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Induced mappings on hyperspaces $F_n^K(X)$

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Abstract. Given a metric continuum X and a positive integer n, $F_n(X)$ denotes the hyperspace of all nonempty subsets of X with at most n points endowed with the Hausdorff metric. For $K \in F_n(X)$, $F_n(K,X)$ denotes the set of elements of $F_n(X)$ containing K and $F_n^K(X)$ denotes the quotient space obtained from $F_n(X)$ by shrinking $F_n(K,X)$ to one point set. Given a map $f\colon X \to Y$ between continua, $f_n\colon F_n(X) \to F_n(Y)$ denotes the induced map defined by $f_n(A) = f(A)$. Let $K \in F_n(X)$, we shall consider the induced map in the natural way $f_{n,K}\colon F_n^K(X) \to F_n^{f(K)}(Y)$. In this paper we consider the maps f, f_n, $f_{n,K}$ for some $K \in F_n(X)$ and $f_{n,K}$ for each $K \in F_n(X)$; and we study relationship between them for the following classes of maps: homeomorphisms, monotone, confluent, light and open maps.

Keywords: continuum; symmetric product; quotient space; hyperspace; induced mapping

Classification: 54B15, 54B20, 54C05, 54C10

1. Introduction

A *continuum* is a nondegenerate compact connected metric space. A *subcontinuum* of a space X is a nonempty closed connected subset of X. Given a continuum X and a positive integer n, we denote by $F_n(X)$ the collection of all nonempty subsets of X with at most n points which is called the *n-fold symmetric product of* X and it is endowed with the Hausdorff metric, see [10, Theorem 0.48, page 23].

Hyperspaces associated with a specific continuum X help to study properties of X from a different point of view. Moreover, given a hyperspace, the topological structure of some quotient spaces obtained from it, also help to study properties of the original space. Many such quotients of hyperspaces have been studied. In particular, given $K \in F_n(X)$, we consider $F_n(K,X)$ the collection of all elements in $F_n(X)$ containing K. In [2], we introduce the hyperspace $F_n^K(X)$ as the quotient space $F_n^K(X) = F_n(X)/F_n(K,X)$, which is obtained from $F_n(X)$ by shrinking

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$F_n(K,X)$ to one point set, endowed with the quotient topology. Observe that for $n=1$, we have that K is a one point set and therefore, the space $F_n^K(X)$ is homeomorphic to X. In [2], we study some topological properties of $F_n^K(X)$ as local connectedness, arcwise connectedness, unicoherence and cut points. In particular, we know that for each continuum X, $n>1$, and every $K\in F_n(X)$, $F_n^K(X)$ is a continuum, see [2, page 660], also, for each continuum X, $n>2$, and $K\in F_n(X)$, $F_n^K(X)$ is unicoherent, see [2, Theorem 7.5, page 675]. In addition, some topological properties of the continuum X are determined by $F_n^K(X)$ but not by $F_n(X)$. For example, it is well known that $F_n(X)$ has no cut points, see [7, Corollary 5, page 289]. However, in [2, Theorem 4.10, page 668], there is a characterization of cut points of X through the cut points of $F_n^K(X)$.

A *map* is a continuous function and a *mapping* means a surjective map. For the purpose of this paper, given a continuum X, $n\geq 1$, and $K\in F_n(X)$

$$\pi^X_{n,K}\colon F_n(X)\to F_n^K(X)$$

denotes the quotient mapping. If $A\in F_n(X)$, $[A]$ denotes $\pi^X_{n,K}(A)$, for convenience, if $A\in F_n(K,X)$, $[A]$ will be denoted by $F^X_{n,K}$. With this notation, we have

$$\pi^X_{n,K}(A)=\begin{cases}[A] & \text{if } A\in F_n(X)-F_n(K,X),\\ F^X_{n,K} & \text{if } A\in F_n(K,X).\end{cases}$$

Given a map $f\colon X\to Y$ between continua and an integer $n\geq 1$, the function $f_n\colon F_n(X)\to F_n(Y)$ given by $f_n(A)=f(A)$ is called the *induced map between the n-fold symmetric products of X and Y*. By [8, Corollary 1.8.23, page 65], f_n is a map. For $n\geq 1$ and $K\in F_n(X)$, we define the natural induced function $f_{n,K}\colon F_n^K(X)\to F_n^{f(K)}(Y)$ given by

$$f_{n,K}([A])=\begin{cases}\pi^Y_{n,f(K)}(f_n(A)) & \text{if } [A]\neq F^X_{n,K},\\ F^Y_{n,f(K)} & \text{if } [A]=F^X_{n,K}.\end{cases}$$

By [3, Theorem 4.3, page 126], $f_{n,K}$ is a mapping and the following diagram is commutative.

$$\begin{tikzcd} F_n(X) \arrow[r, "f_n"] \arrow[d, "\pi^X_{n,K}"'] & F_n(Y) \arrow[d, "\pi^Y_{n,f(K)}"] \\ F_n^K(X) \arrow[r, "f_{n,K}"] & F_n^{f(K)}(Y) \end{tikzcd}$$

If $K=\{p\}$, we will write $F_n^p(X)$, $F_n(p,X)$, $\pi^X_{n,p}$, $F^X_{n,p}$ and $f_{n,p}$ instead of $F_n^{\{p\}}(X)$, $F_n(\{p\},X)$, $\pi^X_{n,\{p\}}$, $F^X_{n,\{p\}}$ and $f_{n,\{p\}}$, respectively.

In the theory of induced maps it is important to know which properties are preserved between the spaces in which these maps are defined. In this way, we are interested in the following general problem.

Problem 1.1. Let $\mathcal{M}$ be a class of maps between continua. Determine the relationships between the following statements:

(1) $f \in \mathcal{M}$;
(2) $f_n \in \mathcal{M}$;
(3) $f_{n,K} \in \mathcal{M}$ for some $K \in F_n(X)$;
(4) $f_{n,K} \in \mathcal{M}$ for each $K \in F_n(X)$.

Related to the above problem, we have that condition (4) always implies (3) and the relations between (1) and (2) have been extensively studied, see for example [1], [4] and [8].

In this paper, we present results for Problem 1.1, when $\mathcal{M}$ is one of the following class: homeomorphisms, open, monotone, confluent and light. For this purpose after preliminaries, Section 3 is devoted to the class of homeomorphisms and we prove that all the statements of Problem 1.1 are equivalent, Corollary 3.3. In Section 4, we study the class of monotone mappings, in particular we show that f is monotone if and only if $f_{n,K}$ is monotone for every $K \in F_n(X)$, Corollary 4.2. In Section 5, we consider confluent mappings and we show that for $n > 2$, $f_{n,K}$ is confluent for each $K \in F_n(X)$ if and only if f is monotone (Corollary 5.2). In Section 6, we focus on light mappings and we show that if $f_{n,K}$ is light for some $K \in F_n(X)$, then f is light, see Theorem 6.2. In Section 7, for $n > 2$, we give properties of f implied by $f_{n,K}$ being open for some $K \in F_n(X)$, see Theorem 7.1 and Corollary 7.3.

Finally, in Section 8, we present some examples that help us to complete the relations of Problem 1.1 in almost all previous sections, however these examples are at the end because some results proved through the paper are used in the examples. At the end of Sections 4 to 7, we include a table which summarizes the results obtained there.

2. Preliminaries

Let X be a continuum with metric d. Let $\varepsilon > 0$ and A a subset of X, we define
$$\mathcal{V}_\varepsilon^X(A) = \{x \in X\colon \text{ there exists } y \in A \text{ such that } d(x,y) < \varepsilon\}$$
and we use the symbol $\mathrm{bd}_X(A)$, $\mathrm{cl}_X(A)$ and $\mathrm{int}_X(A)$ to denote the boundary, closure and the interior of A in X, respectively. In this paper, H denotes the Hausdorff metric.

Given a collection $U_1, \ldots, U_m$ of subsets of X with $1 \leq m \leq n$, $\langle U_1, \ldots, U_m \rangle_n$ denotes the following subset of $F_n(X)$:

$$\left\{ A \in F_n(X) \colon A \subset \bigcup_{i=1}^{m} U_i \text{ and } A \cap U_i \neq \emptyset \text{ for each } i \in \{1, \ldots, m\} \right\}.$$

It is known that the family of all subsets of $F_n(X)$ of the form $\langle U_1, \ldots, U_m \rangle_n$, where each U_i is an open subset of X, forms a basis for a topology of $F_n(X)$, see [10, Theorem 0.11, page 9], called the *Vietoris topology.* The Vietoris topology and the topology induced by the Hausdorff metric coincide, see [10, Theorem 0.13, page 10].

A map $f \colon X \to Y$ between continua is said to be:

- *confluent* if for each subcontinuum B of Y and for each component C of $f^{-1}(B)$, $f(C) = B$;
- *light* if $f^{-1}(y)$ is totally disconnected for each $y \in Y$;
- *monotone* if $f^{-1}(y)$ is connected in X for each $y \in Y$;
- *open* if f is surjetive and $f(U)$ is open in Y for each open subset U of X.

These kinds of maps have been studied in the literature, the reader interested in this topic can consult [9], where the author studied some of the types of maps already mentioned. In [2, page 662], we show that for every $n > 1$ and each $K \in F_n(X)$, $\pi^X_{n,K}$ is a monotone mapping. Also, note that if K has cardinality n then $F_n(K, X)$ is a singleton and hence, $\pi^X_{n,K}$ is a homeomorphism. However, in general $\pi^Y_{n,f(K)}$ is not a homeomorphism, even under assumption that $f^{-1}(f(K)) = K$. For example, if $f \colon [-1, 1] \to [0, 1]$ given by $f(x) = |x|$ and $K_0 = \{-1/2, 1/2\}$, it is easy to see that $\pi^{[0,1]}_{2,f(K_0)}$ is not injective. On the other hand, if $f(K)$ has cardinality n, then K has cardinality n, thus $\pi^X_{n,K}$ and $\pi^Y_{n,f(K)}$ are homeomorphisms and so $f_{n,K}$ is essentially just f_n.

Finally, if $n = 1$, then the map $f_{1,K}$ is essentially the same map as f and hence, Problem 1.1 is trivial. Also, observe that if L is a subset in Y with n elements, under assumption that f is onto, we can find $K \in F_n(X)$ such that $f(K) = L$ and so, the map $f_{n,K}$ is essentially the same as f_n. Therefore, the implications $(4) \Rightarrow (2) \Rightarrow (3)$ in Problem 1.1 are true.

By the above, in this paper we consider $K \in F_n(X)$ such that $f(K)$ has cardinality less than n. Directly from the definitions we have:

Remark 2.1. Every monotone mapping is confluent, see [6, page 123].

Proposition 2.2. *If X is a continuum, $n > 1$, and $K \in F_n(X)$, then $F^K_n(X) - \{F^X_{n,K}\}$ is homeomorphic to $F_n(X) - F_n(K, X)$.*

Proposition 2.3. *If X is a continuum, $n > 1$, and $K \in F_n(X)$, then*

$$\mathrm{int}_{F_n(X)}(F_n(K, X)) = \emptyset.$$

3. Homeomorphisms

This section is dedicated to homeomorphisms and it is shown that the statements of Problem 1.1 are equivalent.

Theorem 3.1. *Let $f\colon X \to Y$ be a map between continua and $n > 1$. Then the following statements are equivalent:*

(1) *f is injective;*
(2) *$f_{n,K}$ is injective for some $K \in F_n(X)$;*
(3) *$f_{n,K}$ is injective for each $K \in F_n(X)$.*

Proof: First we show $[(1) \Rightarrow (3)]$. For this let $K \in F_n(X)$ and let $[A], [B] \in F_n^K(X)$ be two different elements. We consider two cases:
Case (1). Let $[A], [B] \in F_n^K(X) - \{F_{n,K}^X\}$.

In this case, $A, B \notin F_n(K, X)$ and $A \neq B$. Since f is injective, we conclude that $f(K) \not\subset f(A)$, $f(K) \not\subset f(B)$ and $f(A) \neq f(B)$. Hence, $f_n(A) \neq f_n(B)$ and $f_n(A), f_n(B) \notin F_n(f(K), Y)$, which implies that $f_{n,K}([A]) \neq f_{n,K}([B])$.
Case (2). Without loss of generality, suppose that $[A] = F_{n,K}^X$.

In this case $K \subset A$ and $K \not\subset B$. Since f is injective, we have that $f(K) \not\subset f(B)$, and thus

$$f_{n,K}([B]) \neq F_{n,f(K)}^Y = f_{n,K}([A]).$$

Cases (1) and (2) show that $f_{n,K}$ is injective.

Note that $[(3) \Rightarrow (2)]$ is immediate.

For $[(2) \Rightarrow (1)]$, let $K \in F_n(X)$ be such that $f_{n,K}$ is injective. Let $x, y \in X$ be two different points. We consider two cases:
Case (1). Without loss of generality suppose that $\{x\} \in F_n(K, X)$.

In this case $K = \{x\}$, $\{y\} \notin F_n(K, X)$ and $[\{x\}] = F_{n,K}^X \neq [\{y\}]$. By definition of $f_{n,K}$, we obtain that

$$f(\{x\}) = f(K) \not\subset f_n(\{y\}) = f(\{y\}),$$

which implies that $f(x) \neq f(y)$.
Case (2). Let $\{x\}, \{y\} \notin F_n(K, X)$.

In this case $[\{x\}]$ and $[\{y\}]$ are two different elements in $F_n^K(X)$. By definition of $f_{n,K}$,

$$f(\{x\}) = f_n(\{x\}) \neq f_n(\{y\}) = f(\{y\}).$$

Thus, $f(x) \neq f(y)$. Cases (1) and (2) show that f is injective. $\square$

Similar arguments as in Theorem 3.1 can be used to prove the following result.

Theorem 3.2. *Let $f\colon X \to Y$ be a map between continua and $n > 1$. Then the following statements are equivalent:*

(1) *f is surjective;*
(2) *$f_{n,K}$ is surjective for some $K \in F_n(X)$;*
(3) *$f_{n,K}$ is surjective for each $K \in F_n(X)$.*

By [4, Theorem 3.1, page 369], we know that f is a homeomorphism if and only if f_n is a homeomorphism, since $F_n^K(X)$ and $F_n^{f(K)}(Y)$ are compact spaces, by Theorems 3.1 and 3.2, we obtain the following result.

Corollary 3.3. *Let $f\colon X \to Y$ be a map between continua and $n > 1$. Then the following statements are equivalent:*

(1) *f is a homeomorphism;*
(2) *f_n is a homeomorphism;*
(3) *$f_{n,K}$ is a homeomorphism for some $K \in F_n(X)$;*
(4) *$f_{n,K}$ is a homeomorphism for each $K \in F_n(X)$.*

4. Monotone mappings

From now, we will assume that all the functions are mappings, see Theorem 3.2.

Theorem 4.1. *Let $f\colon X \to Y$ be a mapping between continua and $n > 1$. Then f_n is monotone if and only if $f_{n,K}$ is monotone for every $K \in F_n(X)$.*

Proof: Suppose that f_n is monotone and let $K \in F_n(X)$. Since $\pi^Y_{n,f(K)}$ is monotone, by [9, Theorem 5.1, page 29], $\pi^Y_{n,f(K)} \circ f_n$ is monotone. Now, from the fact that $\pi^Y_{n,f(K)} \circ f_n = f_{n,K} \circ \pi^X_{n,K}$, we obtain that $f_{n,K} \circ \pi^X_{n,K}$ is monotone. Finally, by [9, Theorem 5.15, page 32], we conclude that $f_{n,K}$ is monotone.

Now, suppose that $f_{n,K}$ is monotone for every $K \in F_n(X)$. If $B \in F_n(Y)$, then there exists $K_0 \in F_n(X)$ such that $f(K_0) \not\subset B$. This implies that $[B] \neq F^Y_{n,f(K_0)}$. Hence, $f^{-1}_{n,K_0}([B])$ is connected. Since $F^X_{n,K_0} \notin f^{-1}_{n,K_0}([B])$, by Proposition 2.2, we have that

$$(\pi^X_{n,K_0})^{-1}(f^{-1}_{n,K_0}([B])) = f_n^{-1}(B)$$

is a connected subset of X. Therefore, f_n is monotone. □

By [4, Theorem 3.3, page 370], f is monotone if and only if f_n is monotone. Hence, by Theorem 4.1 we obtain the following result.

Corollary 4.2. *Let $f\colon X \to Y$ be a mapping between continua and $n > 1$. Then f is monotone if and only if $f_{n,K}$ is monotone for every $K \in F_n(X)$.*

In order to give an answer to Problem 1.1 for monotone mappings, note that Example 8.2 shows the existence of continua X and Y and a non monotone mapping $f: X \to Y$ such that for every $n > 1$, f_n is not monotone but, there exist $K_1, K_2 \in F_n(X)$ for which f_{n,K_1} is monotone and f_{n,K_2} is not. Hence, for every $n > 1$ we have the following table.

$\Rightarrow$	f monotone	f_n monotone	$f_{n,K}$ monotone for some K	$f_{n,K}$ monotone for each K
f monotone		Yes, [4, Thm. 3.3]	Yes, Corollary 4.2	Yes, Corollary 4.2
f_n monotone	Yes, [4, Thm. 3.3]		Yes, Theorem 4.1	Yes, Theorem 4.1
$f_{n,K}$ monotone for some K	No, Example 8.2	No, Example 8.2		No, Example 8.2
$f_{n,K}$ monotone for each K	Yes, Corollary 4.2	Yes, Theorem 4.1	Yes	

5. Confluent mappings

In order to answer Problem 1.1 for confluent mappings, first of all we will present some results for the case $n > 2$.

Theorem 5.1. *Let $f: X \to Y$ be a mapping between continua, $n > 2$, and $K \in F_n(X)$. If $f_{n,K}$ is confluent, then $f^{-1}(y)$ is a connected subset of X for each $y \in Y - f(K)$.*

Proof: Let $y \in Y - f(K)$. Suppose that $f^{-1}(y)$ is not connected. Let $z \in Y - (\{y\} \cup f(K))$ and let R be a subcontinuum of Y such that

$$\{z\} \subsetneq R \subset Y - (\{y\} \cup f(K)).$$

Following the ideas of the proof of [1, Theorem 6.4, page 1197], applied to the set $\mathcal{K} = \langle \{y\}, R \rangle_n$, it is possible to show the existence of a component $\mathcal{C}$ of $f_n^{-1}(\mathcal{K})$ such that $f_n(\mathcal{C}) \subsetneq \mathcal{K}$. Note that $\mathcal{C} \cap F_n(K, X) = \emptyset$ and $\mathcal{K} \cap F_n(f(K), Y) = \emptyset$. By Proposition 2.2, $\pi^Y_{n,f(K)}(\mathcal{K})$ is a subcontinuum of $F_n^{f(K)}(Y)$ and $\pi^X_{n,K}(\mathcal{C})$ is a component of $f_{n,K}^{-1}(\pi^Y_{n,f(K)}(\mathcal{K}))$. Then,

$$f_{n,K}(\pi^X_{n,K}(\mathcal{C})) = \pi^Y_{n,f(K)}(f_n(\mathcal{C})) = \pi^Y_{n,f(K)}(\mathcal{K}),$$

which implies that $f_n(\mathcal{C}) = \mathcal{K}$, this is a contradiction. Therefore, $f^{-1}(y)$ is a connected subset of X. $\square$

Corollary 5.2. *Let $f: X \to Y$ be a mapping between continua and $n > 2$. Then $f_{n,K}$ is confluent for each $K \in F_n(X)$ if and only if f is monotone.*

Proof: Suppose that $f_{n,K}$ is confluent for each $K \in F_n(X)$. Let $y \in Y$ and choose $K_0 \in F_n(X)$ such that $y \notin f(K_0)$. By Theorem 5.1 (applied to f_{n,K_0}), we have that $f^{-1}(y)$ is a connected subset of X. Hence, f is monotone.

The other implication follows from the fact that every monotone mapping is confluent and Corollary 4.2. □

Now, in order to present solutions of Problem 1.1, for $n = 2$, we have the following result.

Theorem 5.3. *Let $f\colon X \to Y$ be a mapping between continua, $n > 1$, and $K \in F_n(X)$. If f_n is confluent, then $f_{n,K}$ is confluent.*

Proof: Since $\pi^Y_{n,f(K)}$ is monotone, by Remark 2.1, $\pi^Y_{n,f(K)}$ is confluent. Thus, by [9, Theorem 5.4, page 29], $\pi^Y_{n,f(K)} \circ f_n$ is confluent and from the fact that $\pi^Y_{n,f(K)} \circ f_n = f_{n,K} \circ \pi^X_{n,K}$, we obtain that $f_{n,K} \circ \pi^X_{n,K}$ is also confluent. Hence, by [9, Theorem 5.16, page 32], $f_{n,K}$ is confluent. □

Theorem 5.4. *Let $f\colon X \to Y$ be a mapping between continua. If for each $p \in X$, $f_{2,p}$ is confluent, then f is confluent.*

Proof: Let B be a subcontinuum of Y, we may assume that B is a proper subcontinuum of Y. Let $p \in X$ be such that $f(p) \notin B$ and let C be a component of $f^{-1}(B)$. By [1, Lemma 6.1, page 1197], $F_2(C)$ is a component of $f_2^{-1}(F_2(B))$. Since $F_2(p, X) \cap F_2(C) = \emptyset$ and $F_2(f(p), Y) \cap F_2(B) = \emptyset$, by Proposition 2.2, $\pi^Y_{2,f(p)}(F_2(B))$ is a subcontinuum of $F_2^{f(p)}(Y)$ and $\pi^X_{2,p}(F_2(C))$ is a component of $f_{2,p}^{-1}\big(\pi^Y_{2,f(p)}(F_2(B))\big)$. Hence,

$$f_{2,p}\big(\pi^X_{2,p}(F_2(C))\big) = \pi^Y_{2,f(p)}(F_2(B)),$$

and again by Proposition 2.2, we have that $f_2(F_2(C)) = F_2(B)$. Now, we will prove that $f(C) = B$. For this purpose, it suffices to prove that $B \subset f(C)$. If $b \in B$, then $\{b\} \in F_2(B)$, and so there exists $A \in F_2(C)$ such that $f_2(A) = f(A) = \{b\}$. Hence, $b \in \{b\} = f(A) \subset f(C)$. Therefore, $f(C) = B$. This shows that f is confluent. □

By [1, Theorem 6.10, page 1199], if $f\colon X \to Y$ is a mapping between continua such that Y is locally connected, then f is confluent if and only if f_2 is confluent. Hence, by Theorem 5.3 and Theorem 5.4, we obtain the following result.

Corollary 5.5. *Let $f\colon X \to Y$ be a mapping between continua such that Y is locally connected. Then f_2 is confluent if and only if $f_{2,p}$ is confluent for each $p \in X$.*

Question 5.6. Let $f\colon X \to Y$ be a mapping between continua such that Y is not locally connected. If for each $p \in X$, $f_{2,p}$ is confluent, then f_2 is confluent?

In order to present solutions to Problem 1.1 for the class of confluent mappings, we consider the following:

(1) Example 8.1 shows the existence of a mapping between continua $f : X \to Y$ such that f is confluent but for every $n > 2$ and every $K \in F_n(X)$, $f_{n,K}$ is not.
(2) Example 8.2 shows the existence of a mapping between continua $f : X \to Y$ such that for every $n > 1$ there exist $K_0, K_1 \in F_n(X)$ with the properties that f_{n,K_0} is monotone and hence confluent, but f and f_{n,K_1} are not confluent.
(3) Example 8.4 shows the existence of a mapping between continua $f : X \to Y$ such that f is confluent for which there exists $p \in X$ such that f_2 is not confluent and $f_{2,p}$ is not confluent.

Hence, for $n > 2$, we have:

$\Rightarrow$	f confluent	f_n confluent	$f_{n,K}$ confluent for some K	$f_{n,K}$ confluent for each K
f confluent		No, [1, Thm. 6.4]	No, Example 8.1	No, Corollary 5.2
f_n confluent	Yes, [4, Thm. 3.19]		Yes, Theorem 5.3	Yes, Theorem 5.3
$f_{n,K}$ confluent for some K	No, Example 8.2	No, Example 8.2 and [4, Thm. 3.19]		No, Example 8.2
$f_{n,K}$ confluent for each K	No, Corollary 5.2	Yes, Corollary 5.2 and [4, Thm. 3.3]	Yes	

For $n = 2$, we have:

$\Rightarrow$	f confluent	f_2 confluent	$f_{2,p}$ confluent for some p	$f_{2,p}$ confluent for each p
f confluent		No, [4, Ex. 3.18]	?, Question 8.5	No, Example 8.4
f_2 confluent	Yes, [4, Thm. 3.19]		Yes, Theorem 5.3	Yes, Theorem 5.3
$f_{2,p}$ confluent for some p	No, Example 8.2	No, Example 8.2 and [4, Thm. 3.19]		No, Example 8.2
$f_{2,p}$ confluent for each p	Yes, Theorem 5.4	?, Question 5.6	Yes	

6. Light mappings

In this section we present the solution of Problem 1.1 for the class of light mappings.

Lemma 6.1. *Let $f\colon X \to Y$ be a mapping between continua, $n > 1$, and let $K \in F_n(X)$ be such that $f(K)$ has cardinality less than n. If $f_{n,K}$ is light, then $K = f^{-1}(f(K))$.*

Proof: Suppose that there exists $k_0 \in f^{-1}(f(K)) - K$. Let $k \in K$ such that $f(k) = f(k_0)$ and we consider $K_0 = (K - \{k\}) \cup \{k_0\}$. By [2, page 660], $F_n(K_0, X)$ is a continuum and since $\pi^X_{n,K}$ is continuous, $\pi^X_{n,K}(F_n(K_0, X))$ is a nondegenerate subcontinuum of $F^K_n(X)$ such that

$$\pi^X_{n,K}(F_n(K_0, X)) \subset f^{-1}_{n,K}\big(F^Y_{n,f(K)}\big),$$

which implies that $f_{n,K}$ is not light. Therefore $K = f^{-1}(f(K))$. □

Theorem 6.2. *Let $f\colon X \to Y$ be a mapping between continua and $n > 1$. If there exists $K \in F_n(X)$ such that $f_{n,K}$ is light and $f(K)$ has cardinality less than n, then f is light.*

Proof: Let $K \in F_n(X)$ be such that $f_{n,K}$ is light. Let $y \in Y$ and we shall prove that $f^{-1}(y)$ is totally disconnected. We consider two cases:
Case 1. Let $\{y\} \in F_n(f(K), Y)$.
In this case $f(K) = \{y\}$. By Lemma 6.1, $f^{-1}(\{y\}) = f^{-1}(f(K)) = K$. Thus, $f^{-1}(y)$ is totally disconnected.
Case 2. Let $\{y\} \notin F_n(f(K), Y)$.
Observe that, in this case $f(K) \not\subset \{y\}$ and $[\{y\}] \neq F^Y_{n,f(K)}$. Since $f_{n,K}$ is light, $f^{-1}_{n,K}([\{y\}]) = \pi^X_{n,K}(f^{-1}_n(\{y\}))$ is totally disconnected. Moreover, note that $f^{-1}_n(\{y\}) \cap F_n(K, X) = \emptyset$, and by Proposition 2.2, we obtain that $f^{-1}_n(\{y\})$ is totally disconnected. Therefore, $f^{-1}(y)$ is totally disconnected. □

In order to present solutions to Problem 1.1 for the class of light mappings, we consider the following: Example 8.3 shows the existence of a light mapping between continua $f\colon X \to Y$ such that for every $n > 1$, f_n is light and for each $K \in F_n(X)$ such that $f(K)$ has cardinality less than n, $f_{n,K}$ is not. Also, Example 8.2 shows a light mapping between continua $f\colon X \to Y$ such that there exist K_0 and K_1 in $F_n(X)$ for which f_{n,K_1} is light but f_{n,K_0} is not.

$\Rightarrow$	f light	f_n light	$f_{n,K}$ light for some K	$f_{n,K}$ light for each K
f light		Yes, [4, Thm. 3.26]	No, Example 8.3	No, Example 8.3
f_n light	Yes, [4, Thm. 3.26]		No, Example 8.3	No, Example 8.3
$f_{n,K}$ light for some K	Yes, Theorem 6.2	Yes, Theorem 6.2 and [4, Thm. 3.26]		No, Example 8.2
$f_{n,K}$ light for each K	Yes, Theorem 6.2	Yes, Theorem 6.2 and [4, Thm. 3.26]	Yes	

7. Open mappings

In order to answer Problem 1.1 for open mappings, we will present the following results.

Theorem 7.1. *Let $f\colon X \to Y$ be a mapping between continua and $n > 1$. If $K \in F_n(X)$ is such that $f_{n,K}$ is open and $f(K)$ has cardinality less than n, then $K = f^{-1}(f(K))$.*

Proof: Let $K = \{k_1, \ldots, k_m\} \in F_n(X)$ be such that $f_{n,K}$ is open. Without loss of generality, suppose that there exists $a \in X - K$ such that $f(a) = f(k_1)$. Let $A = (K - \{k_1\}) \cup \{a\}$ and choose $z_0 \in Y - f(K)$. Let $f(K) = \{l_1, \ldots, l_s\}$. Note that $f(K) = f(A)$ and $s < n$.

Claim A: There exists an open subset $\mathcal{U}$ in $F_n(X)$ such that $A \in \mathcal{U}$, $\mathcal{U} \cap F_n(K, X) = \emptyset$, $f_n(\mathcal{U}) \cap F_n(f(K), Y) \neq \emptyset$, moreover $f_n(\mathcal{U}) \cup F_n(f(K), Y)$ is not open in $F_n(Y)$.

We shall prove Claim A. Let W_{z_0} be an open subset of Y and for each $i \in \{1, \ldots, s\}$, let V_i be open subsets of Y such that $z_0 \in W_{z_0}$, $l_i \in V_i$, $\mathrm{cl}_Y(V_i) \cap \mathrm{cl}_Y(V_j) = \emptyset$ for each $i, j \in \{1, \ldots, s\}$ with $i \neq j$ and $\mathrm{cl}_Y(W_{z_0}) \cap \mathrm{cl}_Y(V_i) = \emptyset$ for each $i \in \{1, \ldots, s\}$. Since f is continuous, there exist open subsets $U_a, U_2, \ldots, U_m$ of X such that $a \in U_a$, $k_j \in U_j$ for each $j = 2, \ldots, m$, $k_1 \notin \mathrm{cl}_X(U_a \cup U_2 \cup \cdots \cup U_m)$ and $\langle U_a, U_2, \ldots, U_m\rangle_n \subset \langle f^{-1}(V_1), \ldots, f^{-1}(V_s)\rangle_n$. Hence, $\mathcal{U} = \langle U_a, U_2, \ldots, U_m\rangle_n$ is an open subset of $F_n(X)$ such that $A \in \mathcal{U}$, $\mathcal{U} \cap F_n(K, X) = \emptyset$ and $f_n(\mathcal{U}) \cap F_n(f(K), Y) \neq \emptyset$.

Now, we will prove that $f_n(\mathcal{U}) \cup F_n(f(K), Y)$ is not open in $F_n(Y)$. Let $D = \{z_0\} \cup f(K)$. Since $\langle U_1, \ldots, U_m\rangle_n \subset \langle f^{-1}(V_1), \ldots, f^{-1}(V_s)\rangle_n$ and by the election of W_{z_0} and $V_{i's}$, we have that $D \in F_n(F(K), Y) - f_n(\mathcal{U})$. Let $\{y_r\}_{r=1}^{\infty}$ be a sequence in $Y - \{l_1\}$ such that $\lim y_r = l_1$. For each $r \in \mathbb{N}$, let $D_r = \{z_0, y_r, l_2, \ldots, l_s\}$. Note that $\lim D_r = D$ and for each $r \in \mathbb{N}$, we have that $D_r \notin f_n(\mathcal{U}) \cup F_n(f(K), Y)$. This shows that D does not belong to the interior of $f_n(\mathcal{U}) \cup F_n(f(K), Y)$ in $F_n(Y)$ and we finish the proof of the claim.

Since $\mathcal{U} \cap F_n(K, X) = \emptyset$, by Proposition 2.2, $\pi^X_{n,K}(\mathcal{U})$ is an open subset of $F_n^K(X)$, thus

$$f_{n,K}(\pi^X_{n,K}(\mathcal{U})) = \pi^Y_{n,f(K)}(f_n(\mathcal{U}))$$

is an open subset of $F_n^{f(K)}(Y)$. This implies that

$$(\pi^Y_{n,f(k)})^{-1}\big(\pi^Y_{n,f(K)}(f_n(\mathcal{U}))\big) = F_n(f(K), Y) \cup f_n(\mathcal{U})$$

is an open subset of $F_n(Y)$, which contradicts Claim A. This finishes the proof of this result. □

Theorem 7.2. *Let $f\colon X \to Y$ be a mapping between continua and $n > 2$. If $K \in F_n(X)$ is such that $f_{n,K}$ is open, then f is a homeomorphism.*

Proof: Suppose that $K \in F_n(X)$ is such that $f_{n,K}$ is open. If $f(K)$ has cardinality n, then $f_{n,K}$ is essentially f_n. Hence, the result follows from [4, Theorem 3.6, page 371]. Therefore, we may assume that $f(K)$ has cardinality less than n. It is enough to show that f is injective. Let $x, y \in X$ be such that $x \neq y$ and suppose that $z = f(x) = f(y)$.
Claim I. Let $\{x, y\} \cap K \neq \emptyset$.
Suppose that $x, y \in X - K$. Let $y_1, \ldots, y_{n-2} \in Y - (f(K) \cup \{z\})$ such that $y_j \neq y_l$ if $j \neq l$ and $j, l \in \{1, \ldots, n-2\}$ and let $V, V_1, \ldots, V_{n-2}$ be open subsets of Y such that $z \in V$, $y_i \in V_i$, $V \cap V_i = \emptyset$ for each $i \in \{1, \ldots, n-2\}$ and $V_j \cap V_l = \emptyset$ if $j \neq l$ and $j, l \in \{1, \ldots, n-2\}$. Now, let U_x, U_y, $U_1, \ldots, U_{n-2}$ be open subsets of X such that $x \in U_x \subset f^{-1}(V) - K$, $y \in U_y \subset f^{-1}(V) - K$, $U_x \cap U_y = \emptyset$ and $x_i \in U_i \subset f^{-1}(V_i) - K$ such that $f(x_i) = y_i$ for each $i \in \{1, \ldots, n-2\}$. Note that $U_x, U_y, U_1, \ldots, U_{n-2}$ are pairwise disjoint. Let $\mathcal{U} = \langle U_x, U_y, U_1, \ldots, U_{n-2} \rangle_n$. Then, $\mathcal{U}$ is an open subset of $F_n(X)$ such that $\mathcal{U} \cap F_n(K, X) = \emptyset$, thus $\pi^X_{n,K}(\mathcal{U})$ is an open subset of $F^K_n(X)$. Let $\varepsilon > 0$, there exist $s, t \in V_1$ such that $d(s, y_1) < \varepsilon$ and $d(t, y_1) < \varepsilon$. Hence,

$$H(\{z, y_1, \ldots, y_{n-2}\}, \{z, s, t, y_2, \ldots, y_{n-2}\}) < \varepsilon,$$

also $\{z, s, t, y_2, \ldots, y_{n-2}\} \notin f_n(\mathcal{U})$, this shows that $f_n(\mathcal{U})$ is not an open subset of $F_n(Y)$. On the other hand, since $f_{n,K}$ is open, $f_{n,K}(\pi^X_{n,K}(\mathcal{U}))$ is an open subset of $F^{f(K)}_n(Y)$. By Theorem 7.1, it is easy to see that $f_n(\mathcal{U}) \cap F_n(f(K), Y) = \emptyset$, thus

$$\left(\pi^Y_{n,f(K)}\right)^{-1}\left(f_{n,K}(\pi^X_{n,K}(\mathcal{U}))\right) = f_n(\mathcal{U})$$

is an open subset of $F_n(Y)$, which is impossible. Therefore $\{x, y\} \cap K \neq \emptyset$.

By Theorem 7.1 and Claim I, we have that $\{x, y\} \subseteq K$. Assume that $K = \{x, y, k_3, \ldots, k_m\}$, $f(K) = \{l_1, \ldots, l_s\}$ and choose $z_0 \in Y - f(K)$. Note that $f(K - \{x\}) = f(K - \{y\}) = f(K)$ and $s < n$. By Claim A of Theorem 7.1, we can find an open subset $\mathcal{W}$ in $F_n(X)$ such that $K - \{x\} \in \mathcal{W}$, $\mathcal{W} \cap F_n(K, X) = \emptyset$, $f_n(\mathcal{W}) \cap F_n(f(K), Y) \neq \emptyset$ and $f_n(\mathcal{W}) \cup F_n(f(K), Y)$ is not an open subset of $F_n(Y)$. Hence, by the last part of Theorem 7.1, we obtain that $\pi^X_{n,K}(\mathcal{W})$ is an open subset in $F^K_n(X)$ but $f_{n,K}\left(\pi^X_{n,K}(\mathcal{W})\right)$ is not in $F^{f(K)}_n(Y)$. This contradicts the fact that $f_{n,K}$ is open and we conclude that f is injective. $\square$

By Corollary 3.3 and Theorem 7.2, we obtain the following result.

Corollary 7.3. *Let $f\colon X \to Y$ be a mapping between continua and $n > 2$. Then $f_{n,K}$ is open for some $K \in F_n(X)$ if and only if f is a homeomorphism.*

Now, we will present some results related with the case $n = 2$.

Theorem 7.4. *Let $f\colon X \to Y$ be a mapping between continua. If $p \in X$ is such that $f_{2,p}$ is open, then f is open.*

Proof: Let $p \in X$ be such that $f_{2,p}$ is open. Let U be an open subset of X. We consider two cases:
Case I. Let $p \notin U$.
Note that $\langle U\rangle_2$ is an open subset of $F_2(X)$ such that $F_2(p, X) \cap \langle U\rangle_2 = \emptyset$. By Theorem 7.1, it is easy to see that $\langle f(U)\rangle_2 \cap F_2(f(p), Y) = \emptyset$. Thus $\pi^X_{2,p}(\langle U\rangle_2)$ is an open subset of $F_2^p(X)$ and

$$f_{2,p}(\pi^X_{2,p}(\langle U\rangle_2)) = \pi^Y_{2,f(p)}(\langle f(U)\rangle_2)$$

is an open subset of $F_2^{f(p)}(Y)$. Hence, $\langle f(U)\rangle_2$ is an open subset of $F_2(Y)$. By [1, Corollary 5.2, page 1195], $\bigcup\langle f(U)\rangle_2 = f(U)$ is an open subset of Y.
Case II. Let $p \in U$.
Suppose that $f(U)$ is not an open subset of Y. By Theorem 7.1, we have that $f(U) \neq \{f(p)\}$. Note that, if $y \in f(U) - \{f(p)\}$, then there exists $x \in U - \{p\}$ be such that $f(x) = y$ and we can obtain an open subset W of X such that $x \in W$ and $W \subset U - \{p\}$. By Case I, $f(W)$ is an open subset of Y such that $y \in f(W) \subset f(U) - \{f(p)\}$. Therefore, we obtain that $y \in \mathrm{int}_Y(f(U))$. The above shows that $f(p) \in \mathrm{bd}_Y(f(U))$. Then there exists a sequence $\{y_n\}_{n=1}^{\infty}$ contained in $Y - f(U)$ such that $\lim_{n\to\infty} y_n = f(p)$. For each $n \in \mathbb{N}$, there exists $x_n \in X - U$ such that $f(x_n) = y_n$ and, without loss of generality, we may assume that $\lim_{n\to\infty} x_n = x_0 \in X$. Hence, $\lim_{n\to\infty} y_n = f(x_0)$, which implies that $f(x_0) = f(p)$. Finally, by Theorem 7.1, we obtain that $p = x_0$. Then $p \in \mathrm{bd}_X(U)$, this is a contradiction. Therefore, $f(U)$ is an open subset of Y. □

Corollary 7.5. *Let $f\colon X \to Y$ be a mapping between continua. If $f_{2,p}$ is open for each $p \in X$, then f is a homeomorphism.*

Proof: From Theorem 7.1, we obtain that f is injective and therefore, f is a homeomorphism. □

In order to present solutions to Problem 1.1 for the class of open mappings, note that Example 8.1 shows the existence of an open mapping between continua $f\colon X \to Y$ and points $p, q \in X$ such that $f_{2,p}$ is open but $f_{2,q}$ is not, and also, Example 8.3 shows the existence of an open mapping between continua $f\colon X \to Y$ such that for each $p \in X$, $f_{2,p}$ is not open.

For $n > 2$, we have the following:

$\Rightarrow$	f open	f_n open	$f_{n,K}$ open for some K	$f_{n,K}$ open for each K
f open		No, [4, Thm. 3.6]	No, Theorem 7.2	No, Theorem 7.2
f_n open	Yes, [4, Thm. 3.6]		Yes, [4, Thm. 3.6] and Corollary 3.3	Yes, [4, Thm. 3.6] and Corollary 3.3
$f_{n,K}$ open for some K	Yes, Corollary 7.3	Yes, Corollary 7.3 and Corollary 3.3		Yes, Corollary 7.3 and Corollary 3.3
$f_{n,K}$ open for each K	Yes, Corollary 7.3	Yes, Corollary 7.3 and Corollary 3.3	Yes	

For $n = 2$, we have the following:

$\Rightarrow$	f open	f_2 open	$f_{2,p}$ open for some p	$f_{2,p}$ open for each p
f open		Yes, [4, Thm. 3.5]	No, Example 8.3	No, Example 8.3
f_2 open	Yes, [4, Thm. 3.5]		No, Example 8.3	No, Example 8.3
$f_{2,p}$ open for some p	Yes, Theorem 7.4	Yes, Theorem 7.4 and [4, Thm. 3.5]		No, Example 8.1
$f_{2,p}$ open for each p	Yes, Corollary 7.5	Yes, Corollary 7.5 and Corollary 3.3	Yes	

8. Examples

In this section we present some examples which help us to understand the induced mappings studied in this paper from the geometrical point of view, also they show that the converse of some results enunciated in previous sections are not true, and therefore complete the study of induced mappings related with Problem 1.1.

Example 8.1. There exist continua X and Y and a mapping $f\colon X \to Y$, such that:

(1) f is confluent, light and open;
(2) for each $n > 2$, f_n is not confluent;
(3) for each $n > 1$, f_n is light;
(4) for each $n > 2$ and for each $K \in F_n(X)$, $f_{n,K}$ is not confluent;
(5) there exists $p \in X$ such that $f_{2,p}$ is not open;
(6) there exists $q \in X$ such that $f_{2,q}$ is open.

Proof: Let $X = [-1, 1]$, $Y = [0, 1]$ and let $f\colon X \to Y$ be defined by $f(x) = |x|$. It is clear that f is continuous and surjective.

Condition (1) follows immediately from the definition of f.

Condition (2) follows from the fact that f is not monotone and [1, Theorem 6.4, page 1197].

Condition (3) follows from the fact that f is light and [4, Theorem 3.26, page 380].

We shall prove condition (4). Let $n > 2$ and $K \in F_n(X)$. Note that there exists $y \in Y - f(K)$ such that $f^{-1}(y)$ is not connected. Hence, by Theorem 5.1, we obtain that $f_{n,K}$ is not confluent.

We shall prove (5). Suppose that for each $p \in X$, $f_{2,p}$ is open. By Corollary 7.5, we have that f is a homeomorfism, which is a contradiction since f is not injective.

In order to prove condition (6), we will see that $f_{2,0}$ is open. Let Γ be an open subset of $F_2^0(X)$. Then $\mathcal{U} = (\pi_{2,0}^X)^{-1}(\Gamma)$ is an open subset of $F_2(X)$. Since f is open, by [4, Theorem 3.5, page 370], f_2 is open and thus, $f_2(\mathcal{U})$ is an open subset of $F_2(Y)$. If $F_{2,0}^X \in \Gamma$, then $F_2(0,X) \subset \mathcal{U}$ and $F_2(f(0),Y) = F_2(0,Y) \subset f_2(\mathcal{U})$. Hence, $\pi_{2,0}^Y(f_2(\mathcal{U})) = f_{2,0}(\Gamma)$ is an open subset of $F_2^0(Y)$. On the other hand, assume that $F_{2,0}^X \notin \Gamma$. Since $F_2(0,Y) \cap f_2(\mathcal{U}) = \emptyset$, we have that $F_{2,0}^Y \notin f_{2,0}(\Gamma)$. By Proposition 2.2, we have that $f_{2,0}(\Gamma)$ is an open subset of $F_2^0(Y)$. Hence, $f_{2,0}$ is open. □

Example 8.2. There exist continua X and Y and a mapping $f\colon X \to Y$ such that:

(1) f is light but not confluent and hence, it is not monotone;
(2) for each $n > 1$, f_n is not confluent and hence, for each $n > 1$, f_n is not monotone;
(3) there exists $p \in X$ such that for each $n > 1$, $f_{n,p}$ is monotone and confluent, but $f_{n,p}$ is not light;
(4) there exists $K \in F_n(X)$ such that for each $n > 1$, $f_{n,K}$ is light but not confluent and hence, for each $n > 1$, $f_{n,K}$ is not monotone.

Proof: Let $X = [0,1]$ and let $Y = \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 = 1\}$ and let $f\colon X \to Y$ be defined by $f(t) = (\cos(2\pi t), \sin(2\pi t))$. It is clear that f is continuous and surjective.

Condition (1) follows from the definition of f.

Condition (2) holds, from the fact that f is not confluent and [4, Theorem 3.19, page 377].

In order to show condition (3), let $n > 1$ and $p = 0$. We shall prove that $f_{n,0}$ satisfies the required conditions. Let $[B] \in F_n^{f(0)}(Y)$. If $[B] \neq F_{n,f(0)}^Y$, then $B \notin F_n(f(0),Y)$. In this case $f_n^{-1}(B)$ is only one point set and thus, $f_{n,0}^{-1}([B])$ is only one point set. On the other hand, if $[B] = F_{n,f(0)}^Y$, then

$$f_n^{-1}(F_n(f(0),Y)) = F_n(0,X) \cup F_n(1,X),$$

since $F_n(0,X) \cap F_n(1,X) \neq \emptyset$, we have that

$$f_{n,0}^{-1}(F_{n,f(0)}^Y) = \pi_{n,0}^X(F_n(1,X))$$

is connected. This shows that $f_{n,0}$ is monotone and hence, it is confluent. Finally, since

$$f_{n,0}^{-1}(F^Y_{n,f(0)}) = \pi^X_{n,0}(F_n(1,X))$$

is a nondegenerate continuum, we have that $f_{n,0}$ is not light.

We will show (4). Let $K \in F_n(X)$ be such that $\{0,1\} \cap K = \emptyset$ and let $n > 1$. We shall show that $f_{n,K}$ is light. Let $[B] \in F_n^{f(K)}(Y)$ and recall that since f is light, by [4, Theorem 3.26, page 380], we have that f_n is light. Suppose that $[B] = F^Y_{n,f(K)}$, since

$$f_n^{-1}(F_n(f(K),Y)) = F_n(K,X),$$

$f_{n,K}^{-1}(F^Y_{n,f(K)}) = F^X_{n,K}$. On the other hand, if $[B] \neq F^Y_{n,f(K)}$, since f_n is light, by Proposition 2.2, $f_{n,K}^{-1}([B])$ is totally disconnected. Hence, $f_{n,K}$ is light.

Now, let $K = \{x,y\}$ be such that $x, y \in X - \{0\}$ and $x \neq y$. By the above, $f_{n,K}$ is light. In order to prove that $f_{n,K}$ is not monotone, note that $f(x) \neq f(y)$ and $f(K) = \{f(x), f(y)\}$. Hence, $f_{2,K}$ is essentially f_2. We will see that f_2 is not confluent. Let $M = f([0,1/3]) \cup f([2/3,1])$. Note that M is a subcontinuum of Y and $N_1 = F_2([0,1/3])$ is a component of $f_2^{-1}(F_2(M))$ in $F_2(X)$ such that $f_2(N_1)$ is a proper subcontinuum of $F_2(M)$. Therefore, f_2 is not confluent. This proves that $f_{2,K}$ is not confluent and hence, it is not monotone. For $n > 2$, since $f(0) \in Y - f(K)$ is such that $f^{-1}(f(0)) = \{0,1\}$, by Theorem 5.1, we have that $f_{n,K}$ is not confluent and hence, it is not monotone. □

Example 8.3. There exist continua X and Y and a mapping $f\colon X \to Y$ such that:

(1) f is light and open;
(2) for each $n > 1$, f_n is light;
(3) for each $n > 1$ and for each $K \in F_n(X)$ such that $f(K)$ has cardinality less than n, $f_{n,K}$ is not light;
(4) f_2 is open;
(5) for each $p \in X$, $f_{2,p}$ is not open.

Proof: Let

$$X = \left\{\left(x, \sin\left(\frac{1}{x}\right)\right) \in \mathbb{R}^2\colon\ 0 < x \leq \frac{2}{\pi}\right\} \cup (\{0\} \times [-1,1]),$$

$Y = [-1,1]$ and let $f\colon X \to Y$ be defined by $f((x,y)) = y$. It is clear that f is continuous and surjective.

Condition (1) follows from the definition of f.

Condition (2) holds from condition (1) and [4, Theorem 3.26, page 380].

We shall show (3). Let $n > 1$ and $K \in F_n(X)$. Since $f^{-1}(f(K)) \neq K$, by Lemma 6.1, we obtain that $f_{n,K}$ is not light.

Conditions (4) follows from condition (1) and [4, Theorem 3.5, page 370].

In order to prove condition (5), let $p \in X$, since $f^{-1}(f(p)) \neq \{p\}$, by Theorem 7.1, we conclude that $f_{2,p}$ is not open. $\square$

It is important to mention that the properties of the next example are a consequence of the results and the main ideas used in the proof of [4, Example 3.18, page 376].

Example 8.4. There exist continua X and Y and a mapping $f\colon X \to Y$ such that:

(1) f is confluent;
(2) f_2 is not confluent;
(3) there exists $p \in X$ such that $f_{2,p}$ is not confluent.

Proof: In the complex plane, let S^1 be the unit circle centered at the origin. Let $X = S^1 \cup I \cup J$, where I and J are two rays converging each one to one half of S^1 as it is shown in Figure 1 (see also Figure 1 of [4, Example 3.18, page 376] and compare with [5, Example 5.1]). Let f be the restriction of the complex function $z \to z^2$ to X and let $Y = f(X)$.

By [4, Example 3.18, page 376], conditions (1) and (2) hold.

In order to proof condition (3), let $p \in I \cup J$. Without loss of generality, suppose that $p \in I$. Let $\alpha\colon [0,\infty) \to f(I)$ and $\beta\colon [0,\infty) \to f(J)$ be parametrizations of $f(I)$ and $f(J)$ by arc length respectively. Let $t_0 \in [0,\infty)$ be such that $\alpha(t_0) = f(p)$ and we define $\alpha'\colon [t_0 + 1/16, \infty) \to f(I)$ by $\alpha'(t) = \alpha(t)$. Let

$$\mathcal{A}' = \Big\{\Big\{\alpha'(t), \alpha'\Big(t + \frac{1}{16}\Big)\Big\} \colon t \in \Big[t_0 + \frac{1}{16}, \infty\Big)\Big\},$$

$$\mathcal{B} = \Big\{\Big\{\beta(t), \beta\Big(t + \frac{1}{16}\Big)\Big\} \colon t \in [0,\infty)\Big\}$$

and

$$\mathcal{K}' = \mathrm{cl}_{F_2(Y)}(\mathcal{A}') \cup \mathrm{cl}_{F_2(Y)}(\mathcal{B}).$$

Compare $\mathcal{A}'$ and $\mathcal{K}'$ with $\mathcal{A}$ and $\mathcal{K}$ defined in the proof of [4, Example 3.18, page 376]. Similar arguments as in the proof of [4, Example 3.18, page 376], can be used to show $\mathcal{K}'$ is a subcontinuum of $F_2(Y)$ and that no component of $f_2^{-1}(\mathcal{K}')$ goes onto $\mathcal{K}'$ under f_2.

By Proposition 2.2 and the fact that $\mathcal{K}' \cap F_2(f(p), Y) = \emptyset$, we obtain that no component of $f_{2,p}^{-1}(\pi_{2,p}^X(\mathcal{K}'))$ goes onto $\pi_{2,p}^X(\mathcal{K}')$ under $f_{2,p}$. Hence, $f_{2,p}$ is not confluent. $\square$

Related to (4) of Problem 1.1, we do not know if $f_{2,p}$ is confluent when $p \in S^1$ in the last example. In this way, we have the following question.

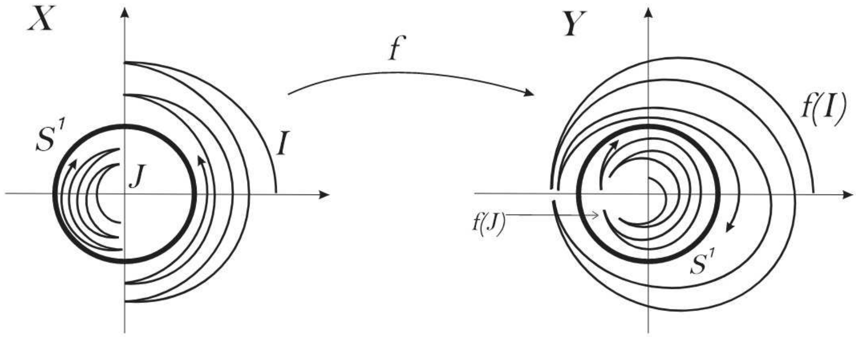

Figure 1.

Question 8.5. Let $f\colon X \to Y$ be a mapping between continua. If f is confluent, does there exist $p \in X$ such that $f_{2,p}$ is confluent?

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