On the range of some elementary operators

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Abstract. Let $L(H)$ denote the algebra of all bounded linear operators on a complex infinite dimensional Hilbert space H. For $A, B \in L(H)$, the generalized derivation $\delta_{A,B}$ and the multiplication operator $M_{A,B}$ are defined on $L(H)$ by $\delta_{A,B}(X) = AX - XB$ and $M_{A,B}(X) = AXB$. In this paper, we give a characterization of bounded operators A and B such that the range of $M_{A,B}$ is closed. We present some sufficient conditions for $\delta_{A,B}$ to have closed range. Some related results are also given.

Keywords: generalized derivation; elementary operator; generalized inverse; Kato spectrum

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Introduction

Let H denote a separable infinite dimensional complex Hilbert space, and let $L(H)$ denote the algebra of all bounded linear operators on H. For operators A and B in $L(H)$, let L_A and $R_B \in L(L(H))$ denote respectively the operators

$$L_A(X) = AX, \qquad R_B(X) = XB$$

of left multiplication by A and right multiplication by B. The most important examples of elementary operators are the generalized derivations $\delta_{A,B}$, and the multiplications operators $M_{A,B}$ defined on $L(H)$ by

$$\delta_{A,B}(X) = (L_A - R_B)(X) = AX - XB,$$
$$M_{A,B}(X) = (L_A \circ R_B)(X) = AXB.$$

The purpose of this paper is to study the cases in which the range of multiplications $M_{A,B}$ and the range of generalized derivations $\delta_{A,B}$ are norm closed in $L(H)$.

In [1], J. Anderson and C. Foias proved that if A and B are scalar operators on a Banach space, then $R(\delta_{A,B})$, the range of $\delta_{A,B}$, is norm closed if and only if $\sigma(A) \cap \sigma(B)$ contains no limit point of $\sigma(A) \cup \sigma(B)$, where $\sigma(A)$ denote the

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spectrum of A. In particular, if $A \in L(H)$ is similar to a normal operator, then $\delta_A = \delta_{A,A}$ has closed range if and only if $\sigma(A)$ is finite.

J. Stampfli in [12] proved that if A is hyponormal, then $R(\delta_A)$ is closed if and only if $\sigma(A)$ is finite. Subsequently, C. Apostol and J. Stampfli studied inner derivation δ_A with closed range for the cases in which the operator A is compact, nilpotent or a weighted shift, see [3]. C. Apostol characterized the Hilbert space operators which induce inner derivations having closed range, see [2]. More precisely, he showed that the range of the derivation δ_A is closed if and only if A is similar to a Jordan operator. A. Fialkow and A. Herrero in [7] extended Apostol's results to the Calkin algebra $\mathcal{C}(H) = L(H)/K(H)$. In this paper, we are interested in the following problem: Under what conditions on A and B, the range of either $\delta_{A,B}$ or $M_{A,B}$ is norm closed in $L(H)$? This problem is raised by L. Fialkow in [9, Problem 3.9] for the multiplication operator.

Our principal objective in this paper is to study the closedness of the range of the basic elementary operator $M_{A,B}$ and the generalized derivation $\delta_{A,B}$ in terms of spectral or structural properties of operators A and B. We prove that the range of $M_{A,B}$ is closed if and only if the ranges of A and B are closed. We shall present a sufficient condition for $\delta_{A,B}$ to have closed range. Some related results are also given.

We shall use the following notations and terminology. For a bounded linear operator T, we shall denote the kernel, the orthogonal complement of the kernel, the range of T by $\ker(T)$, $\ker^{\perp}(T)$ and $R(T)$, respectively. The spectrum, the approximate point spectrum, the surjectivity spectrum and the resolvent set of T will be denoted by $\sigma(T)$, $\sigma_\pi(T)$, $\sigma_\delta(T)$ and $\varrho(T)$. The approximate point spectrum and the approximate defect spectrum of A are defined as follows:

$$\sigma_\pi(A) = \{\lambda \in \mathbb{C} \colon\ A - \lambda \ \text{ is not bounded below}\},$$
$$\sigma_\delta(A) = \{\lambda \in \mathbb{C} \colon\ A - \lambda \ \text{ is not surjective}\}.$$

1. The multiplication operator $M_{A,B} = L_A \circ R_B$

Definition 1.1. Let E be a Banach space. An operator $A \in L(E)$ possesses a generalized inverse, if there exists an operator $T \in L(E)$ such that

$$ATA = A \qquad \text{and} \qquad TAT = T.$$

In this case we will say that T is the generalized inverse of A.

Before we state and prove our first principal result, we require additional preliminary lemmas.

Lemma 1.2 ([5]). *For every operator $A \in L(H)$ on a Hilbert space H, the following assertions are equivalent:*

(i) *A has a generalized inverse.*
(ii) *$R(A)$ is closed,* $\ker A$ *and $R(A)$ are complemented subspaces of H.*
(iii) *There exists an operator $T \in L(H)$ such that $ATA = A$.*

Remark 1.3. Let H be a Hilbert space and $A \in L(H)$. Note that A has a generalized inverse if and only if the range of A is closed. It is clear that T is a generalized inverse of A if and only if T^* is a generalized inverse of A^*. Then, the range of A is closed if and only if the range of A^* is closed.

Lemma 1.4. *Let $A, B \in L(H)$ be operators on a Hilbert space H. Then, the range of $M_{A,B}$ is closed if and only if the range M_{B^*,A^*} is closed.*

Proof: Since $L(H)$ is a $\mathcal{C}^*$-algebra we have the following equality:

$$\|AX_nB - Y\|_{L(H)} = \|(AX_nB - Y)^*\|_{L(H)} = \|B^*X_n^*A^* - Y^*\|_{L(H)}.$$

Then $AX_nB \longrightarrow Y$ if and only if $B^*X_n^*A^* \longrightarrow Y^*$. □

Now we are in a position to prove our first principal theorem.

Theorem 1.5. *Let A and B be nonzero operators in $L(H)$. Then, the range of $M_{A,B}$ is closed if and only if the ranges of A and B are closed.*

Proof: Suppose that the range of $M_{A,B}$ is closed. We show firstly that the range of A is closed. Let $(x_n)_n$ be a sequence in H such that $Ax_n \longrightarrow y$ as $n \longrightarrow \infty$. We have B is a nonzero operator, then there exists $u \in H$ such that $\|B^*u\|_H = 1$. We wish to define a sequence of operators $(X_n)_n$ and Y in $L(H)$ where

$$X_n = x_n \otimes u \qquad \text{and} \qquad Y = y \otimes B^*u.$$

Then, it is easily verified that

$$\begin{aligned} AX_nBz - Yz &= A(\langle Bz, u\rangle x_n) - \langle z, B^*u\rangle y \\ &= \langle z, B^*u\rangle Ax_n - \langle z, B^*u\rangle y \\ &= \langle z, B^*u\rangle (Ax_n - y). \end{aligned}$$

Hence, we get

$$\|AX_nB - Y\|_{L(H)} \leq \|Ax_n - y\|_H.$$

It follows that $AX_nB \longrightarrow Y$, then there exists $X \in L(H)$ such that $AXB = Y$. In particular, if we choose $x = XBB^*u$, then we have $Ax = y$.

To prove that the range of B is closed, we will use Lemma 1.4. So, the closedness of the range of $M_{A,B}$ implies that the range of M_{B^*,A^*} is closed, and by the previous argument we deduce that the range of B^* is closed, and therefore the range B is closed by Remark 1.3. Consequently, the ranges of A and B are closed.

Conversely, suppose that the ranges of A and B are closed. It follows from Lemma 1.2 that A and B have generalized inverses A' and B' respectively. There holds that

$$M_{A,B}M_{A',B'}M_{A,B} = M_{A,B},$$
$$M_{A',B'}M_{A,B}M_{A',B'} = M_{A',B'}.$$

It results that $M_{A',B'}$ is a generalized inverse of $M_{A,B}$. Hence, Lemma 1.2 asserts that the range of $M_{A,B}$ is closed, which completes the proof of the theorem. □

In the following proposition we prove some commutativity results for the multiplication operator $M_{A,B}$.

Proposition 1.6. *Let A, B and $X \in L(H)$. If A is one-one and B has dense range, then $M_{A^2,B^2}(X) = M_{A^3,B^3}(X)$ if and only if $M_{A,B}(X) = X$.*

PROOF: Suppose that $M_{A^2,B^2}(X) = M_{A^3,B^3}(X)$, then we have $A^2XB^2 = A^3XB^3$, or equivalently $A(A^2XB^2 - AXB)B = 0$. Since A is one-one and B has dense range it follows that $A^2XB^2 - AXB = 0$. Hence, $A(AXB - X)B = 0$ and A is one-one and B has dense range implies that $M_{A,B}(X) = X$. The reverse implication is obvious. □

Remark 1.7. We obtain the following result:
Given Hilbert space operators A, B and $X \in L(H)$. If A is one-one and B has dense range and $M_{A^n,B^n}(X) = M_{A^{n+1},B^{n+1}}(X)$ for some integer $n \geq 1$, then $M_{A,B}(X) = X$.

2. The generalized derivation $\delta_{A,B} = L_A - R_B$

Given an operator $A \in L(E)$ on a Banach space E, the Kato resolvent of A is defined as

$$\varrho_K(A) = \left\{\lambda \in \mathbb{C}\colon\ R(A-\lambda) \ \text{ is closed and } \ \ker(A-\lambda) \subset \bigcap_{n=1}^{\infty} R(A-\lambda)^n\right\}.$$

The Kato spectrum of A is $\sigma_K(A) = \mathbb{C} \setminus \varrho_K(A)$. We can easily show that

$$\sigma_K(A) \subset \sigma_\pi(A) \cap \sigma_\delta(A).$$

If H is a Hilbert space, the condition $\mu \in \varrho_K(A)$ is equivalent to the existence of a neighborhood $\mathcal{U}$ of μ and an operator-valued analytic function $S\colon \mathcal{U} \longrightarrow L(H)$

such that $S(\lambda)$ is a generalized inverse of $(A-\lambda)$, that is, for every $\lambda \in \mathcal{U}$ we have

$$(A-\lambda)S(\lambda)(A-\lambda) = (A-\lambda) \qquad \text{and} \qquad S(\lambda)(A-\lambda)S(\lambda) = S(\lambda).$$

In other words, $\lambda \longmapsto A-\lambda$ has local analytic generalized inverses in $\varrho_K(A)$, see [4] and [10] for more details. Then, there exists a global analytic generalized inverse in the Kato resolvent.

Now we state some results about the Kato spectrum, which will be needed later.

Proposition 2.1. *If H is a Hilbert space, there exists an operator-valued analytic function $S\colon \varrho_K(A) \longrightarrow L(H)$ such that*

$$(A-\lambda)S(\lambda)(A-\lambda) = (A-\lambda) \qquad \textit{and} \qquad S(\lambda)(A-\lambda)S(\lambda) = S(\lambda)$$

for all $\lambda \in \varrho_K(A)$. Then the following assertions hold:

i) *If $A \in L(H)$, $\sigma_K(A)$ is a nonempty compact subset of $\mathbb{C}$ and we have*

$$\partial\sigma(A) \subset \sigma_K(A) \subset \sigma(A).$$

ii) *If $A,B \in L(H)$, then $\sigma_K(A) = \sigma_K(A^*)^*$ and $\sigma_K(A \oplus B) = \sigma_K(A) \cup \sigma_K(B)$.*

iii) *If G is a connected component of $\varrho_K(A)$ and $G \cap \varrho(A) \neq \emptyset$, then $G \subset \varrho(A)$.*

iv) *$\sigma_K(L_A) = \sigma_K(A)$ and $\sigma_K(R_A) = \sigma_K(A)$.*

Proof: The assertions (i), (ii) and (iii) have been proved in [8, Chapter 3].

(iv) To establish the remaining assertion, note that

$$\lambda \in \varrho_K(L_A) \Leftrightarrow \lambda \in \varrho_K(A)$$

is equivalent to

$$0 \in \varrho_K(L_B) \Leftrightarrow 0 \in \varrho_K(B)$$

where $B = A-\lambda$. The closedness of the range of L_B is equivalent to the closedness of the range of B, hence it is sufficient to prove that for all positive integer n we have

$$\ker(L_B) \subset R(L_B^n) \Leftrightarrow \ker(B) \subset \mathcal{R}(B^n).$$

We consider the implication $(\Rightarrow)$. Suppose that $\ker(L_B) \subset R(L_B^n)$ for fixed $n \geq 1$. Let $x \in \ker(B)$ and $x' \in H$ such that $\|x'\| = 1$, and $x \otimes x'$ the rank-one operator defined by $x \otimes x'(z) = \langle z, x'\rangle x$. It follows that $x \otimes x' \in \ker(L_B)$. By hypothesis there exists $Y \in L(H)$ such that

$$x \otimes x' = L_B^n(Y) = L_{B^n}(Y) = B^n Y.$$

This implies that $x = B^n Y(x')$, as required.

Conversely, let $X \in \ker(L_B)$ and n is a positive integer. Then, we have $R(X) \subset \ker B \subset R(B^n)$, it follows by Douglas factorization theorem [9, Theorem 7.1] that there exists a bounded operator C such that $X = B^n C = L_B^n(C)$. This completes the proof. □

Let J be a rectifiable Jordan curve, then the Jordan curve theorem asserts that $\mathbb{C} \setminus J$ consists of exactly two disjoint connected components V (the bounded component) and U (the unbounded component), in addition $\partial V = \partial U = J$. If K and L are compact subsets of $\mathbb{C}$, we say that K and L are separated by the curve J if one of the following conditions holds:

- ∘ $K \subset V$ and $L \subset U$, or
- ∘ $L \subset V$ and $K \subset U$.

The following theorem is the second main result of this paper.

Theorem 2.2. *Let $A, B \in L(H)$. If $\sigma_K(A)$ and $\sigma_K(B)$ are separated by a rectifiable Jordan curve, then δ_{AB} has a generalized inverse. Hence the range of δ_{AB} is closed, furthermore $R(\delta_{AB})$ and $\ker(\delta_{AB})$ are complemented subspaces.*

Proof: The main ingredient in the proof of this result is to use a generalized inverse for certain operators. Let J be a rectifiable Jordan curve that separates $\sigma_K(A)$ and $\sigma_K(B)$. By using the same notations as above, suppose that we have

$$\sigma_K(A) \subset V \qquad \text{and} \qquad \sigma_K(B) \subset U.$$

There exists a global analytic operator-valued function $S_B(\lambda)$ defined on $\varrho_K(B)$ such that

$$(B-\lambda)S_B(\lambda)(B-\lambda) = (B-\lambda) \qquad \text{for all } \lambda \in \varrho_K(B).$$

In particular $S_B(\lambda)$ is analytic in a neighborhood of the bounded region $\overline{V}$ whose boundary is the curve J, then we deduce from Cauchy's theorem that

$$\int_J S_B(\lambda)\,\mathrm{d}\lambda = 0.$$

It is easily seen that $\sigma(A) \subset V$. Indeed $\overline{U} = U \cup J$ is contained in a connected component G of $\varrho_K(A)$ and G is unbounded, then necessarily $\varrho(A) \cap G \neq \emptyset$. It follows from Proposition 2.1 (iii) that $G \subset \varrho(A)$, hence $\sigma(A) \subset \mathbb{C} \setminus G \subset V$. In particular, $(A-\lambda)^{-1}$ is well defined for every $\lambda \in J$. Let us define the bounded operator

$$R(Y) = \frac{1}{2\pi\mathrm{i}} \int_J (A-\lambda)^{-1} Y S_B(\lambda)\,\mathrm{d}\lambda \qquad \text{for all } Y \in \mathcal{L}(H),$$

where J is positively oriented. A straight computation shows that

$$\begin{aligned}
R\delta_{AB}(X) &= \frac{1}{2\pi\mathrm{i}}\int_J (A-\lambda)^{-1}[(A-\lambda)X - X(B-\lambda)]S_B(\lambda)\,\mathrm{d}\lambda\\
&= \frac{1}{2\pi\mathrm{i}}\int_J XS_B(\lambda)\,\mathrm{d}\lambda - \frac{1}{2\pi\mathrm{i}}\int_J (A-\lambda)^{-1}X(B-\lambda)S_B(\lambda)\,\mathrm{d}\lambda\\
&= 0 - \frac{1}{2\pi\mathrm{i}}\int_J (A-\lambda)^{-1}X(B-\lambda)S_B(\lambda)\,\mathrm{d}\lambda\,,\\
\delta_{AB}R\delta_{AB}(X) &= \frac{-1}{2\pi\mathrm{i}}\int_J (A-\lambda+\lambda)(A-\lambda)^{-1}X(B-\lambda)S_B(\lambda)\,\mathrm{d}\lambda\\
&\quad + \frac{1}{2\pi\mathrm{i}}\int_J (A-\lambda)^{-1}X(B-\lambda)S_B(\lambda)(B-\lambda+\lambda)\,\mathrm{d}\lambda\\
&= \frac{-1}{2\pi\mathrm{i}}\int_J X(B-\lambda)S_B(\lambda)\,\mathrm{d}\lambda + \frac{1}{2\pi\mathrm{i}}\int_J (A-\lambda)^{-1}X(B-\lambda)\,\mathrm{d}\lambda\\
&= 0 + \frac{1}{2\pi\mathrm{i}}\int_J (A-\lambda)^{-1}XB\,\mathrm{d}\lambda - \frac{1}{2\pi\mathrm{i}}\int_J \lambda(A-\lambda)^{-1}X\,\mathrm{d}\lambda\,.
\end{aligned}$$

By using Cauchy's theorem and the fact that $\lambda \longmapsto (B-\lambda)S_B(\lambda)$ is analytic in a neighborhood of the bounded region $\overline{V}$, one obtains

$$\int_J (B-\lambda)S_B(\lambda)\,\mathrm{d}\lambda = 0.$$

Then, it follows that

$$\delta_{AB}R\delta_{AB}(X) = \left(\frac{1}{2\pi\mathrm{i}}\int_J (A-\lambda)^{-1}\,\mathrm{d}\lambda\right)XB - \left(\frac{1}{2\pi\mathrm{i}}\int_J \lambda(A-\lambda)^{-1}\,\mathrm{d}\lambda\right)X.$$

Since the Jordan curve J surrounds $\sigma(A)$, it results from Dunford functional calculus, see [11], that

$$\frac{1}{2\pi\mathrm{i}}\int_J (A-\lambda)^{-1}\,\mathrm{d}\lambda = -1, \qquad \frac{1}{2\pi\mathrm{i}}\int_J \lambda(A-\lambda)^{-1}\,\mathrm{d}\lambda = -A.$$

Consequently, $\delta_{A,B}R\delta_{A,B} = \delta_{A,B}$. We conclude that $\delta_{A,B}$ has a generalized inverse.

In the case in which $\sigma_K(B) \subset V$ and $\sigma_K(A) \subset U$, by minor modifications of the previous argument we prove that $\delta_{A,B}$ has a generalized inverse. We shall prove it otherwise.

There exists a global analytic generalized inverse $S_A(\lambda)$ defined in $\varrho_K(A)$, in particular it is analytic in the bounded region $\overline{V}$, then Cauchy's theorem implies $\int_J S_A(\lambda)\,\mathrm{d}\lambda = 0$.

On the other hand, J surrounds necessarily $\sigma(B)$. So, we can use Dunford functional calculus for B. Hence, let us define the bounded linear operator

$$T(Y) = \frac{-1}{2\pi\mathrm{i}} \int_J S_A(\lambda) Y (B-\lambda)^{-1}\,\mathrm{d}\lambda \qquad \text{for all } \; Y \in L(H),$$

where J is positively oriented. Then, a simple calculation shows that

$$\delta_{A,B} T \delta_{A,B} = \delta_{A,B}.$$

Thus, we may conclude that $\delta_{A,B}$ has a generalized inverse if the associated Kato spectrum of A and B are separated by a certain rectifiable Jordan curve. This completes the proof. □

Remark 2.3. C. Davis and P. Rosenthal in [6] proved the following equalities

$$\sigma_\pi(\delta_{A,B}) = \sigma_\pi(A) - \sigma_\delta(B),$$
$$\sigma_\delta(\delta_{A,B}) = \sigma_\delta(A) - \sigma_\pi(B).$$

Those equalities can be used as sufficient conditions to get information about the closedness of the range of $\delta_{A,B}$. Indeed

$\sigma_\delta(A) \cap \sigma_\pi(B) = \emptyset \Leftrightarrow 0 \notin \sigma_\delta(\delta_{A,B}) \Leftrightarrow \delta_{A,B}$ is surjective (the range is trivially closed)

$\sigma_\pi(A) \cap \sigma_\delta(B) = \emptyset \Leftrightarrow 0 \notin \sigma_\pi(\delta_{A,B}) \Leftrightarrow \delta_{A,B}$ has closed range and injective.

The next example is used to show the closedness of $R(\delta_{A,B})$ by virtue of the previous theorem. However, the criterion of C. Davis and P. Rosenthal is not enough to guarantee that $\delta_{A,B}$ has closed range.

Example 2.4. Let U be the forward shift defined in a suitable Hilbert space H. We define the operator $A = U^* \oplus U$ in the Hilbert space $H \oplus H$. Let $B \in L(H \oplus H)$ be an arbitrary nonzero operator with spectrum included in the open unit disc D. It is well known that

$$\sigma(U) = \sigma(U^*) = \sigma_\pi(U^*) = \overline{D} \;\; \text{(the closed unite disk)},$$

we have also

$$\sigma_\pi(U) = \partial D \;\; \text{(the unit circle)}.$$

It is easily seen that

$$\partial D = \partial\sigma(U^*) \subset \sigma_K(U^*) \subset \sigma_\pi(U^*) \cap \sigma_\delta(U^*),$$
$$\partial D \subset \sigma_K(U^*) \subset \overline{D} \cap \sigma_\pi(U)^* = \partial D.$$

It follows that

$$\sigma_K(U^*) = \sigma_K(U) = \partial D.$$

Then, we get

$$\sigma_K(A) = \sigma_K(U^* \oplus U) = \sigma_K(U^*) \cup \sigma_K(U) = \partial D,$$
$$\sigma_\pi(A) = \sigma_\pi(U^* \oplus U) = \sigma_\pi(U^*) \cup \sigma_\pi(U) = \overline{D},$$
$$\sigma_\delta(A) = \sigma_\delta(U^*) \cup \sigma_\delta(U) = \overline{D}.$$

Furthermore, we obtain

$$\sigma_K(A) = \partial D \qquad \text{and} \qquad \sigma_\pi(A) = \sigma_\delta(A) = \overline{D}.$$

Consequently, we have

$$\sigma_\delta(A) \cap \sigma_\pi(B) = \sigma_\pi(B) \neq \emptyset \qquad \text{and} \qquad \sigma_\pi(A) \cap \sigma_\delta(B) = \sigma_\delta(B) \neq \emptyset.$$

Note that the range of $\delta_{A,B}$ is not trivial ($\neq L(H \oplus H)$), and Davis–Rosenthal's criterion asserts that either the range of $\delta_{A,B}$ is not closed or $\delta_{A,B}$ is not injective. Hence, Davis–Rosenthal's criterion cannot guarantee the closedness of the range of $\delta_{A,B}$.

On the contrary, the spectrum of B is contained in the open unit disk, then we can find an open disk D' with center 0 and radius strictly lower than 1 such that

$$\sigma_K(B) \subset \sigma(B) \subset D'.$$

Hence, the Jordan curve $\partial D'$ separates $\sigma_K(A)$ and $\sigma_K(B)$. It results from Theorem 2.2 that the range of $\delta_{A,B}$ is closed, and furthermore $\mathcal{R}(\delta_{A,B})$ and $\ker \delta_{A,B}$ have topological complements in $L(H \oplus H)$.

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