA+(1-t)\frac{A+B}{2}\Big\|+\Big\|tB+(1-t)\frac{A+B}{2}\Big\| \leq \|A\|+\|B\|$$

for any $t \in \mathbb{R}$. If we replace B by $\mathrm{e}^{\mathrm{i}\theta} B$, we infer that

$$\begin{aligned}\|A+\mathrm{e}^{\mathrm{i}\theta}B\| &\le \left\|tA+(1-t)\frac{A+\mathrm{e}^{\mathrm{i}\theta}B}{2}\right\| + \left\|t\,\mathrm{e}^{\mathrm{i}\theta}B+(1-t)\frac{A+\mathrm{e}^{\mathrm{i}\theta}B}{2}\right\| \\ &= \frac{1}{2}(\|(1+t)A+(1-t)\,\mathrm{e}^{\mathrm{i}\theta}B\| + \|(1-t)A+(1+t)\,\mathrm{e}^{\mathrm{i}\theta}B\|).\end{aligned}$$

From this we can write,

$$\begin{aligned}\frac{1}{2}\|A+\mathrm{e}^{\mathrm{i}\theta}B\| &\le \frac{1}{4}(\|(1+t)A+(1-t)\,\mathrm{e}^{\mathrm{i}\theta}B\| + \|(1-t)A+(1+t)\,\mathrm{e}^{\mathrm{i}\theta}B\|) \\ &\le \frac{1}{2}\left(\omega\left(\begin{bmatrix} O & (1+t)A \\ (1-t)B^* & O\end{bmatrix}\right) + \omega\left(\begin{bmatrix} O & (1-t)A \\ (1+t)B^* & O\end{bmatrix}\right)\right)\end{aligned}$$

i.e.,

$$\frac{1}{2}\|A+\mathrm{e}^{\mathrm{i}\theta}B\| \le \frac{1}{2}\left(\omega\left(\begin{bmatrix} O & (1+t)A \\ (1-t)B^* & O\end{bmatrix}\right) + \omega\left(\begin{bmatrix} O & (1-t)A \\ (1+t)B^* & O\end{bmatrix}\right)\right)$$

for any $t \in \mathbb{R}$. Now, if we take supremum over $\theta \in \mathbb{R}$, we obtain

$$\omega\left(\begin{bmatrix} O & A \\ B^* & O\end{bmatrix}\right) \le \frac{1}{2}\left(\omega\left(\begin{bmatrix} O & (1+t)A \\ (1-t)B^* & O\end{bmatrix}\right) + \omega\left(\begin{bmatrix} O & (1-t)A \\ (1+t)B^* & O\end{bmatrix}\right)\right).$$

We deduce the desired result by substituting B by B^*. □

Assume that $0 \le t \le 1$. We can write

$$\begin{aligned}&\omega\left(\begin{bmatrix} O & (1+t)A \\ (1-t)B & O\end{bmatrix}\right) + \omega\left(\begin{bmatrix} O & (1-t)A \\ (1+t)B & O\end{bmatrix}\right) \\ &\qquad\le \frac{(1+t)\|A\|+(1-t)\|B\|}{2} + \frac{(1-t)\|A\|+(1+t)\|B\|}{2} \\ &\qquad= \|A\|+\|B\|,\end{aligned}$$

due to (2.8). We can write from the above observation:

Corollary 2.1. *Let $A, B \in \mathscr{B}(\mathscr{H})$. Then*

$$\begin{aligned}\omega\left(\begin{bmatrix} O & A \\ B & O\end{bmatrix}\right) &\le \frac{1}{2}\left(\int_0^1 \omega\left(\begin{bmatrix} O & (1+t)A \\ (1-t)B & O\end{bmatrix}\right)\mathrm{d}t \right.\\ &\qquad\left. + \int_0^1 \omega\left(\begin{bmatrix} O & (1-t)A \\ (1+t)B & O\end{bmatrix}\right)\mathrm{d}t\right) \\ &\le \frac{1}{2}(\|A\|+\|B\|).\end{aligned}$$

References

[1] Bhatia R., *Positive Definite Matrices*, Princeton Series in Applied Mathematics, Princeton University Press, Princeton, 2007.

[2] Furuichi S., Moradi H. R., Sababheh M., *New inequalities for interpolational operator means*, J. Math. Inequal. **15** (2021), no. 1, 107–116.

[3] El-Haddad M., Kittaneh F., *Numerical radius inequalities for Hilbert space operators. II*, Studia Math. **182** (2007), no. 2, 133–140.

[4] Hirzallah O., Kittaneh F., Shebrawi K., *Numerical radius inequalities for certain* 2×2 *operator matrices*, Integral Equations Operator Theory **71** (2011), no. 2, 129–147.

[5] Kittaneh F., *Notes on some inequalities for Hilbert space operators*, Publ. Res. Inst. Math. Sci. **24** (1988), no. 2, 283–293.

[6] Kittaneh F., *Numerical radius inequalities for Hilbert space operators*, Studia Math. **168** (2005), no. 1, 73–80.

[7] Kittaneh F., Moradi H. R., *Cauchy–Schwarz type inequalities and applications to numerical radius inequalities*, Math. Inequal. Appl. **23** (2020), no. 3, 1117–1125.

[8] Moradi H. R., Sababheh M., *More accurate numerical radius inequalities (II)*, Linear Multilinear Algebra **69** (2021), no. 5, 921–933.

[9] Omidvar M. E., Moradi H. R., *Better bounds on the numerical radii of Hilbert space operators*, Linear Algebra Appl. **604** (2020), 265–277.

[10] Omidvar M. E., Moradi H. R., *New estimates for the numerical radius of Hilbert space operators*, Linear Multilinear Algebra **69** (2021), no. 5, 946–956.

[11] Sababheh M., *Convexity and matrix means*, Linear Algebra Appl. **506** (2016), 588–602.

[12] Sababheh M., Conde C., Moradi H. R., *A convex-block approach for numerical radius inequalities*, arXiv:2302.06777v1 [math.FA] (2023), 17 pages.

[13] Sababheh M., Moradi H. R., *More accurate numerical radius inequalities (I)*, Linear Multilinear Algebra. **69** (2021), no. 10, 1964–1973.

[14] Sababheh M., Moradi H. R., Heydarbeygi Z., *Buzano, Kreĭn and Cauchy–Schwarz inequalities*, Oper. Matrices. **16** (2022), no. 1, 239–250.

[15] Sheybani S., Sababheh M., Moradi H. R., *Weighted inequalities for the numerical radius*, Vietnam J. Math. **51** (2023), no. 2, 363–377.

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