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Sergio Macías

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On the set function $\wp$

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In memoriam Professor Alfredo R. L. Macías

Abstract. Inspired by the work that Professor Janusz R. Prajs did on homogeneous metric continua in his paper (2010) and the version of his work for Hausdorff continua with the uniform property of Effros done by this author, we introduce a new set function, $\wp$, and present properties of it.

Keywords: additivity; almost connected im kleinen; analytic set; aposyndetic continuum; atomic map; continuum; decomposable continuum; G_δ set; hyperspace; indecomposable continuum; monotone map; property of Kelley; set function $\mathcal{K}$; set function $\mathcal{T}$; set function $\wp$; set functions continuous on continua; uniform property of Effros; upper semicontinuous function

Classification: 54B20, 54C60, 54F16

1. Introduction

Inspired by the work that Professor Janusz R. Prajs did on homogeneous metric continua [20] and the version of his work for Hausdorff continua with the uniform property of Effros done by this author [12], see also [15, Section 3.4], we introduce the set function $\wp$ and present properties of it. Our main motivation is [12, Lemma 5.7] or [15, Lemma 3.4.7].

The paper is divided in nine sections. After this introduction, we provide definitions in Section 2. In Section 3, we present properties of $\wp$. For example, in Theorem 3.8, we show that if X is a continuum such that the image of each singleton under $\wp$ is X, then X is indecomposable, and in Example 3.10, we present a decomposable metric continuum showing that the converse implication of Theorem 3.8 is not true. In Theorem 3.13, we extend the additivity property to families of nonempty closed subsets whose union is closed. In Theorem 3.19, we give a sufficient condition to have that $\wp$ is idempotent on closed sets. Assuming the continuity of Professor Jones' set function $\mathcal{T}$, if A is a subcontinuum of X, then we show that the continua W in the definition of $\wp(A)$ may be taken with connected interior (Theorem 3.28). We present a sufficient condition for the aposyndesis of a continuum (Theorem 3.30), we observe that Example 3.10 also

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shows that the converse of this result is not true. We give a sufficient condition for the colocal connectedness of a continuum (Theorem 3.33) and present an example of a continuum proving the converse of this theorem is not true (Example 3.34). We prove that, for the class of locally connected continua, $\wp$ coincides with Professor Jones' set function $\mathcal{K}$ (Theorem 3.38). In Section 4, we present relationships between maps and $\wp$. In Theorem 4.1, we show that a homeomorphism and $\wp$ commute. In Theorem 4.2, we give a formula for the image of a nonempty closed subset of a continuum X under $\wp$ in terms of an atomic open map. We prove that if we have an atomic open map between continua X and Y, then the cardinality of the family of $\wp$-closed sets of X is equal to the cardinality of the family of $\wp$-closed sets of Y (Corollary 4.12). We also mention that the set function $\wp$ may be used to define upper semicontinuous functions $\wp_f$ and do dynamical systems as it is done in [2], (Remark 4.13). In Section 5, we show that $\wp$ is always upper semicontinuous (Theorem 5.1) and that $\wp$ is continuous for the class of locally connected metric continua (Theorem 5.2). In Section 6, we work with the class of continua with the property of Kelley. We show that for the class of $\mathcal{T}$-symmetric continua with the property of Kelley, $\wp$ is idempotent on closed sets (Theorem 6.2). In Theorem 6.5, we prove that the family of images of singletons under $\wp$ forms a decomposition of a continuum with the property of Kelley. We present a proof of Professor Janusz J. Prajs' mutual aposyndesis decomposition theorem for the class of continua with the uniform property of Effros using the set function $\wp$ (Theorem 6.10). We give a sufficient condition on a homogeneous continuum with the uniform property of Effros to have that Professor F. Burton Jones' aposyndetic decomposition coincides with Prajs' decomposition (Theorem 6.11). We end this section with a theorem about the dimension of the images of the singletons under $\wp$ for homogeneous metric continua that are not aposyndetic (Corollary 6.13). In Section 7, we consider generalized inverse limits of metric homogeneous continua to exhibit an infinite dimensional homogeneous compact metric space having a continuous decomposition whose quotient space is a mutually aposyndetic metric continuum, the homogeneous compact metric space also contains an infinite dimensional homogeneous continuum (Theorem 7.5). In Section 8, we work with the class of metric continua. We show that the family of $\wp$-closed sets of a metric continuum X is a G_δ subset of the hyperspace of nonempty closed subsets, 2^X, of X (Theorem 8.1). We also prove that, for a metric continuum X, $\wp(2^X)$ is an analytic set (Theorem 8.2). Section 9 contains a result of independent interest, we show that every continuum with the uniform property of Effros is an Effros continuum (Theorem 9.6), the reverse implication is already known.

2. Definitions

If Z is a Hausdorff topological space, given a subset A of Z, the interior of A is denoted by $\mathrm{Int}(A)$, the boundary of A is denoted by $\mathrm{Bd}(A)$, and the closure of A by $\mathrm{Cl}(A)$.

A *compactum* is a compact Hausdorff space with more than one point. A *subcompactum* of a Hausdorff topological space Z is a compactum contained in Z. A *continuum* is a connected compactum. A *subcontinuum* of a Hausdorff topological space Z is a continuum contained in Z. A subcontinuum W of a continuum X is a *strong continuum domain of* X provided that $\mathrm{Int}(W)$ is a nonempty connected open subset of X and $W = \mathrm{Cl}(\mathrm{Int}(W))$. A continuum is *decomposable* if it is the union of two of its proper subcontinua. A continuum is *indecomposable* if it is not decomposable. A subcontinuum K of a continuum X is *terminal* provided that for every subcontinuum L of X such that $L \cap K \neq \emptyset$, we have that either $L \subset K$ or $K \subset L$. A continuum X is *almost connected im kleinen at a point* p if for each open subset U of X, having p, there exists a subcontinuum W of X such that $\mathrm{Int}(W) \neq \emptyset$ and $W \subset U$. A continuum X is *colocally connected* provided that each point x of X has a local basis of open subsets whose complements are connected.

A continuum X is *semi-aposyndetic* if for each pair of distinct points p and q of X, there exists a subcontinuum W of X such that $\{p, q\} \cap \mathrm{Int}(W) \neq \emptyset$ and $\{p, q\} \cap (X \setminus W) \neq \emptyset$. Let X be a continuum and let p and q be two distinct points of X. Then X is *aposyndetic at p with respect to q* provided that there exists a subcontinuum W of X such that $p \in \mathrm{Int}(W) \subset W \subset X \setminus \{q\}$. We say that X is *aposyndetic at p* if X is aposyndetic at p with respect to each point $q \in X \setminus \{p\}$. Also, we have that X is *aposyndetic* if it is aposyndetic at each of its points. A continuum X is *mutually aposyndetic* provided that for each pair p and q of distinct points of X, there exist two disjoint subcontinua W_1 and W_2 of X such that $p \in \mathrm{Int}(W_1)$ and $q \in \mathrm{Int}(W_2)$.

A *map* is a continuous function. A surjective map $f\colon X \twoheadrightarrow Y$ between continua is *monotone* provided that $f^{-1}(C)$ is connected for every connected subset C of Y. The map f is *atomic* if for each subcontinuum L of X such that $f(L)$ is nondegenerate, then $L = f^{-1}(f(L))$.

Remark 2.1. Note that atomic maps between continua are monotone [14, Theorem 8.1.24]. Observe that the proof given in [14, Theorem 8.1.24] is valid for Hausdorff continua since it only uses the boundary bumping theorem which is true for Hausdorff continua, a proof of it may be found in [15, Theorem 1.4.36].

Let X be a compactum and let $\mathcal{F}$ be a family of nonempty subsets of X. We say that the group of homeomorphisms $H(X)$ *respects* $\mathcal{F}$, provided that for each pair of elements F and F' of $\mathcal{F}$ and every $h \in H(X)$, either $h(F) = F'$ or $h(F) \cap F' = \emptyset$.

F. Burton Jones defined what are now known as the set functions $\mathcal{T}$ and $\mathcal{K}$, see [10]. He used them to study aposyndesis on metric continua. These functions have been used to study continua, $\mathcal{T}$ has been used more that $\mathcal{K}$, see [15]. The set functions $\mathcal{T}$ and $\mathcal{K}$ are considered "dual" functions. Next, we present their definition.

Given a continuum X, we define the *set function* $\mathcal{T}$ as follows: If A is a subset of X, then

$$\mathcal{T}(A) = X \setminus \{x \in X \colon \text{ there exists a subcontinuum } W \text{ of } X \text{ such that } x \in \operatorname{Int}(W) \subset W \subset X \setminus A\}.$$

Observe that for any subset A of X, $\mathcal{T}(A)$ is a closed subset of X and $A \subset \mathcal{T}(A)$. Note that, if W is a subcontinuum of X, then $\mathcal{T}(W)$ is a subcontinuum of X [15, Theorem 2.1.27]. Also, by [15, Corollary 2.1.14], $\mathcal{T}(\emptyset) = \emptyset$.

For the continuum X, the *set function* $\mathcal{K}$ is defined as follows: If A is a subset of X, then

$$\mathcal{K}(A) = \bigcap \{W \colon W \text{ is a subcontinuum of } X \text{ such that } A \subset \operatorname{Int}(W)\}.$$

We also have that for any nonempty subset A of X, $\mathcal{K}(A)$ is a closed subset of X, $A \subset \mathcal{K}(A)$ and $\mathcal{K}(\emptyset) = \emptyset$. It is not necessarily true that if L is a subcontinuum of X, then $\mathcal{K}(L)$ is connected, see [8, Example 4].

Next, we introduce the set function that is the objective of our study, which is inspired by [12, Lemma 5.7], see also [15, Lemma 3.4.7].

Given a continuum X, we define the *set function* $\wp$ as follows: If A is a subset of X, then

$$\wp(A) = \bigcap \{\mathcal{T}(W) \colon W \text{ is a subcontinuum of } X \text{ and } A \subset \operatorname{Int}(W)\}.$$

Note that it is not necessarily true that $\wp(\emptyset) = \emptyset$, see Theorem 3.8 and Example 3.11.

Let X be a continuum and let $\mathcal{R} \in \{\mathcal{T}, \mathcal{K}, \wp\}$. Then X is *point $\mathcal{R}$-symmetric* provided that for each pair of points p and q of X, we have that $p \in \mathcal{R}(\{q\})$ if and only if $q \in \mathcal{R}(\{p\})$. Also, X is *$\mathcal{R}$-symmetric* if given two nonempty closed subsets A and B of X, $A \cap \mathcal{R}(B) = \emptyset$ if and only if $\mathcal{R}(A) \cap B = \emptyset$. We say that X is *$\mathcal{R}$-additive* provided that given two nonempty closed subsets A and B of X, $\mathcal{R}(A \cup B) = \mathcal{R}(A) \cup \mathcal{R}(B)$. The set function $\mathcal{R}$ is *idempotent* if $\mathcal{R}^2(A) = \mathcal{R}(A)$ for all subsets A of X. Now, $\mathcal{R}$ is *idempotent on closed sets* (*idempotent on continua*, *idempotent on singletons*, respectively) if for each nonempty closed subset (continuum, singleton, respectively) A of X, $\mathcal{R}^2(A) = \mathcal{R}(A)$.

Let X be a continuum. We consider the following *hyperspaces* of X:

$$2^X = \{A \subset X\colon A \text{ is closed and nonempty}\},$$
$$\mathcal{C}_1(X) = \{A \in 2^X\colon A \text{ is connected}\},$$
$$\mathcal{F}_1(X) = \{\{x\}\colon x \in X\},$$

with the *Vietoris topology*, defined below. If $K_1, \dots, K_m$ are nonempty subsets of X, let

$$\langle K_1, \dots, K_m\rangle = \left\{A \in 2^X\colon A \subset \bigcup_{j=1}^{m} K_j \text{ and } A \cap K_j \neq \emptyset \text{ for all } j \in \{1, \dots, m\}\right\}.$$

A base for the Vietoris topology for 2^X is given by the sets of the form $\langle U_1, \dots, U_m\rangle$, where $U_1, \dots, U_m$ are open subsets of X, see [19, Theorem (0.11)].

Let X be a continuum and let $\mathcal{R} \in \{\mathcal{T}, \mathcal{K}, \wp\}$. A nonempty closed subset A of X is an $\mathcal{R}$*-closed set* provided that $\mathcal{R}(A) = A$. The family of $\wp$-closed sets is denoted by $\mathfrak{P}(X)$. Note that we may restrict $\mathcal{R}$ to 2^X to have a function $\mathcal{R}\colon 2^X \to 2^X$ (for $\wp$ see Proposition 3.1). Since 2^X has the Vietoris topology, we may consider the continuity of $\mathcal{R}$. We say that X is a continuum for which $\mathcal{R}$ *is continuous* if $\mathcal{R}\colon 2^X \to 2^X$ is continuous.

In order to present the definition of the uniform property of Effros, we need to recall the definition of a uniformity.

Let Z be a Hausdorff space. If V and W are subsets of $Z \times Z$, then

$$-V = \{(z', z) \in Z \times Z\colon (z, z') \in V\}$$

and

$$V + W = \{(z, z'')\colon \text{ there exists } z' \in Z \text{ such that } (z, z') \in V \text{ and } (z', z'') \in W\}.$$

We write $1V = V$ and for each positive integer n, $(n+1)V = nV + 1V$.

The diagonal of Z is the set $\Delta_Z = \{(z, z)\colon z \in Z\}$. An *entourage* of Δ_Z is a subset V of $Z \times Z$ such that $\Delta_Z \subset V$ and $V = -V$. The family of all entourages of the diagonal of Z is denoted by $\mathfrak{D}_Z$. If $V \in \mathfrak{D}_Z$ and $(z, z') \in V$, then we write $\varrho_Z(z, z') < V$. If $V \in \mathfrak{D}_Z$ and $z \in Z$, then $B(z, V) = \{z' \in Z\colon \varrho_Z(z, z') < V\}$. We have that if z, z' and z'' are points of Z, and V and W belong to $\mathfrak{D}_Z$ then the following hold, see [4, page 426]:

(i) $\varrho_Z(z, z) < V$.
(ii) $\varrho_Z(z, z') < V$ if and only if $\varrho_Z(z', z) < V$.
(iii) If $\varrho_Z(z, z') < V$ and $\varrho_Z(z', z'') < W$, then $\varrho_Z(z, z'') < V + W$.

Let Z be a Tychonoff space. A *uniformity* on Z is a subfamily $\mathfrak{U}$ of $\mathfrak{D}_Z \setminus \{\Delta_Z\}$ such that:

(1) If $V \in \mathfrak{U}$, $W \in \mathfrak{D}_Z$ and $V \subset W$, then $W \in \mathfrak{U}$.
(2) If V and W belong to $\mathfrak{U}$, then $V \cap W \in \mathfrak{U}$.
(3) For every $V \in \mathfrak{U}$, there exists $W \in \mathfrak{U}$ such that $2W \subset V$.
(4) $\bigcap\{V\colon V \in \mathfrak{U}\} = \Delta_Z$.

A *uniform space* is a pair $(Z, \mathfrak{U})$ consisting of a nonempty set Z and a uniformity on the set Z. For any uniformity $\mathfrak{U}$ on a set Z, the family $\mathfrak{O} = \{G \subset Z\colon$ for every $z \in G$, there exists $V \in \mathfrak{U}$ such that $B(z, V) \subset G\}$ is a topology on the set Z, see [4, 8.1.1]. The topology $\mathfrak{O}$ is called the *topology induced by the uniformity* $\mathfrak{U}$.

Remark 2.2. Note that by [4, 8.3.13], for every compactum Z, there exists a unique uniformity $\mathfrak{U}_Z$ on Z that induces the original topology of Z.

Let Z_1 and Z_2 be Tychonoff spaces, let $\mathfrak{U}$ and $\mathfrak{U}'$ be uniformities inducing the topology of Z_1 and Z_2, respectively. Let $g\colon Z_1 \twoheadrightarrow Z_2$ be a surjective uniformly continuous function. Then g is *uniformly completely regular with respect to* $\mathfrak{U}$ *and* $\mathfrak{U}'$ provided that for each $U \in \mathfrak{U}$, there exists $V \in \mathfrak{U}'$ such that if z_2 and z_2' are two points of Z_2 with $\varrho_{Z_2}(z_2, z_2') < V$, there exists a homeomorphism $h\colon g^{-1}(z_2) \twoheadrightarrow g^{-1}(z_2')$ such that $\varrho_{Z_1}(z, h(z)) < U$ for all points $z \in g^{-1}(z_2)$. Information about this class of maps may be found in [15, Section 1.5].

Many of the properties presented in the paper are similar to the ones that the set functions $\mathcal{K}$ and $\mathcal{T}$ have. We include the details for the convenience of the reader.

3. General properties

We present properties of the set function $\wp$. Note that from the definition of $\wp$, we have:

Proposition 3.1. *Let X be a continuum. If A is a nonempty subset of X, then $\wp(A)$ is a closed subset of X and $A \subset \wp(A)$.*

As a consequence of the definition of the set functions $\mathcal{K}$ and $\wp$, we obtain:

Proposition 3.2. *Let X be a continuum. If A is a nonempty subset of X, then $\mathcal{K}(A) \subset \wp(A)$.*

Proposition 3.3. *If X is a hereditarily unicoherent continuum, then $\wp(A)$ is a subcontinuum of X for every nonempty subset A of X.*

Proof: Let A be a nonempty subset of X. By [15, Theorem 2.1.27], if W is a subcontinuum of X, then $\mathcal{T}(W)$ is a subcontinuum of X. Hence, since X is hereditarily unicoherent, $\wp(A)$ is connected. Therefore, $\wp(A)$ is a subcontinuum of X (Proposition 3.1). $\square$

Lemma 3.4. *If X is a continuum and A and B are nonempty subsets of X such that $A \subset B$, then $\wp(A) \subset \wp(B)$.*

Proof: Let x be an element of $X \setminus \wp(B)$. Then there exists a subcontinuum W of X such that $B \subset \mathrm{Int}(W)$ and $x \in X \setminus \mathcal{T}(W)$. Since $A \subset B$, $A \subset \mathrm{Int}(W)$ and $x \in X \setminus \mathcal{T}(W)$. Hence, x belongs to $X \setminus \wp(A)$. □

Corollary 3.5. *If X is a continuum and A and B are nonempty subsets of X, then $\wp(A) \cup \wp(B) \subset \wp(A \cup B)$.*

Theorem 3.6. *Let X be a continuum and let A be a nonempty subset of X. If B is a nonempty subset of X and $B \subset \mathrm{Int}(\wp(A))$, then $\wp(B) \subset \wp(A)$.*

Proof: Let B be a nonempty subset of X such that $B \subset \mathrm{Int}(\wp(A))$. Let z be a point of $X \setminus \wp(A)$. Then there exists a subcontinuum W of X such that $A \subset \mathrm{Int}(W)$ and $z \in X \setminus \mathcal{T}(W)$. Note that $\wp(A) \subset W$. Since $B \subset \mathrm{Int}(\wp(A))$, we have that $B \subset \mathrm{Int}(W)$. Thus, z belongs to $X \setminus \wp(B)$. Therefore, $\wp(B) \subset \wp(A)$. □

Example 3.7. Let X be the cone over the Cantor set. Note that, since every subcontinuum of X with nonempty interior contains the vertex of the cone, we have that $\wp(\{x\}) = X$ for all points x of X, see [15, Example 2.1.15]. Thus, $\wp(A) = X$ for each nonempty subset A of X.

Theorem 3.8. *If X is an indecomposable continuum, then $\wp(\{x\}) = X$ for all points x in X. In particular, $\wp(A) = X$ for every nonempty subset A of X. We also have that $\wp(\emptyset) = X$.*

Proof: If X is an indecomposable continuum, then for each proper subcontinuum W of X, $\mathrm{Int}(W) = \emptyset$, see [15, Corollary 1.4.35]. Hence, $\wp(\{x\}) = X$ for each point x of X. If A is a nonempty subset of X, by Lemma 3.4, $\wp(A) = X$. To prove that $\wp(\emptyset) = X$, note that, since $\emptyset$ is a subset of the interior for every proper subcontinuum W of X, which is empty, see [15, Corollary 1.4.35], and $\mathcal{T}(W) = X$, see [15, Theorem 2.1.44], we obtain that $\wp(\emptyset) = X$. □

Note that Example 3.7 is a hereditarily decomposable and hereditarily unicoherent metric continuum that proves that the converse implication of Theorem 3.8 is not true. In the following example we present a family of decomposable metric continua for which the converse implication of Theorem 3.8 is not valid. This family also shows that the converse implication of Theorems 3.29 and 3.30 are not true either. Next lemma is used in Example 3.10.

Lemma 3.9. *If X and Y are continua, then $X \times Y$ is a decomposable continuum.*

Proof: It is clear that $X \times Y$ is a continuum. Let U be an open subset of X such that $\mathrm{Cl}(U) \neq X$. Let y_1 be a point of Y, and let

$$L = (\mathrm{Cl}(U) \times Y) \cup (X \times \{y_1\}).$$

Then L is a proper subcontinuum of $X \times Y$ such that $\mathrm{Int}(L) \neq \emptyset$. Hence, by [15, Lemma 1.4.34], $X \times Y$ is a decomposable continuum. □

Example 3.10. A metric continuum X is *chainable* provided that for each $\varepsilon > 0$, there exists a finite open cover $\mathcal{U} = \{U_1, \dots, U_m\}$ of X such that $U_j \cap U_k \neq \emptyset$ if and only if $|j - k| \leq 1$ and $\mathrm{diam}(U_j) < \varepsilon$ for all $j \in \{1, \dots, m\}$. C. L. Hagopian proved that if X is an indecomposable chainable metric continuum, then for every pair of subcontinua W_1 and W_2 of $X \times X$ such that $\mathrm{Int}(W_j) \neq \emptyset$, $j \in \{1, 2\}$ (Lemma 3.9), we have that $W_1 \cap W_2 \neq \emptyset$, see [7, Theorem 10]. Hence, if X is an indecomposable chainable metric continuum and W is a subcontinuum of $X \times X$ such that $\mathrm{Int}(W) \neq \emptyset$, then $\mathcal{T}_{X \times X}(W) = X \times X$. Let X be an indecomposable chainable metric continuum. Then $X \times X$ is a decomposable continuum and for each point (x_1, x_2) of $X \times X$, we have that $\wp_{X \times X}(\{(x_1, x_2)\}) = X \times X$.

In the next example, we give a decomposable continuum Z for which the image of the empty set under $\wp$ is nonempty.

Example 3.11. Let $Z = Z_1 \cup Z_2$ be a continuum, where Z_1 and Z_2 are indecomposable continua such that $Z_1 \cap Z_2 = \{p\}$. Let W be a subcontinuum. If $W \cap Z_2 = \emptyset$, then $\mathcal{T}(W) = Z_1$, see [15, Theorem 2.1.44]. Similarly, if $W \cap Z_1 = \emptyset$, then $\mathcal{T}(W) = Z_2$. Note that if $W \cap Z_1 \neq \emptyset$ and $W \cap Z_2 \neq \emptyset$, then $\mathcal{T}(W) = Z$. Hence, $\wp(\emptyset) = \{p\}$.

Theorem 3.12. *Let X be a continuum and let Γ be a filterbase of nonempty closed subsets of X. Then $\wp\big(\bigcap \Gamma\big) = \bigcap\{\wp(G) \colon G \in \Gamma\}$.*

Proof: Note that, by Lemma 3.4, we have that $\wp\big(\bigcap \Gamma\big) \subset \bigcap\{\wp(G) \colon G \in \Gamma\}$. Let $x \in X \setminus \wp\big(\bigcap \Gamma\big)$. By definition, there exists a subcontinuum W of X such that $\bigcap \Gamma \subset \mathrm{Int}(W)$ and $x \in X \setminus \mathcal{T}(W)$. Since $X \setminus \mathrm{Int}(W)$ is compact, there exist $G_1, \dots, G_m$ in Γ such that $X \setminus \mathrm{Int}(W) \subset \bigcup_{j=1}^m X \setminus G_j$. Hence, $\bigcap_{j=1}^m G_j \subset \mathrm{Int}(W)$. Since Γ is a filterbase, there exists G_0 in Γ such that $G_0 \subset \bigcap_{j=1}^m G_j$. Hence, $G_0 \subset \mathrm{Int}(W)$ and $x \in X \setminus \mathcal{T}(W)$. Thus, $x \in X \setminus \wp(G_0)$ and $x \in X \setminus \bigcap\{\wp(G) \colon G \in \Gamma\}$. Therefore, $\wp\big(\bigcap \Gamma\big) = \bigcap\{\wp(G) \colon G \in \Gamma\}$. □

The following theorem extends the definition of $\wp$-additivity to certain families of nonempty closed sets.

Theorem 3.13. *Let X be a continuum. Then X is $\wp$-additive if and only if for each collection Γ of nonempty closed subsets of X whose union is a closed subset of X, we have that $\wp\big(\bigcup \Gamma\big) = \bigcup\{\wp(G) \colon G \in \Gamma\}$.*

Proof: Suppose X is $\wp$-additive and let Γ be a collection of nonempty closed subsets of X whose union is a closed subset of X. By Lemma 3.4, we have that $\bigcup\{\wp(G) \colon G \in \Gamma\} \subset \wp\big(\bigcup \Gamma\big)$. Suppose $x \in X \setminus \bigcup\{\wp(G) \colon G \in \Gamma\}$. Then for each $G \in \Gamma$, let $\mathcal{F}(G) = \{A \subset X\colon A$ is closed and $G \subset \mathrm{Int}(A)\}$. Then $\mathcal{F}(G)$ is a filterbase of closed subsets of X. Since $\bigcap\{A\colon A \in \mathcal{F}(G)\} = G$, by Theorem 3.12, $\wp(G) = \bigcap\{\wp(A)\colon A \in \mathcal{F}(G)\}$.

Hence, for each $G \in \Gamma$, $x \in X \setminus \bigcap\{\wp(A)\colon A \in \mathcal{F}(G)\}$. Thus, for every $G \in \Gamma$, there exists $A_G \in \mathcal{F}(G)$ such that $x \in X \setminus \wp(A_G)$. Note that $\{\mathrm{Int}(A_G)\colon G \in \Gamma\}$ is an open cover of $\bigcup\{G\colon G \in \Gamma\}$. Since this set is compact, there exist $G_1, \dots, G_m$ in Γ such that $\bigcup\{G\colon G \in \Gamma\} \subset \bigcup_{j=1}^m \mathrm{Int}(A_{G_j})$. Since, by hypothesis and mathematical induction, $\wp\big(\bigcup_{j=1}^m A_{G_j}\big) = \bigcup_{j=1}^m \wp(A_{G_j})$, we obtain that $\wp\big(\bigcup\{G\colon\ G \in \Gamma\}\big) \subset \bigcup_{j=1}^m \wp(A_{G_j})$. Now, since we have that for every $j \in \{1, \dots, m\}$, $x \in X \setminus \wp(A_{G_j})$, it follows that $x \in X \setminus \wp\big(\bigcup\{G\colon G \in \Gamma\}\big)$. Thus, we obtain that $\wp\big(\bigcup\{G\colon G \in \Gamma\}\big) \subset \bigcup\{\wp(G)\colon G \in \Gamma\}$. Therefore, $\wp\big(\bigcup \Gamma\big) = \bigcup\{\wp(G)\colon G \in \Gamma\}$. The other implication is clear. □

Theorem 3.14. *Let X be a continuum and let A be a nonempty closed subset of X. If $x \in X \setminus \wp(A)$, then there exists an open subset U of X such that $A \subset U$ and $x \in X \setminus \wp(\mathrm{Cl}(U))$.*

Proof: Let $x \in X \setminus \wp(A)$. Then, by definition, there exists a subcontinuum W of X such that $A \subset \mathrm{Int}(W)$ and $x \in X \setminus \mathcal{T}(W)$. Since X is a normal space, there exists an open subset U of X such that $A \subset U \subset \mathrm{Cl}(U) \subset \mathrm{Int}(W)$. Hence, U is an open subset of X containing A, $\mathrm{Cl}(U) \subset \mathrm{Int}(W)$ and $x \in X \setminus \mathcal{T}(W)$. Thus, $x \in X \setminus \wp(\mathrm{Cl}(U))$. □

Observe that the next theorem is a consequence of Theorem 3.13. We present a proof using the definition of $\wp$-additivity and Theorem 3.14.

Theorem 3.15. *If X is a $\wp$-additive continuum and A is a nonempty closed subset of X, then $\wp(A) = \bigcup\{\wp(\{a\})\colon a \in A\}$.*

Proof: Let A be a closed subset of X. We know that $\bigcup\{\wp(\{a\})\colon a \in A\} \subset \wp(A)$ (Lemma 3.4). Let $x \in X \setminus \bigcup\{\wp(\{a\})\colon a \in A\}$. Then for each point a of A, by Theorem 3.14, there exists an open subset U_a of X such that $a \in U_a$ and $x \in X \setminus \wp(\mathrm{Cl}(U_a))$. Since A is a compact set, there exist $a_1, \dots, a_n$ in A such that $A \subset \bigcup_{j=1}^n U_{a_j}$. Note that this implies that $\wp(A) \subset \wp\big(\bigcup_{j=1}^n \mathrm{Cl}(U_{a_j})\big) = \bigcup_{j=1}^n \wp(\mathrm{Cl}(U_{a_j})) \subset X \setminus \{x\}$ (the equality is true by $\wp$-additivity and mathematical induction). Hence, $x \in X \setminus \wp(A)$ and $\wp(A) = \bigcup\{\wp(\{a\})\colon a \in A\}$. □

Theorem 3.16. *Let X be a $\wp$-additive continuum. If $\wp$ is idempotent on singletons, then $\wp$ is idempotent on closed sets.*

Proof: Let A be a nonempty closed subset of X. Since X is $\wp$-additive, by Theorem 3.15, we have that $\wp(A) = \bigcup\{\wp(\{a\})\colon a \in A\}$. Thus, $\wp^2(A) = \wp(\bigcup\{\wp(\{a\})\colon a \in A\}) = \bigcup\{\wp^2(\{a\})\colon a \in A\} = \bigcup\{\wp(\{a\})\colon a \in A\} = \wp(A)$. The second equality is true by $\wp$-additivity (Theorem 3.13) and the third one holds by the idempotency on singletons of $\wp$. $\square$

Corollary 3.17. *Let X be a $\wp$-additive continuum. If $\wp$ is idempotent on continua, then $\wp$ is idempotent on closed sets.*

Theorem 3.18. *If X is a $\mathcal{T}$-additive continuum and W is a subcontinuum of X, then $\wp(W) = \bigcup\{\wp(\{w\})\colon w \in W\}$.*

Proof: We know that $\bigcup\{\wp(\{w\})\colon w \in W\} \subset \wp(W)$ (Lemma 3.4). Let $x \in X \setminus \bigcup\{\wp(\{w\})\colon w \in W\}$. Then $x \in \bigcap\{X \setminus \wp(\{w\})\colon w \in W\}$. For each point $w \in W$, by definition, there exists a subcontinuum M_w of X such that $w \in \operatorname{Int}(M_w)$ and $x \in X \setminus \mathcal{T}(M_w)$. Then the family $\{\operatorname{Int}(M_w)\colon w \in W\}$ is an open cover of W. Since W is compact, there exist $w_1, \dots, w_n$ in W such that $W \subset \bigcup_{j=1}^{n} \operatorname{Int}(M_{w_j})$. Let $M = \bigcup_{j=1}^{n} M_{w_j}$. Thus, M is a subcontinuum of X, $W \subset \operatorname{Int}(M)$ and $x \in X \setminus \mathcal{T}(M)$ (since X is $\mathcal{T}$-additive, $\mathcal{T}(M) = \mathcal{T}\big(\bigcup_{j=1}^{n} M_{w_j}\big) = \bigcup_{j=1}^{n} \mathcal{T}(M_{w_j})$). Hence, $x \in X \setminus \wp(W)$. Therefore, $\wp(W) = \bigcup\{\wp(\{w\})\colon w \in W\}$. $\square$

The next theorem gives us a sufficient condition for the idempotency of $\wp$ on closed sets.

Theorem 3.19. *Let X be a continuum. If for each nonempty closed subset K of X and every subcontinuum L of X such that $\operatorname{Int}(L) \neq \emptyset$ and $K \cap L = \emptyset$, there exists a subcontinuum W of X such that $L \subset \operatorname{Int}(W)$ and $K \cap \mathcal{T}(W) = \emptyset$, then $\wp$ is idempotent on closed sets.*

Proof: Let A be a nonempty closed subset of X. By Proposition 3.1, $\wp(A) \subset \wp^2(A)$. Let x be a point in $X \setminus \wp(A)$. Then there exists a subcontinuum L of X such that $x \in X \setminus \mathcal{T}(L)$ and $A \subset \operatorname{Int}(L)$. Note that $\wp(A) \subset L$. By hypothesis, there exists a subcontinuum W of X such that $x \in X \setminus \mathcal{T}(W)$ and $L \subset \operatorname{Int}(W)$. Hence, W is a subcontinuum of X such that $\wp(A) \subset \operatorname{Int}(W)$ and $x \in X \setminus \mathcal{T}(W)$. Thus, $x \in X \setminus \wp^2(A)$. Therefore, $\wp(A) = \wp^2(A)$, and $\wp$ is idempotent on closed sets. $\square$

Theorem 3.20. *If X is a $\wp$-symmetric continuum, then X is $\wp$-additive.*

Proof: Suppose X is a $\wp$-symmetric continuum. Let A and B be two nonempty closed subsets of X. We know that $\wp(A) \cup \wp(B) \subset \wp(A \cup B)$ (Lemma 3.4). Let $p \in \wp(A \cup B)$. Since X is $\wp$-symmetric, we have that $\wp(\{p\}) \cap (A \cup B) \neq \emptyset$. Hence, either $\wp(\{p\}) \cap A \neq \emptyset$ or $\wp(\{p\}) \cap B \neq \emptyset$. Thus, by $\wp$-symmetry, either $p \in \wp(A)$ or $p \in \wp(B)$. Hence, $p \in \wp(A) \cup \wp(B)$. Therefore, $\wp(A) \cup \wp(B) = \wp(A \cup B)$, and X is $\wp$-additive. $\square$

Theorem 3.21. *A continuum X is $\wp$-symmetric if and only if X is point $\wp$-symmetric and $\wp$-additive.*

Proof: Suppose X is $\wp$-symmetric. Then, by Theorem 3.20, X is $\wp$-additive. It is clear that each $\wp$-symmetric continuum is point $\wp$-symmetric.

Now, assume that X is a point $\wp$-symmetric and $\wp$-additive continuum. Let A and B be two nonempty closed subsets of X such that $A \cap \wp(B) = \emptyset$ and $\wp(A) \cap B \neq \emptyset$. Let $x \in \wp(A) \cap B$. Then $x \in \wp(A)$ and $x \in B$. Note that $x \in X \setminus A$, otherwise, x would be an element of $A \cap B \subset A \cap \wp(B)$, a contradiction. Since X is $\wp$-additive, by Theorem 3.15, $\wp(A) = \bigcup\{\wp(\{a\})\colon a \in A\}$. Thus, there exists a point $a \in A$ such that $x \in \wp(\{a\})$. Hence, since X is point $\wp$-symmetric, $a \in \wp(\{x\})$. Consequently, since $x \in B$, we have that $a \in \wp(\{x\}) \subset \wp(B)$. Thus, $a \in A \cap \wp(B)$, a contradiction to our assumption. Therefore, $\wp(A) \cap B = \emptyset$, and X is $\wp$-symmetric. □

Theorem 3.22. *Let X be a continuum and let p and q be two points of X. If $q \in \mathcal{T}(\{p\})$, then $p \in \wp(\{q\})$.*

Proof: Suppose that $p \in X \setminus \wp(\{q\})$. Then there exists a subcontinuum W of X such that $q \in \operatorname{Int}(W)$ and $p \in X \setminus \mathcal{T}(W) \subset X \setminus W$. Hence, $q \in X \setminus \mathcal{T}(\{p\})$. □

Theorem 3.23. *If X is a point $\wp$-symmetric continuum, then $\mathcal{T}(\{x\}) \subset \wp(\{x\})$ for all elements x of X.*

Proof: Let x be a point of X. If $q \in \mathcal{T}(\{x\})$, by Theorem 3.22, we have that $x \in \wp(\{q\})$. Since X is point $\wp$-symmetric, we obtain that $q \in \wp(\{x\})$. Therefore, $\mathcal{T}(\{x\}) \subset \wp(\{x\})$. □

Theorem 3.24. *If X is a point $\mathcal{T}$-symmetric continuum, then $\mathcal{T}(\{x\}) \subset \wp(\{x\})$ for each point x of X.*

Proof: Let x be a point of X. If $q \in \mathcal{T}(\{x\})$, then $x \in \mathcal{T}(\{q\})$ (X is point $\mathcal{T}$-symmetric). Hence, by Theorem 3.22, we have that $q \in \wp(\{x\})$. Therefore, $\mathcal{T}(\{x\}) \subset \wp(\{x\})$. □

Theorem 3.25. *If X is a $\mathcal{T}$-symmetric continuum, then*

$$\wp(A) = \bigcap\{\mathcal{K}(W)\colon W \text{ is a subcontinuum of } X \text{ and } A \subset \operatorname{Int}(W)\}.$$

Proof: Let X be a $\mathcal{T}$-symmetric continuum. Let W be a subcontinuum of X. Then, by [15, Theorem 7.7.1], we have that $\mathcal{K}(W) = \bigcup\{\mathcal{K}(\{w\})\colon w \in W\}$. Since X is $\mathcal{T}$-symmetric, X is point $\mathcal{T}$-symmetric. Hence, by [15, Corollary 7.7.4], we know that $\mathcal{K}(\{x\}) = \mathcal{T}(\{x\})$ for all points x of X. Thus, $\mathcal{K}(W) = \bigcup\{\mathcal{T}(\{w\})\colon w \in W\}$. By [15, Theorem 2.2.11], X is $\mathcal{T}$-additive. Hence, by [15, Corollary 2.2.13], we have that $\mathcal{T}(W) = \bigcup\{\mathcal{T}(\{w\})\colon w \in W\}$. Therefore, $\mathcal{T}(W) = \mathcal{K}(W)$. Now, the theorem follows from the definition of $\wp$. □

Corollary 3.26. *Let X be a continuum for which $\mathcal{T}$ is continuous. Then*

$$\wp(A) = \bigcap\{\mathcal{K}(W)\colon\ W \text{ is a subcontinuum of } X \text{ and } A \subset \mathrm{Int}(W)\}.$$

Proof: Since X is a continuum for which $\mathcal{T}$ is continuous, by [15, Corollary 5.1.16], X is $\mathcal{T}$-symmetric. Now the corollary follows from Theorem 3.25. □

Theorem 3.27. *Let X be a continuum for which $\mathcal{T}$ is continuous. Then $\mathcal{T}(A) \subset \wp(A)$ for each nonempty closed subset A of X.*

Proof: Let X be a continuum or which $\mathcal{T}$ is continuous. Then X is $\mathcal{T}$-additive [15, Theorem 5.1.10] and point $\mathcal{T}$-symmetric [15, Corollary 5.1.16]. Since X is point $\mathcal{T}$-symmetric, by Theorem 3.24, we have that $\mathcal{T}(\{x\}) \subset \wp(\{x\})$ for every element x of X. Thus, if A is a nonempty closed subset of X, then, by $\mathcal{T}$-additivity and [15, Corollary 2.2.13], we have that $\mathcal{T}(A) = \bigcup\{\mathcal{T}(\{a\})\colon\ a \in A\} \subset \bigcup\{\wp(\{a\})\colon\ a \in A\} \subset \wp(A)$ (Lemma 3.4). □

Theorem 3.28. *Let X be a continuum for which $\mathcal{T}$ is continuous and let A be a subcontinuum of X. If $x \in X \setminus \wp(A)$, then there exists a strong continuum domain W of X such that $x \in X \setminus \mathcal{T}(W)$ and $A \subset \mathrm{Int}(W)$.*

Proof: Let $x \in X \setminus \wp(A)$. Then, by definition, there exists a subcontinuum L of X such that $x \in X \setminus \mathcal{T}(L)$ and $A \subset \mathrm{Int}(L)$. Hence, $\mathcal{T}(\{x\}) \cap A = \emptyset$. Since $\mathcal{T}$ is continuous, by [15, Theorem 5.1.6], $\mathcal{T}$ is idempotent. Thus, since $\mathcal{T}(\{x\}) \cap A = \emptyset$ for each point a of A, there exists a strong continuum domain W_a of X such that $a \in \mathrm{Int}(W_a) \subset W_a \subset X \setminus \{x\}$, see [15, Corollary 2.3.7]. Observe that $\{\mathrm{Int}(W_a)\colon\ a \in A\}$ is an open cover of A. Since A is compact, there exist $a_1, \dots, a_m$ in A such that $A \subset \bigcup_{j=1}^{m} \mathrm{Int}(W_{a_j})$. Let $U = \bigcup_{j=1}^{m} \mathrm{Int}(W_{a_j})$. Since A and each $\mathrm{Int}(W_{a_j})$, $j \in \{1, \dots, m\}$, are connected, we have that U is an open connected subset of X. Hence, $W = \mathrm{Cl}(U)$ is a strong continuum domain of X such that $A \subset \mathrm{Int}(W)$ and $x \in X \setminus W$. Note that, by [15, Theorem 5.1.13], $W = \mathcal{T}(W)$. Therefore $A \subset \mathrm{Int}(W)$ and $x \in X \setminus \mathcal{T}(W)$. □

Theorem 3.29. *Let X be a continuum. If for each pair of distinct points x_1 and x_2 of X, we have that either $x_1 \in X \setminus \wp(\{x_2\})$ or $x_2 \in X \setminus \wp(\{x_1\})$, then X is semi-aposyndetic.*

Proof: Let x_1 and x_2 be two distinct points of X and assume that $x_2 \in X \setminus \wp(\{x_1\})$. Then, by definition, there exists a subcontinuum W of X such that $x_2 \in X \setminus \mathcal{T}(W)$ and $x_1 \in \mathrm{Int}(W)$. Hence, $\mathcal{T}(W)$ is a subcontinuum of X [15, Theorem 2.1.27] such that $\{x_1, x_2\} \cap \mathrm{Int}(\mathcal{T}(W)) \neq \emptyset$ and $\{x_1, x_2\} \cap X \setminus \mathcal{T}(W) \neq \emptyset$. Therefore, X is semi-aposyndetic. □

Theorem 3.30. *Let X be a continuum. If $\wp(\{x\}) = \{x\}$ for all points x of X, then X is aposyndetic.*

PROOF: Let x and x' be two distinct points of X. Since $\wp(\{x\}) = \{x\}$, by definition, there exists a subcontinuum W of X such that $x \in \mathrm{Int}(W)$ and $x' \in X \setminus \mathcal{T}(W)$. In particular, $x' \in X \setminus W$. Hence, X is aposyndetic at x with respect to x'. Since x and x' are arbitrary points of X, we have that X is aposyndetic. □

Remark 3.31. Note that Example 3.10 presents a family of metric aposyndetic continua for which the converse of Theorems 3.29 and 3.30 are not true. Recall that, by [15, Corollary 4.1.6], the product of two continua is aposyndetic. Hence, the product of two continua is semi-aposyndetic.

Theorem 3.32. *Let X be a continuum and let A be a nonempty closed subset of X. If p belongs to $X \setminus \wp(A)$, then there exists a nonempty closed subset M of X, a subcontinuum W of X and a nonempty open subset Q of X such that $X \setminus M = \mathrm{Int}(\mathcal{T}(W))$, p is in $(M \cap Q) \setminus \mathcal{T}(W)$, $\wp(A) \cap Bd(Q) = \emptyset$ and $M \cap Q \cap A = \emptyset$.*

PROOF: Let p be an element of $X \setminus \wp(A)$. By definition, there exists a subcontinuum W of X such that $A \subset \mathrm{Int}(W)$ and p belongs to $X \setminus \mathcal{T}(W)$. Since X is a regular space, there exists an open subset Q of X such that $p \in Q \subset \mathrm{Cl}(Q) \subset X \setminus \mathcal{T}(W)$. Let $M = X \setminus \mathrm{Int}(\mathcal{T}(W))$. Then p is in $M \cap Q$, $\wp(A) \cap Bd(Q) = \emptyset$ and $M \cap Q \cap A = \emptyset$. □

Theorem 3.33. *Let X be a continuum. If $\wp(A) = A$ for each nonempty closed subset A of X, then X is colocally connected.*

PROOF: Let x be a point of X and let U be an open subset of X containing x. Then x does not belong to $X \setminus U$ and, by hypothesis, $\wp(X \setminus U) = X \setminus U$. Thus, by definition, there exists a subcontinuum W of X such that $X \setminus U \subset \mathrm{Int}(W) \subset W \subset \mathcal{T}(W)$, and x is in $X \setminus \mathcal{T}(W)$. Let $V = X \setminus \mathcal{T}(W)$. Then V is an open subset of X, x belongs to V, $V \subset U$ and $X \setminus V = \mathcal{T}(W)$ is connected [15, Theorem 2.1.27]. Therefore, X is colocally connected. □

The next example shows that the converse implication of Theorem 3.33 is not true.

Example 3.34. Let X be the suspension over the Cantor set with verteces ν^+ and ν^-. Then X is a colocally connected metric continuum. Note that if W is a subcontinuum of X that contains $\{\nu^+, \nu^-\}$ in its interior, then $\mathcal{T}(W) = X$. Hence, $\wp(\{\nu^+, \nu^-\}) = X$.

Corollary 3.35. *Let X be a continuum. If $\wp(A) = A$ for each nonempty closed subset A of X, then $\wp(A) = \mathcal{K}(A)$ for all nonempty closed subsets A of X.*

Proof: Suppose that $\wp(A) = A$ for each nonempty closed subset A of X. Then, by Theorem 3.33, X is colocally connected. Hence, by [6, Theorem 26], $\mathcal{K}(A) = A$ for every nonempty closed subset A of X. Therefore, $\wp(A) = \mathcal{K}(A)$ for all nonempty closed subsets A of X. □

Theorem 3.36. *Let X be a hereditarily unicoherent continuum. If A is a nonempty closed subset of X and U is an open subset of X such that $\wp(A) \subset U$, then there exists a subcontinuum W of X such that $A \subset \mathrm{Int}(W) \subset W \subset \mathcal{T}(W) \subset U$.*

Proof: Let A be a nonempty closed subset of X and let U be an open subset of X such that $\wp(A) \subset U$. Let

$$\mathcal{W} = \{\mathcal{T}(W)\colon W \text{ is a subcontinuum of } X \text{ and } A \subset \mathrm{Int}(W)\}.$$

Since $\wp(A) \subset U$, by definition, $\mathcal{W}$ is a family of subcontinua, see [15, Theorem 2.1.27] of X such that $\{X \setminus \mathcal{T}(W)\colon \mathcal{T}(W) \in \mathcal{W}\}$ is an open cover of the compact subset $X \setminus U$ of X. Hence, there exists $\mathcal{T}(W_1), \dots, \mathcal{T}(W_l)$ in $\mathcal{W}$ such that $X \setminus U \subset \bigcup_{j=1}^{l}(X \setminus \mathcal{T}(W_j))$. Thus, $\bigcap_{j=1}^{l} \mathcal{T}(W_j) \subset U$. Observe that, by definition, $A \subset \mathrm{Int}(W_j)$ for all $j \in \{1, \dots, l\}$. Let $W = \bigcap_{j=1}^{l} W_j$. Since X is hereditarily unicoherent, W is a subcontinuum of X. Hence, $A \subset \mathrm{Int}(W) \subset W \subset \mathcal{T}(W) \subset \bigcap_{j=1}^{l} \mathcal{T}(W_j) \subset U$. □

Theorem 3.37. *Let X be a hereditarily unicoherent continuum. If A is a nonempty closed subset of X, then $\wp(A) = \mathcal{T} \circ \mathcal{K}(A)$.*

Proof: Let A be a nonempty closed subset of X. Let

$$\mathcal{W} = \{\mathcal{T}(W)\colon W \text{ is a subcontinuum of } X \text{ and } A \subset \mathrm{Int}(W)\}$$

and let W_1 and W_2 be two subcontinua of X such that $A \subset \mathrm{Int}(W_j)$, $j \in \{1, 2\}$. Hence, $\mathcal{T}(W_j)$ belongs to $\mathcal{W}$, $j \in \{1, 2\}$. Since X is hereditarily unicoherent, $W_1 \cap W_2$ is a subcontinuum of X and $A \subset \mathrm{Int}(W_1 \cap W_2)$. Thus, $\mathcal{T}(W_1 \cap W_2)$ belongs to $\mathcal{W}$. Since $\mathcal{T}(W_1 \cap W_2) \subset \mathcal{T}(W_1) \cap \mathcal{T}(W_2)$, we have that the family $\mathcal{W}$ is a filter base of closed subsets of X. Hence, by [15, Lemma 2.2.9], we have that

$$\begin{aligned}\wp(A) &= \bigcap\{\mathcal{T}(W)\colon W \text{ is a subcontinuum of } X \text{ and } A \subset \mathrm{Int}(W)\}\\ &= \mathcal{T}\Big(\bigcap\{W\colon W \text{ is a subcontinuum of } X \text{ and } A \subset \mathrm{Int}(W)\}\Big)\\ &= \mathcal{T} \circ \mathcal{K}(A).\end{aligned}$$

□

Theorem 3.38. *If X is a locally connected continuum, then $\wp(A) = \mathcal{K}(A)$ for all nonempty subsets A of X.*

Proof: Suppose X is a locally connected continuum and let A be a nonempty subset of X. By [15, Theorem 2.1.37], we have that $\mathcal{T}(B) = B$ for each nonempty closed subset of X. Hence,

$$\begin{aligned}\wp(A) &= \bigcap\{\mathcal{T}(W)\colon\ W \text{ is a subcontinuum of } X \text{ and } A \subset \mathrm{Int}(W)\}\\ &= \bigcap\{W\colon\ W \text{ is a subcontinuum of } X \text{ and } A \subset \mathrm{Int}(W)\} = \mathcal{K}(A).\end{aligned}$$

□

4. Set function $\wp$ and maps

We present relationships between the images of $\wp$ and the images of maps.

Theorem 4.1. *Let X and Y be continua, and let $f\colon X \twoheadrightarrow Y$ be a homeomorphism. Then $f\wp_X(A) = \wp_Y f(A)$ for all nonempty subsets A of X.*

Proof: Let A be a nonempty subset of X. Note that, since f is a homeomorphism, we have that by [15, Lemma 3.3.1]:

$$\begin{aligned}&\{\mathcal{T}_Y(M)\colon\ M \text{ is a subcontinuum of } Y \text{ such that } f(A) \subset \mathrm{Int}(M)\}\\ &= \{f\mathcal{T}_X(W)\colon\ W \text{ is a subcontinuum of } X \text{ such that } A \subset \mathrm{Int}(W)\}.\end{aligned}$$

Hence,

$$\begin{aligned}f\wp_X(A) &= f\Big(\bigcap\{\mathcal{T}_X(W)\colon\ W \text{ is a subcontinuum of } X \text{ such that } A \subset \mathrm{Int}(W)\}\Big)\\ &= \bigcap\{f\mathcal{T}_X(W)\colon\ W \text{ is a subcontinuum of } X \text{ such that } A \subset \mathrm{Int}(W)\}\\ &= \bigcap\{\mathcal{T}_Y(M)\colon\ M \text{ is a subcontinuum of } Y \text{ such that } f(A) \subset \mathrm{Int}(M)\}\\ &= \wp_Y f(A).\end{aligned}$$

□

Theorem 4.2. *Let X and Y be continua, and let $f\colon X \twoheadrightarrow Y$ be an atomic open map. Then $\wp_X(A) = f^{-1}\wp_Y f(A)$ for all nonempty closed subsets A of X.*

Proof: Let A be a nonempty closed subset of X. Let $x \in X \setminus f^{-1}\wp_Y f(A)$. Then $f(x) \in Y \setminus \wp_Y f(A)$. Thus, by definition, there exists a subcontinuum W of Y such that $f(x) \in Y \setminus \mathcal{T}_Y(W)$ and $f(A) \subset \mathrm{Int}_Y(W)$. Hence, since each atomic map is monotone (Remark 2.1), $f^{-1}(W)$ is a subcontinuum of X. Also, by the monotoneity of f, we have that $f^{-1}\mathcal{T}_Y(W) = \mathcal{T}_X f^{-1}(W)$, see [15, Theorem 2.1.51 (e)]. Thus, $x \in f^{-1}f(x) \subset X \setminus f^{-1}\mathcal{T}_Y(W) = X \setminus \mathcal{T}_X f^{-1}(W)$ and $A \subset f^{-1}f(A) \subset \mathrm{Int}_X(f^{-1}(W))$. Therefore, $x \in X \setminus \wp_X(A)$.

Now, let $x \in X \setminus \wp_X(A)$. Then, by definition, there exists a subcontinuum W of X such that x is in $X \setminus \mathcal{T}_X(W)$ and $A \subset \mathrm{Int}_X(W)$. Note that, since f

is an atomic map, f is monotone (Remark 2.1), and by [14, Theorem 8.1.25][1] $f^{-1}f(x)$ is a terminal subcontinuum of X. Observe that, by [15, Lemma 1.4.51], $f^{-1}f(x)$ is nowhere dense. Thus, $f^{-1}f(x) \cap \mathcal{T}_X(W) = \emptyset$. Since f is an open map and $\mathrm{Int}_X(W) \neq \emptyset$, $f(W)$ is nondegenerate. Hence, $W = f^{-1}f(W)$. We claim that $f(x) \in Y \setminus \mathcal{T}_Y f(W)$. Suppose that $f(x) \in \mathcal{T}_Y f(W)$. Then $x \in f^{-1}f(x) \subset f^{-1}\mathcal{T}_Y f(W) = \mathcal{T}_X f^{-1}f(W) = \mathcal{T}_X(W)$ (the first equality is true by [15, Theorem 2.1.51 (e)], because f is monotone and open), a contradiction. Thus, $f(x) \in Y \setminus \mathcal{T}_Y f(W)$. Hence, $\mathcal{T}_Y f(W)$ is a subcontinuum of Y such that $f(x) \in Y \setminus \mathcal{T}_Y f(W)$ and $f(A) \subset \mathrm{Int}_Y(f(W))$. This implies that $f(x) \in Y \setminus \wp_Y f(A)$. Thus, $x \in f^{-1}f(x) \subset X \setminus f^{-1}\wp_Y f(A)$. Therefore, $\wp_X(A) = f^{-1}\wp_Y f(A)$. □

Theorem 4.3. *Let X and Y be continua and let $f\colon X \twoheadrightarrow Y$ be an atomic map. Then for each element x in X, $f^{-1}f(x) \subset \wp(\{x\})$.*

Proof: Let $x \in X$ and let $z \in X \setminus \wp(\{x\})$. Then, by definition, there exists a subcontinuum W of X such that $x \in \mathrm{Int}_X(W)$ and $z \in X \setminus \mathcal{T}_X(W)$. Since f is an atomic map, $f^{-1}f(x)$ is a terminal subcontinuum of X, see [14, Theorem 8.1.25] and footnote 1 on this page below. Hence, by [15, Lemma 1.4.51], $\mathrm{Int}_X(f^{-1}f(x)) = \emptyset$. Thus, $f^{-1}f(x) \subset W$. Therefore, $z \in X \setminus f^{-1}f(x)$, and $f^{-1}f(x) \subset \wp(\{x\})$. □

Theorem 4.4. *Let X and Y be continua, where Y is mutually aposyndetic, and let $f\colon X \twoheadrightarrow Y$ be a monotone map. Then for each element x in X, $\wp(\{x\}) \subset f^{-1}f(x)$.*

Proof: Let x be a point of X and let $x' \in X \setminus f^{-1}(f(x))$. Then $f(x) \neq f(x')$. Hence, since Y is mutually aposyndetic, there exist two disjoint subcontinua W and W' of Y such that $f(x) \in \mathrm{Int}_Y(W)$ and $f(x') \in \mathrm{Int}_Y(W')$. Since f is monotone, $f^{-1}(W)$ and $f^{-1}(W')$ are two disjoint subcontinua of X such that $x \in \mathrm{Int}_X(f^{-1}(W))$ and $x' \in \mathrm{Int}_X(f^{-1}(W'))$. Thus, $x' \in X \setminus \mathcal{T}(W)$ and $x' \in X \setminus \wp(\{x\})$. Therefore, $\wp(\{x\}) \subset f^{-1}f(x)$. □

Corollary 4.5. *Let X and Y be continua, where Y is mutually aposyndetic. If $f\colon X \twoheadrightarrow Y$ be an atomic map, then for each element x in X, $f^{-1}f(x) = \wp(\{x\})$.*

Proof: Let x be an element of X. By Theorem 4.3, we have that $f^{-1}f(x) \subset \wp(\{x\})$. Since atomic maps are monotone (Remark 2.1) and Y is mutually aposyndetic, by Theorem 4.4, we obtain that $\wp(\{x\}) \subset f^{-1}f(x)$. Therefore, $f^{-1}f(x) = \wp(\{x\})$. □

Theorem 4.6. *Let X and Y be continua. If $f\colon X \twoheadrightarrow Y$ is a monotone and open map, then $f\wp_X(A) \subset \wp_Y f(A)$ for every nonempty closed subset A of X.*

[1] The proof of [14, Theorem 8.1.25] only uses [14, Theorem 8.1.24] (Remark 2.1).

Proof: Let A be a nonempty closed subset of X and let

$$\mathcal{B} = \{\mathcal{T}_Y f(M)\colon\ M \text{ is a subcontinuum of } X \text{ such that } A \subset \mathrm{Int}_X(M)\},$$

and

$$\mathcal{C} = \{\mathcal{T}_Y(L)\colon\ L \text{ is a subcontinuum of } Y \text{ such that } f(A) \subset \mathrm{Int}_Y(L)\}.$$

We show that $\mathcal{B} = \mathcal{C}$. Let M be a subcontinuum of X such that $A \subset \mathrm{Int}_X(M)$. Then $\mathcal{T}_Y f(M) \in \mathcal{B}$. Since f is open, $f(A) \subset \mathrm{Int}_Y(f(M))$. Hence, $\mathcal{T}_Y f(M) \in \mathcal{C}$. Next, let $\mathcal{T}_Y(L) \in \mathcal{C}$. Then L is a subcontinuum of Y and $f(A) \subset \mathrm{Int}_Y(L)$. Since f is monotone, $f^{-1}(L)$ is a subcontinuum of X and $A \subset f^{-1}f(A) \subset \mathrm{Int}_X(f^{-1}(L))$. Thus, $\mathcal{T}_Y(L) = \mathcal{T}_Y f f^{-1}(L) \in \mathcal{B}$. Therefore, $\mathcal{B} = \mathcal{C}$. Now, we have that

$$\begin{aligned}
f\wp_X(A) &= f\Big(\bigcap\{\mathcal{T}_X(M)\colon\ M \text{ is a subcontinuum of } X \text{ such that } A \subset \mathrm{Int}_X(M)\}\Big)\\
&\subset \bigcap\{f\mathcal{T}_X(M)\colon\ M \text{ is a subcontinuum of } X \text{ such that } A \subset \mathrm{Int}_X(M)\}\\
&\subset \bigcap\{\mathcal{T}_Y f(M)\colon\ M \text{ is a subcontinuum of } X \text{ such that } A \subset \mathrm{Int}_X(M)\}\\
&= \bigcap\{\mathcal{T}_Y(L)\colon\ L \text{ is a subcontinuum of } Y \text{ such that } f(A) \subset \mathrm{Int}_Y(L)\}\\
&= \wp_Y f(A).
\end{aligned}$$

The second inclusion follows from [15, Theorem 2.1.51 (b)]. □

Theorem 4.7. *Let X and Y be continua and let $f\colon X \twoheadrightarrow Y$ be a monotone map. Then $\wp_X f^{-1}(B) \subset f^{-1}\wp_Y(B)$ for all nonempty closed subsets B of Y.*

Proof: Let B be a nonempty closed subset of Y. Note that if W is a subcontinuum of Y and $B \subset \mathrm{Int}_Y(W)$, then, since f is a monotone map, $f^{-1}(W)$ is a subcontinuum of X, and $f^{-1}(B) \subset \mathrm{Int}_X(f^{-1}(W))$. Hence,

$$\begin{aligned}
\wp_X f^{-1}(B) &= \bigcap\{\mathcal{T}_X(W)\colon\ W \text{ is a subcontinuum of } X \text{ and } f^{-1}(B) \subset \mathrm{Int}_X(W)\}\\
&\subset \bigcap\{f^{-1}\mathcal{T}_Y(L)\colon\ L \text{ is subcontinuum of } Y \text{ and } B \subset \mathrm{Int}_Y(L)\}\\
&= f^{-1}\Big(\bigcap\{\mathcal{T}_Y(L)\colon\ L \text{ is subcontinuum of } Y \text{ and } B \subset \mathrm{Int}_Y(L)\}\Big)\\
&= f^{-1}\wp_Y(B).
\end{aligned}$$

□

Since atomic maps are monotone (Remark 2.1), from Theorems 4.6 and 4.7, we obtain:

Corollary 4.8. *Let X and Y be continua and let $f\colon X \twoheadrightarrow Y$ be an atomic map. Then the following hold:*

(1) $\wp_X f^{-1}(B) \subset f^{-1}\wp_Y(B)$ *for all nonempty closed subsets B of Y.*

(2) *If f is also open, then $f\wp_X(A) \subset \wp_Y f(A)$ for every nonempty closed subset A of X.*

The following three results relate $\wp$-closed sets with monotone and atomic open maps.

Theorem 4.9. *Let X and Y be continua and let $f\colon X \twoheadrightarrow Y$ be a monotone map. If $B \in \mathfrak{P}(Y)$, then $f^{-1}(B) \in \mathfrak{P}(X)$.*

Proof: Let $B \in \mathfrak{P}(Y)$. By Theorem 4.7, we have that $\wp_X f^{-1}(B) \subset f^{-1}\wp_Y(B)$. Since $B \in \mathfrak{P}(Y)$, $\wp_Y(B) = B$. Thus, $\wp_X f^{-1}(B) \subset f^{-1}(B)$. Hence, by Proposition 3.1, $\wp_X f^{-1}(B) = f^{-1}(B)$. Therefore, $f^{-1}(B) \in \mathfrak{P}_X$. □

Theorem 4.10. *Let X and Y be continua and let $f\colon X \twoheadrightarrow Y$ be an atomic open map. If $A \in \mathfrak{P}(X)$, then $f(A) \in \mathfrak{P}(Y)$.*

Proof: Note that, by Remark 2.1, f is a monotone map and, by [14, Theorem 8.1.25], see footnote 1, $f^{-1}f(x)$ is a terminal subcontinuum of X for all elements x of X. Let $A \in \mathfrak{P}(X)$. Then, by Theorem 4.2, $\wp_X(A) = f^{-1}\wp_Y f(A)$. Since $A \in \mathfrak{P}(X)$, $\wp_X(A) = A$. Hence, $A = f^{-1}\wp_Y f(A)$. Thus, $f(A) = ff^{-1}\wp_Y f(A) = \wp_Y f(A)$. Therefore, $f(A) \in \mathfrak{P}(Y)$. □

Corollary 4.11. *Let X and Y be continua and let $f\colon X \twoheadrightarrow Y$ be an atomic open map. Then $A \in \mathfrak{P}(X)$ if and only if $f(A) \in \mathfrak{P}(Y)$.*

If A is a nonempty set, then $|A|$ denotes the cardinality of A.

Corollary 4.12. *Let X and Y be continua. If $f\colon X \twoheadrightarrow Y$ is an atomic open map, then $|\mathfrak{P}(X)| = |\mathfrak{P}(Y)|$.*

Remark 4.13. By Theorem 5.1, $\wp$ is an upper semicontinuous function. Hence, we can do dynamical systems involving the set function $\wp$, on metric continua, in the way it is done in [2]. To this end, we define the appropriate upper semicontinous functions: If $f\colon X \to X$ is a map, let $\wp_f\colon X \to 2^X$ be given by $\wp_f(x) = \wp(\{f(x)\})$. When f is the identity map 1_X of X, $\wp_{1_X}$ becomes a *coselection*; i.e., $x \in \wp_{1_X}(x)$ for all points x of X. For a general map f, $\wp_f$ is called *$\wp$-f-coselection*. Thus, the results of [2, Sections 3 and 4] are true for the coselection $\wp_{1_X}$ and the results of [2, Sections 3 and 5] are valid for $\wp$-f-coselections $\wp_f$. By Theorem 4.1, the results of [2, Section 6] are true for $\wp$-f-coselections $\wp_f$, when f is a homeomorphism.

5. Continuity of $\wp$

We start with showing $\wp$ is always upper semicontinuous.

Theorem 5.1. *If X is a continuum, then $\wp$ is upper semicontinuous.*

Proof: Let U be an open subset of X, and let $\mathcal{U} = \{A \in 2^X\colon \wp(A) \in \langle U\rangle\}$. We show that $\mathcal{U}$ is an open subset of 2^X. Let $B \in \mathrm{Cl}_{2^X}(2^X \setminus \mathcal{U})$. Then there exists a net $\{B_\lambda\}_{\lambda\in\Lambda}$ of elements of $2^X \setminus \mathcal{U}$ converging to B, see [4, 1.6.3] and [18, Theorem 4]. Note that for each $\lambda \in \Lambda$, $\wp(B_\lambda)\setminus U \neq \emptyset$. Let $x_\lambda \in \wp(B_\lambda)\setminus U$. Without loss of generality, we assume that the net $\{x_\lambda\}_{\lambda\in\Lambda}$ converges to an element x of X, see [4, 3.1.23 and 1.6.1]. Observe that $x \in X\setminus U$. We assert that $x \in \wp(B)$. Suppose that $x \in X \setminus \wp(B)$. Then, by definition, there exists a subcontinuum W of X such that $B \subset \mathrm{Int}(W)$ and $x \in X \setminus \mathcal{T}(W)$. Since $\{B_\lambda\}_{\lambda\in\Lambda}$ converges to B and $\{x_\lambda\}_{\lambda\in\Lambda}$ converges to x, there exists $\lambda_0 \in \Lambda$ such that if $\lambda \geq \lambda_0$, we have that $B_\lambda \subset \mathrm{Int}(W)$ and $x_\lambda \in X \setminus \mathcal{T}(W)$. Let $\lambda \geq \lambda_0$. Then $B_\lambda \subset \mathrm{Int}(W)$ and $x_\lambda \in X \setminus \mathcal{T}(W)$. Hence, $x_\lambda \in X \setminus \wp(B_\lambda)$, a contradiction. Thus, $x \in \wp(B) \setminus U$. Hence, $B \in 2^X \setminus \mathcal{U}$. Therefore, $2^X \setminus \mathcal{U}$ is a closed subset of 2^X and $\mathcal{U}$ an open subset of 2^X. □

Theorem 5.2. *If X is a locally connected metric continuum, then $\wp$ is continuous.*

Proof: Let X be a locally connected metric continuum. Then, by Theorem 3.38, $\wp(A) = \mathcal{K}(A)$ for all nonempty closed subsets A of X. Now, by [6, Theorem 36], $\mathcal{K}$ is continuous. Therefore, $\wp$ is continuous. □

Theorem 5.3. *Let X be a hereditarily unicoherent continuum. If $\mathcal{T}$ is continuous and $\mathcal{K}$ is continuous at each $\{x\}$ for all points x of X, then $\wp$ is continuous.*

Proof: Let X be a hereditarily unicoherent continuum such that $\mathcal{K}$ is continuous at each $\{x\}$ for every element x of X. Thus, by [5, Theorem 7], $\mathcal{K}$ is continuous. Now, suppose $\mathcal{T}$ is continuous also. Hence, we have that, by Theorem 3.37, $\wp = \mathcal{T} \circ \mathcal{K}$. Thus, $\wp$ is the composition of two continuous functions. Therefore, $\wp$ is continuous. □

6. Property of Kelley

We present properties of the set function $\wp$ for continua with the property of Kelley. Then we introduce the uniform property of Effros and prove Professor Janusz R. Prajs' mutually aposyndetic decomposition theorem for this class of continua.

We begin with the pointwise version of the property of Kelley given by W. J. Charatonik, see [3], and W. Makuchowski, see [17]. A global definition of the property of Kelley may be found in [15, Definition 1.6.17].

A continuum X has the *property of Kelley at a point* x_0 if for every subcontinuum L of X containing x_0 and each open subset $\mathcal{U}$ of $\mathcal{C}_1(X)$ containing L, there exists an open subset $K(x_0, L, \mathcal{U})$ of X such that $x_0 \in K(x_0, L, \mathcal{U})$ and if

$x \in K(x_0, L, \mathcal{U})$, then there exists a subcontinuum M of X such that $x \in M$ and $M \in \mathcal{U}$. The open set $K(x_0, L, \mathcal{U})$ is called a *Kelley set* for $\mathcal{U}$, L and x_0. The continuum X has the *property of Kelley* if it has this property at each of its points.

Lemma 6.1. *Let X be a continuum and let A be a nonempty subset of X. Then the following hold:*

(1) *If $x \in X \setminus \wp(A)$, then $A \cap \mathcal{T}(\{x\}) = \emptyset$.*
(2) *If X is $\mathcal{T}$-symmetric, B is a nonempty closed subset of X, X has the property of Kelley and $B \cap \mathcal{T}(A) = \emptyset$, then $A \cap \wp(B) = \emptyset$.*
(3) *If X is $\mathcal{T}$-symmetric, B is a nonempty closed subset of X and $A \cap \wp(B) = \emptyset$, then $A \cap \mathcal{T}(B) = \emptyset$.*

Proof: Suppose $x \in X \setminus \wp(A)$. Then there exists a subcontinuum W of X such that $A \subset \mathrm{Int}(W)$ and $x \in X \setminus \mathcal{T}(W) \subset X \setminus W$. Therefore, $A \cap \mathcal{T}(\{x\}) = \emptyset$.

Assume X has the property of Kelley and $B \cap \mathcal{T}(A) = \emptyset$. Then there exists a subcontinuum W of X such that $B \subset \mathrm{Int}(W) \subset W \subset X \setminus A$. Since X has the property of Kelley, by [15, Corollary 1.6.21], there exists a subcontinuum L of X such that $W \subset \mathrm{Int}(L)$ and $L \cap A = \emptyset$. Hence, $W \cap \mathcal{T}(A) = \emptyset$. Since X is $\mathcal{T}$-symmetric, we have that $A \cap \mathcal{T}(W) = \emptyset$. Therefore, $A \cap \wp(B) = \emptyset$.

Suppose X is a $\mathcal{T}$-symmetric continuum and $A \cap \wp(B) = \emptyset$. Let a be a point of A. Then $a \in X \setminus \wp(B)$. Hence, by part (1) of this lemma, we have that $B \cap \mathcal{T}(\{a\}) = \emptyset$. Since X is $\mathcal{T}$-symmetric, $\{a\} \cap \mathcal{T}(B) = \emptyset$. Thus, since a is an arbitrary element of A, we obtain that $A \cap \mathcal{T}(B) = \emptyset$. □

Theorem 6.2. *If X is a $\mathcal{T}$-symmetric continuum with the property of Kelley, then $\wp$ is idempotent on closed sets.*

Proof: If X is an indecomposable continuum, then $\wp(A) = X$ for each nonempty closed subset A of X (Theorem 3.8). Hence, $\wp(\wp(A)) = \wp(X) = X = \wp(A)$ for all nonempty closed subsets A of X.

Let X be a decomposable continuum with the property of Kelley and let K be a nonempty closed subset of X and let L be a subcontinuum of X such that $\mathrm{Int}(L) \neq \emptyset$ and $L \cap K = \emptyset$. Since X has the property of Kelley, by [15, Corollary 1.6.21], there exists a subcontinuum W of X such that $L \subset \mathrm{Int}(W)$ and $W \cap K = \emptyset$. Applying [15, Corollary 1.6.21] again, there exists a subcontinuum M of X such that $W \subset \mathrm{Int}(M)$ and $M \cap K = \emptyset$. Hence, $W \cap \mathcal{T}(K) = \emptyset$. Since X is $\mathcal{T}$-symmetric, $K \cap \mathcal{T}(W) = \emptyset$. Therefore, by Theorem 3.19, $\wp$ is idempotent on closed sets. □

Theorem 6.3. *Let X be a continuum with the property of Kelley and let W_1 and W_2 be disjoint subcontinua of X with nonempty interior. Then $\wp(W_1) \cap \wp(W_2) = \emptyset$.*

Proof: By [15, Corollary 1.6.21], there exists a subcontinuum M_1 of X such that $W_1 \subset \operatorname{Int}(M_1)$ and $W_2 \cap M_1 = \emptyset$. This implies that $\mathcal{T}(W_2) \cap \mathcal{T}(M_1) = \emptyset$, see [15, Theorem 2.1.67]. Since $\wp(W_1) \subset \mathcal{T}(M_1)$, we have that $\wp(W_1) \cap \mathcal{T}(W_2) = \emptyset$. Similarly, by [15, Corollary 1.6.21], there exists a subcontinuum M_2 of X such that $W_2 \subset \operatorname{Int}(M_2)$ and $\wp(W_1) \cap \mathcal{T}(M_2) = \emptyset$. Therefore, $\wp(W_1) \cap \wp(W_2) = \emptyset$. □

Theorem 6.4. *Let X be a decomposable continuum with the property of Kelley. If x_1 and x_2 are two points of X such that $\wp(\{x_1\}) \cap \wp(\{x_2\}) \neq \emptyset$, then $\{x_1, x_2\} \subset \wp(\{x_1\}) \cap \wp(\{x_2\})$.*

Proof: Let x_1 and x_2 be two points of X such that $\wp(\{x_1\}) \cap \wp(\{x_2\}) \neq \emptyset$. We show that $x_1 \in \wp(\{x_2\})$. Let $x_3 \in \wp(\{x_1\}) \cap \wp(\{x_2\})$. Suppose $x_1 \in X \setminus \wp(\{x_2\})$. Then there exists a subcontinuum W_2 of X such that $x_1 \in X \setminus \mathcal{T}(W_2)$ and $x_2 \in \operatorname{Int}(W_2)$. Since $x_1 \in X \setminus \mathcal{T}(W_2)$, there exists a subcontinuum W_1 of X such that $x_1 \in \operatorname{Int}(W_1) \subset W_1 \subset X \setminus W_2$. Thus, $W_1 \cap W_2 = \emptyset$. Hence, since X has the property of Kelley, $\mathcal{T}(W_1) \cap \mathcal{T}(W_2) = \emptyset$, see [15, Theorem 2.1.67]. Now, since $x_3 \in \wp(\{x_1\}) \cap \wp(\{x_2\})$, we have that $x_3 \in \mathcal{T}(W_1) \cap \mathcal{T}(W_2)$, a contradiction. Thus, $x_1 \in \wp(\{x_2\})$. Similarly, $x_2 \in \wp(\{x_1\})$. Therefore, $\{x_1, x_2\} \subset \wp(\{x_1\}) \cap \wp(\{x_2\})$. □

Theorem 6.5. *If X is a decomposable continuum with the property of Kelley, then $\mathcal{Q} = \{\wp(\{x\})\colon x \in X\}$ is a decomposition of X.*

Proof: Let x_1 and x_2 be two points of X such that $\wp(\{x_1\}) \cap \wp(\{x_2\}) \neq \emptyset$. We show that $\wp(\{x_1\}) \subset \wp(\{x_2\})$. Suppose there exists $x_3 \in \wp(\{x_1\}) \setminus \wp(\{x_2\})$. Then there exists a subcontinuum W_2 of X such that $x_2 \in \operatorname{Int}(W_2)$ and $x_3 \in X \setminus \mathcal{T}(W_2)$. Note that, since X has the property of Kelley $\mathcal{T}^2(W_2) = \mathcal{T}(W_2)$ [15, Theorem 2.3.18]. Since $x_3 \in X \setminus \mathcal{T}^2(W_2)$, there exists a subcontinuum W_3 of X such that $x_3 \in \operatorname{Int}(W_3)$ and $W_3 \cap \mathcal{T}(W_2) = \emptyset$. Hence, by [15, Theorem 2.1.67], we have that $\emptyset = \mathcal{T}(W_3) \cap \mathcal{T}^2(W_2) = \mathcal{T}(W_3) \cap \mathcal{T}(W_2)$. Now, observe that, by Theorem 6.4, we have that $x_1 \in \mathcal{T}(W_2)$. Hence, W_3 is a subcontinuum of X such that $x_3 \in \operatorname{Int}(W_3)$ and $x_1 \in X \setminus \mathcal{T}(W_3)$. Thus, $x_1 \in X \setminus \wp(\{x_3\})$. A contradiction to the fact that, by Theorem 6.4, $x_1 \in \wp(\{x_3\})$. Therefore, $\wp(\{x_1\}) \subset \wp(\{x_2\})$. Similarly, $\wp(\{x_2\}) \subset \wp(\{x_1\})$. Therefore, $\wp(\{x_1\}) = \wp(\{x_2\})$ and $\mathcal{Q}$ is a decomposition of X. □

Corollary 6.6. *If X is a continuum with the property of Kelley, then X is point $\wp$-symmetric.*

The following is [12, Lemma 5.8], see also [15, Lemma 3.4.8], it is used in the proof of Theorem 6.10. We include the modifications to the original proof for completeness.

Lemma 6.7. *Let X be a decomposable continuum with the property of Kelley. If W is a subcontinuum of X with nonempty interior and $\mathcal{Q} = \{\wp(\{x\})\colon x \in X\}$, then $\mathcal{T}(W) = q^{-1}q\mathcal{T}(W)$, where $q\colon X \twoheadrightarrow X/\mathcal{Q}$ is the quotient map.*

Proof: Observe that, by Theorem 6.5, $\mathcal{Q}$ is a decomposition of X. Note that, for every point x of X, $\wp(\{x\}) = q^{-1}q(x)$. Let W be a subcontinuum of X with nonempty interior [15, Theorem 1.4.34]. Clearly, $\mathcal{T}(W) \subset q^{-1}q\mathcal{T}(W)$. Let $x \in X \setminus \mathcal{T}(W)$. Since X has the property of Kelley, $\mathcal{T}$ is idempotent on closed sets [15, Theorem 2.3.18]. Hence, $x \in X \setminus \mathcal{T}^2(W)$. Thus, there exists a subcontinuum W_x of X such that $x \in \mathrm{Int}(W_x) \subset W_x \subset X \setminus \mathcal{T}(W)$. By [15, Corollary 1.6.21], there exists a subcontinuum K_W of X such that $\mathcal{T}(W) \subset \mathrm{Int}(K_W)$ and $W_x \cap K_W = \emptyset$. By [15, Theorem 2.1.67], $\mathcal{T}(W_x) \cap \mathcal{T}(K_W) = \emptyset$. This implies that $\wp(\{x\}) \cap \mathcal{T}(W) = \emptyset$. Therefore, $\{q(x)\} \cap q\mathcal{T}(W) = \emptyset$. Thus, $q^{-1}q(x) \cap q^{-1}q\mathcal{T}(W) = \emptyset$, and $x \in X \setminus q^{-1}q\mathcal{T}(W)$. Therefore, $\mathcal{T}(W) = q^{-1}q\mathcal{T}(W)$. $\square$

Let Z be a Tychonoff space and let $\mathfrak{U}$ be a uniformity that induces the topology of Z. Then Z has the *uniform property of Effros with respect to* $\mathfrak{U}$, provided that for each $U \in \mathfrak{U}$, there exists $V \in \mathfrak{U}$ such that if z_1 and z_2 are two points of Z with $\varrho_Z(z_1, z_2) < V$, there exists a homeomorphism $h\colon Z \twoheadrightarrow Z$ such that $h(z_1) = z_2$ and $\varrho_Z(z, h(z)) < U$ for all $z \in Z$. The entourage V is called an *Effros entourage for* U. A homeomorphism $h\colon Z \twoheadrightarrow Z$ satisfying $\varrho_Z(z, h(z)) < U$ for all $z \in Z$, is called a *U-homeomorphism*.

The following theorem is [12, Theorem 4.1], see also [15, Theorem 1.4.59].

Theorem 6.8. *Let Z be a connected Tychonoff space and let $\mathfrak{U}$ be a uniformity that induces the topology of Z. If Z has the uniform property of Effros with respect to $\mathfrak{U}$, then Z is a homogeneous space.*

The next result is [13, Theorem 5.11], see also [15, Theorem 1.6.22].

Theorem 6.9. *If X is a continuum with the uniform property of Effros, then X has the property of Kelley.*

We are ready to state and prove Professor Janusz R. Prajs' mutual aposyndesis decomposition theorem from [12, Theorem 5.9], see also [15, Theorem 3.4.9], in terms of the set function $\wp$. The fact that $X/\mathcal{Q}$ has the uniform property of Effros follows from [16, Theorem 4.6]. We include all the details for the convenience of the reader.

Theorem 6.10. *Let X be a decomposable continuum with the uniform property of Effros. If $\mathcal{Q} = \{\wp(\{x\})\colon x \in X\}$, then the following hold:*

1. *$\mathcal{Q}$ is a continuous decomposition of X.*
2. *The elements of $\mathcal{Q}$ are homogeneous and mutually homeomorphic closed subsets of X.*

(3) *The quotient map $q\colon X \twoheadrightarrow X/\mathcal{Q}$ is uniformly completely regular.*
(4) *The quotient space $X/\mathcal{Q}$ is a mutually aposyndetic homogeneous continuum with the uniform property of Effros.*

Proof: By Theorem 6.8, X is a homogeneous continuum. By Theorem 6.9, X has the property of Kelley. Note that, by Theorem 6.5, $\mathcal{Q}$ is a decomposition of X, and by Theorem 4.1, the group of homeomorphisms of X respects $\mathcal{Q}$. Hence, by [12, Theorem 4.2], or [15, Theorem 3.3.6], $\mathcal{Q}$ is a continuous decomposition of X, the elements of $\mathcal{Q}$ are homogeneous and mutually homeomorphic closed subsets of X, the quotient map is uniformly completely regular and the quotient space is a homogeneous continuum.

We show that $X/\mathcal{Q}$ has the uniform property of Effros. Let $\mathcal{U} \in \mathfrak{U}_{X/\mathcal{Q}}$ (Remark 2.2). Since the quotient map q is continuous, we have that $(q \times q)^{-1}(\mathcal{U}) \in \mathfrak{U}_X$. Since X has the uniform property of Effros, there exists an Effros entourage V for $(q \times q)^{-1}(\mathcal{U})$. Since q is an open map, see [15, Corollary 1.1.24], we have that $(q \times q)(V) \in \mathfrak{U}_{X/\mathcal{Q}}$. We prove that $(q \times q)(V)$ is an Effros entourage for $\mathcal{U}$. Let χ_1 and χ_2 be two elements of $X/\mathcal{Q}$ such that $\varrho_{X/\mathcal{Q}}(\chi_1, \chi_2) < (q \times q)(V)$. Then there exist two elements x_1 and x_2 in X such that $q(x_1) = \chi_1$, $q(x_2) = \chi_2$ and $\varrho_X(x_1, x_2) < V$. Since V is an Effros entourage for $(q\times q)^{-1}(\mathcal{U})$, there exists a $(q\times q)^{-1}(\mathcal{U})$-homeomorphism $h\colon X \twoheadrightarrow X$ such that $h(x_1) = x_2$. As in the proof of [12, Theorem 4.2], or [15, Theorem 3.3.6], it can be seen that the map $\zeta\colon X/\mathcal{Q} \twoheadrightarrow X/\mathcal{Q}$ given by $\zeta(\chi) = q\big(h(q^{-1}(\chi))\big)$ is a homeomorphism and $\zeta(\chi_1) = \chi_2$. We prove that ζ is a $\mathcal{U}$-homeomorphism. Let χ be a point of $X/\mathcal{Q}$ and let $x \in q^{-1}(\chi)$. Since h is a $(q \times q)^{-1}(\mathcal{U})$-homeomorphism, we have that $\varrho_X(x, h(x)) < (q \times q)^{-1}(\mathcal{U})$; i.e., $(x, h(x)) \in (q \times q)^{-1}(\mathcal{U})$. Hence, $(q \times q)(x, h(x)) \in \mathcal{U}$. Thus, since $q(x) = \chi$, we obtain that $\varrho_{X/\mathcal{Q}}(\chi, \zeta(\chi)) < \mathcal{U}$. Therefore, ζ is $\mathcal{U}$-homeomorphism. Hence, $(q \times q)(V)$ is an Effros entourage for $\mathcal{U}$ and $X/\mathcal{Q}$ has the uniform property of Effros.

We need to prove that $X/\mathcal{Q}$ is mutually aposyndetic. To this end, let χ_1 and χ_2 be two distinct elements of $X/\mathcal{Q}$. Let x_1 and x_2 be points of X such that $q(x_1) = \chi_1$ and $q(x_2) = \chi_2$. Note that, $\wp(\{x_1\}) \cap \wp(\{x_2\}) = \emptyset$ (Theorem 6.5). Hence, there exist two subcontinua W_1 and W_2 of X such that $x_j \in \mathrm{Int}_X(W_j)$, $j \in \{1, 2\}$, and $\mathcal{T}(W_1) \cap \mathcal{T}(W_2) = \emptyset$. Since $\mathcal{Q}$ is a continuous decomposition, q is open [15, Corollary 1.1.24]. Thus, by Lemma 6.7, $q\mathcal{T}_X(W_1) \cap q\mathcal{T}_X(W_2) = \emptyset$, $\chi_1 \in \mathrm{Int}_{X/\mathcal{Q}}(q\mathcal{T}_X(W_1))$ and $\chi_2 \in \mathrm{Int}_{X/\mathcal{Q}}(q\mathcal{T}_X(W_2))$. Therefore, $X/\mathcal{Q}$ is mutually aposyndetic. □

The following result is [12, Theorem 5.10], see also [15, Theorem 3.4.11]. It gives us a sufficient condition to have that Jones' decomposition coincides with Prajs' decomposition. We include the modifications to the original proof for the convenience of the reader.

Theorem 6.11. *Let X be a continuum with the uniform property of Effros. If $\mathcal{G} = \{\mathcal{T}_X(\{x\})\colon x \in X\}$ is Jones' decomposition* [15, Theorem 3.3.8], $\mathcal{Q} = \{\wp_X(\{x\})\colon x \in X\}$ *is Prajs' decomposition* (Theorem 6.10), *and $X/\mathcal{G}$ is a mutually aposyndetic continuum, then $\mathcal{T}_X(\{x\}) = \wp_X(\{x\})$ for each $x \in X$. In particular, $\mathcal{G} = \mathcal{Q}$.*

Proof: By Theorem 6.8, X is a homogeneous continuum. By [15, Theorem 3.3.8], $\mathcal{G}$ is a continuous decomposition of X. Let x_1 be a point of X. Let $x_2 \in X \setminus \mathcal{T}_X(\{x_1\})$. Then, since $\mathcal{G}$ is a decomposition, we have that $\mathcal{T}_X(\{x_1\}) \cap \mathcal{T}_X(\{x_2\}) = \emptyset$. Let $q\colon X \twoheadrightarrow X/\mathcal{G}$ be the quotient map. Note that $q(x_1) \neq q(x_2)$. Since $X/\mathcal{G}$ is mutually aposyndetic, there exist two disjoint subcontinua $\mathcal{W}_1$ and $\mathcal{W}_2$ of $X/\mathcal{G}$ such that $q(x_1) \in \mathrm{Int}_{X/\mathcal{G}}(\mathcal{W}_1)$ and $q(x_2) \in \mathrm{Int}_{X/\mathcal{G}}(\mathcal{W}_2)$. Then, since q is monotone, $q^{-1}(\mathcal{W}_1)$ and $q^{-1}(\mathcal{W}_2)$ are two disjoint subcontinua of X such that $x_1 \in \mathrm{Int}_X(q^{-1}(\mathcal{W}_1))$ and $x_2 \in \mathrm{Int}_X(q^{-1}(\mathcal{W}_2))$. Since X has the uniform property of Effros, by Theorem 6.9, X has the property of Kelley. Hence, since $q^{-1}(\mathcal{W}_1) \cap q^{-1}(\mathcal{W}_2) = \emptyset$, by [15, Theorem 2.1.67], $\mathcal{T}_X(q^{-1}(\mathcal{W}_1)) \cap \mathcal{T}_X(q^{-1}(\mathcal{W}_2)) = \emptyset$. Thus, $x_1 \in X \setminus \wp_X(\{x_2\})$ and $x_2 \in X \setminus \wp_X(\{x_1\})$. This implies that $\wp_X(\{x_1\}) \subset \mathcal{T}_X(\{x_1\})$. Since $\mathcal{G}$ is a decomposition of X, X is point $\mathcal{T}$-symmetric. Hence, by Theorem 3.24, $\mathcal{T}_X(\{x_1\}) \subset \wp(\{x_1\})$. Therefore, $\mathcal{T}_X(\{x_1\}) = \wp(\{x_1\})$. □

We end this section with a theorem about the dimension of the images of the singletons under $\wp$ for homogeneous metric continua that are not aposyndetic.

Theorem 6.12. *Let X be a metric homogeneous continuum that is not aposyndetic and let x_0 be an element of X. If K is a component of $\wp(\{x_0\})$, then $\dim(K) = \dim(X)$.*

Proof: Let X be a metric homogeneous continuum that is not aposyndetic. Then, by [15, Theorem 2.1.34], there exists an element x' in X such that $\mathcal{T}(\{x'\}) \neq \{x'\}$. Hence, by homogeneity, $\mathcal{T}(\{x\}) \neq \{x\}$ for any $x \in X$, see [15, Lemma 3.3.1]. Let K be a component of $\wp(\{x_0\})$ and let $z \in K$. Since $\mathcal{Q} = \{\wp_X(\{x\})\colon x \in X\}$ is a decomposition of X (Theorem 6.10), we have that $\wp(\{x_0\}) = \wp(\{z\})$. By Theorem 3.23, we have that $\mathcal{T}(\{z\}) \subset \wp(\{z\})$. Since $\mathcal{T}(\{z\})$ is connected, we obtain that $\mathcal{T}(\{z\}) \subset K$. Now, by [21, Corollary 9], $\dim(\mathcal{T}(\{z\})) = \dim(X)$. Therefore, $\dim(K) = \dim(X)$. □

Corollary 6.13. *If X is a metric homogeneous continuum that is not aposyndetic, then for each point x of X, $\dim(\wp(\{x\})) = \dim(X)$.*

Remark 6.14. Note that if X is a homogeneous metric continuum and $\mathcal{G}$ is Jones' aposyndetic decomposition of X, then $\dim(X/\mathcal{G}) = 1$, see [22, Theorem 3].

Hence, we have the following:

Question 6.15. If X is a homogeneous metric continuum and $\mathcal{Q}$ is Prajs' mutual aposyndetic decomposition of X, then is $\dim(X/\mathcal{Q}) = 1$?

7. Generalized inverse limits

Let X and Y be compacta and let $f : X \to 2^Y$ be a function. Then f is *upper semicontinuous at a point x of X* provided that for each open subset U of Y with $f(x) \subset U$, there exists an open subset V of X such that if $x' \in V$, then $f(x') \subset U$. We say that f is *upper semicontinuous* if f is upper semicontinuous at each point of X.

Let X, Y and Z be compacta and let $f\colon X \to 2^Y$ and $g\colon Y \to 2^Z$ be upper semicontinuous functions. Then the *composition* of f and g, denoted $g \circ f$, is defined by

$$(g \circ f)(x) = \{z \in Z\colon \text{ there exists } y \in Y \text{ such that } y \in f(x) \text{ and } z \in g(y)\}.$$

Let X be a compactum. Then X^ω denotes the product of countably many copies of X with the product topology. The symbol Δ_X^ω denotes the diagonal of X^ω; i.e., $\Delta_X^\omega = \{(x, x, x, \ldots)\colon x \in X\}$.

Let X be a compactum. If $f\colon X \to 2^X$ is an upper semicontinuous function, then its *generalized inverse limit* is

$$\varprojlim f = \{(x_n)_{n=1}^\omega \in X^\omega\colon x_n \in f(x_{n+1}) \text{ for each positive integer } n\}.$$

The function f is called the *bonding function.* For each positive integer n, $\pi_n\colon \varprojlim f \to X$ is the projection map.

Let X be a compactum and let $\mathcal{G}$ be an upper semicontinuous decomposition of X. Then $f_{\mathcal{G}}\colon X \to 2^X$ is given by $f_{\mathcal{G}}(x) = G_x$, where G_x is the unique element of $\mathcal{G}$ such that $x \in G_x$.

The following two results are [15, Lemmas 1.2.26 and 1.2.27].

Lemma 7.1. *If X is a compactum and $\mathcal{G}$ is an upper semicontinuous decomposition of X, then $f_{\mathcal{G}}$ is an upper semicontinuous function.*

Lemma 7.2. *If X is a compactum and $\mathcal{G} = \{G_x\colon x \in X\}$ is an upper semicontinuous decomposition of X, then $\varprojlim f_{\mathcal{G}} = \bigcup\{G_x^\omega\colon x \in X\}$. Moreover, $\Delta_X^\omega \subset \varprojlim f_{\mathcal{G}}$.*

Remark 7.3. Let X be a metric continuum and let $\mathcal{G}$ be an upper semicontinuous decomposition of X. If $\mathcal{G}^\star = \{K\colon K \text{ is a component of } G \text{ for some } G \in \mathcal{G}\}$, then $\mathcal{G}^\star$ is an upper semicontinuous decomposition of X, see [15, Theorem 1.1.35] and $\varprojlim f_{\mathcal{G}^\star}$ is a metric subcontinuum of $\varprojlim f_{\mathcal{G}}$, see [15, Theorem 1.4.76].

The next result is [15, Theorem 1.4.78].

Theorem 7.4. *If X is a metric continuum and $\mathcal{G} = \{G_x\colon x \in X\}$ is a continuous decomposition of X, then $\mathcal{G}^* = \{G_x^\omega\colon x \in X\}$ is a continuous decomposition of $\varprojlim f_{\mathcal{G}}$. Moreover, $\varprojlim f_{\mathcal{G}}/\mathcal{G}^*$ is homeomorphic to $X/\mathcal{G}$.*

Let X be a decomposable homogeneous nonmutualy aposyndetic metric continuum and let $\mathcal{L} = \{\wp_{1_X}(x)\colon x \in X\}$ (Remark 4.13). Let $\mathcal{L}^\star = \{P_x\colon P_x$ is a component of $\wp(\{x\})$ for some $x \in X\}$.

Theorem 7.5. *Let X be a decomposable homogeneous nonmutualy aposyndetic metric continuum and let $\mathcal{L} = \{\wp_{1_X}(x)\colon x \in X\}$. If $\wp_{1_X}\colon X \to 2^X$ is given by $\wp_{1_X}(x) = \wp(\{x\})$, then $\varprojlim(\wp_{1_X})_{\mathcal{L}} = \bigcup\{\wp(\{x\})^\omega\colon x \in X\}$ is an infinite-dimensional homogeneous compact metric space. Moreover, if $\mathfrak{L} = \{P_x^\omega\colon P_x \in \mathcal{L}^\star\}$* (Remark 7.3), *then $\mathfrak{L}$ is a continuous decomposition of $\varprojlim(\wp_{1_X})_{\mathcal{L}}$, $\varprojlim(\wp_{1_X})_{\mathcal{L}^\star}$ is an infinite dimensional homogeneous subcontinuum of $\varprojlim(\wp_{1_X})_{\mathcal{L}}$ and $\varprojlim(\wp_{1_X})_{\mathcal{L}}/\mathfrak{L}$ is homeomorphic to $X/\mathcal{L}$. In particular, $\varprojlim(\wp_{1_X})_{\mathcal{L}}/\mathfrak{L}$ is mutually aposyndetic metric continuum.*

Proof: By [14, Theorem 4.2.31], X has the property of Effros. By Theorem 6.10, $\mathcal{L}$ is a continuous decomposition of X. Note that by [9, Theorem 3.2], $\varprojlim(\wp_{1_X})_{\mathcal{L}}$ is a nonempty compact metric space. Also, by Lemma 7.2, we have that $\varprojlim(\wp_{1_X})_{\mathcal{L}} = \bigcup\{\wp(\{x\})^\omega\colon x \in X\}$. Hence, $\varprojlim(\wp_{1_X})_{\mathcal{L}}$ is infinite-dimensional. We prove that $\varprojlim(\wp_{1_X})_{\mathcal{L}}$ is a homogeneous space. Let $(x_n)_{n=1}^\omega$ and $(z_n)_{n=1}^\omega$ be two elements of $\varprojlim(\wp_{1_X})_{\mathcal{L}}$. Since X is homogeneous for each positive integer n, there exists a homeomorphism $h_n\colon X \twoheadrightarrow X$ satisfying that $h_n(x_n) = z_n$. Let $h\colon \varprojlim(\wp_{1_X})_{\mathcal{L}} \twoheadrightarrow \varprojlim(\wp_{1_X})_{\mathcal{L}}$ be given by $h((w_n)_{n=1}^\omega) = (h_n(w_n))_{n=1}^\omega$. By Theorem 4.1, h is a well defined homeomorphism and holds $h((x_n)_{n=1}^\omega) = (z_n)_{n=1}^\omega$. Therefore, $\varprojlim(\wp_{1_X})_{\mathcal{L}}$ is homogeneous. By Theorem 7.4, $\mathfrak{L}$ is a continuous decomposition of $\varprojlim(\wp_{1_X})_{\mathcal{L}}$. By [9, Theorem 4.7], $\varprojlim(\wp_{1_X})_{\mathcal{L}^\star}$ is a subcontinuum of $\varprojlim(\wp_{1_X})_{\mathcal{L}}$. The proof of the homogeneity of $\varprojlim(\wp_{1_X})_{\mathcal{L}^\star}$ is similar to the one already given for $\varprojlim(\wp_{1_X})_{\mathcal{L}}$. Since given a point x of X, we have that $P_x^\omega \subset \varprojlim(\wp_{1_X})_{\mathcal{L}^\star}$, we obtain that $\varprojlim(\wp_{1_X})_{\mathcal{L}^\star}$ is infinite dimensional. By Theorem 7.4, $\varprojlim(\wp_{1_X})_{\mathcal{L}}/\mathfrak{L}$ is homeomorphic to $X/\mathcal{L}$. Therefore, $\varprojlim(\wp_{1_X})_{\mathcal{L}}/\mathfrak{L}$ is a mutually aposyndetic metric continuum. $\square$

8. Two topological properties

In this section, we work with metric continua. We prove a couple of results, one regarding the family of $\wp$-closed sets and the other regarding the family of images of nonempty closed sets under $\wp$.

If (X,d) is a metric continuum, A is nonempty closed subset of X, and $\varepsilon > 0$, then let $\mathcal{V}^d_\varepsilon(A) = \{x \in X\colon$ there exists $a \in A$ such that $d(x,a) < \varepsilon\}$. If B is another nonempty closed subset of X, then

$$\mathcal{H}(A,B) = \inf\{\varepsilon > 0\colon\ A \subset \mathcal{V}_\varepsilon(B) \text{ and } B \subset \mathcal{V}_\varepsilon(A)\}.$$

$\mathcal{H}$ is called the *Hausdorff metric* induced by d, see [14, Theorem 1.8.3]. Recall that if V is an open subset of X, then $\langle X, V\rangle = \{A \in 2^X\colon\ A \cap V \neq \emptyset\}$ is an open subset of 2^X, see [19, Theorem (0.11)].

The next theorem tells us that the family of $\wp$-closed sets of a metric continuum X is a G_δ subset of 2^X (compare with [15, Theorem 4.1.11]). Recall that the family of $\wp$-closed sets of a continuum X is denoted by $\mathfrak{P}(X)$.

Theorem 8.1. *If X is a metric continuum, then $\mathfrak{P}(X)$ is a G_δ subset of 2^X. Hence, $\mathfrak{P}(X)$ is completely metrizable.*

Proof: Given $\varepsilon > 0$ define

$$\Theta_\varepsilon = \{A \in 2^X\colon\ \wp(A) \subset \mathcal{V}_\varepsilon(A)\}.$$

Note that if $A \in \Theta_\varepsilon$, then $\mathcal{H}(A, \wp(A)) < \varepsilon$. If $A \in \Theta_\varepsilon$, then define

$$\mathcal{M}_\varepsilon(A) = \bigcup_{0<s<1} \big(\mathcal{V}^{\mathcal{H}}_{s\varepsilon}(A) \cap \{B \in 2^X\colon\ \wp(B) \subset \mathcal{V}_{(1-s)\varepsilon}(A)\}\big).$$

Note that $A \in \mathcal{M}_\varepsilon(A)$ and $\mathcal{M}_\varepsilon(A)$ is an open subset of 2^X (the second set is open in 2^X since $\wp$ is upper semicontinuous by Theorem 5.1).

We show that $\mathcal{M}_\varepsilon(A) \subset \Theta_\varepsilon$. Let $B \in \mathcal{M}_\varepsilon(A)$. Then there exists $s \in (0,1)$ such that $\mathcal{H}(B,A) < s\varepsilon$ and $\wp(B) \subset \mathcal{V}_{(1-s)\varepsilon}(A)$. We need to prove that $\wp(B) \subset \mathcal{V}_\varepsilon(B)$. Let $x \in \wp(B)$. Then, since $\wp(B) \subset \mathcal{V}_{(1-s)\varepsilon}(A)$, there exists $a \in A$ such that $d(x,a) < (1-s)\varepsilon$. Hence, since $\mathcal{H}(B,A) < s\varepsilon$, there exists $b \in B$ such that $d(b,a) < s\varepsilon$. By the triangle inequality, we obtain that $d(x,b) \leq d(x,a)+d(a,b) < (1-s)\varepsilon + s\varepsilon = \varepsilon$. Thus, $x \in \mathcal{V}^d_\varepsilon(B)$, and $\wp(B) \subset \mathcal{V}^d_\varepsilon(B)$. Hence, A is an interior point of Θ_ε. Thus, Θ_ε is open.

Observe that

$$\mathfrak{P}(X) = \bigcap_{n=1}^{\infty} \Theta_{1/n}.$$

Therefore, $\mathfrak{P}(X)$ is a G_δ subset of 2^X. Since 2^X is a compact metric space [19, Theorem (0.8)], we have that, by [23, Theorem 24.12], $\mathfrak{P}(X)$ is completely metrizable. □

A *Polish space* is a separable completely metrizable space. Let Z be a topological space. A subset A of Z is an *analytic set* provided that there exist a polish space Y and a map $f\colon Y \to Z$ such that $f(Y) = A$.

The following theorem shows that for each metric continuum X, $\wp(2^X)$ is an analytic set in 2^X (compare with [15, Theorem 6.1.3]).

Theorem 8.2. *If X is a metric continuum, then $\wp(2^X)$ is an analytic set.*

Proof: Let $\mathcal{F} = \{(A,B) \in 2^X \times 2^X \colon \wp(A) = B\}$. We prove that $\mathcal{F}$ is a G_δ-set. To this end, let $\mathcal{M} = \{(A,B) \in 2^X \times 2^X \colon B \subset \wp(A)\}$ and let $\mathcal{N} = \{(A,B) \in 2^X \times 2^X \colon \wp(A) \subset B\}$. We claim that $\mathcal{M}$ is a closed subset of $2^X \times 2^X$. To see this, let $(C,D) \in (2^X \times 2^X) \setminus \mathcal{M}$. Then there exists x in $D \setminus \wp(C)$. Since X is a regular space, there exist two disjoint open subsets U and V of X such that x belongs to U and $\wp(C) \subset V$. Since $\wp$ is upper semicontinuous (Theorem 5.1), there exists an open subset $\mathcal{U}$ of 2^X such that $C \in \mathcal{U}$ and $\wp(E) \in \langle V \rangle$; i.e., $\wp(E) \subset V$ for each $E \in \mathcal{U}$. Let $\mathcal{W} = \mathcal{U} \times \langle X, V \rangle$. Note that $(C,D) \in \mathcal{W}$ and $\mathcal{W}$ is an open subset of $2^X \times 2^X$. Also, if (E,F) is in $\mathcal{W}$, then $\wp(E) \subset V$ and $F \cap V \neq \emptyset$. Thus, $F \setminus \wp(E) \neq \emptyset$ and (E,F) belongs to $(2^X \times 2^X) \setminus \mathcal{M}$. Therefore, $\mathcal{W} \subset (2^X \times 2^X) \setminus \mathcal{M}$, and $\mathcal{M}$ is a closed subset of $2^X \times 2^X$.

Now, we show that $\mathcal{N}$ is a G_δ set. For each $n \in \mathbb{N}$, let

$$\mathcal{L}_n = \mathrm{Cl}_{2^X \times 2^X}\big(\{(A,B) \in 2^X \times 2^X \colon \mathcal{V}_{1/n}(a) \cap B = \emptyset \text{ for some } a \in \wp(A)\}\big).$$

We claim that $\mathcal{N} = (2^X \times 2^X) \setminus \bigcup_{n=1}^{\infty} \mathcal{L}_n$. To prove this, let (A,B) be an element of $(2^X \times 2^X) \setminus \mathcal{N}$. Then there exists $x \in \wp(A) \setminus B$. Let $n_0 \in \mathbb{N}$ be such that $\mathcal{V}_{1/n_0}(x) \cap B = \emptyset$. Hence, $(A,B) \in \mathcal{L}_{n_0} \subset \bigcup_{n=1}^{\infty} \mathcal{L}_n$. Therefore, $(2^X \times 2^X) \setminus \bigcup_{n=1}^{\infty} \mathcal{L}_n \subset \mathcal{N}$.

Next, assume that $(A,B) \in \mathcal{L}_n$ for some $n \in \mathbb{N}$. Let $\{(A_m, B_m)\}_{m=1}^{\infty}$ be a sequence in $\{(C,D) \in 2^X \times 2^X \colon \mathcal{V}_{1/n}(c) \cap D = \emptyset \text{ for some } c \in \wp(C)\}$ converging to (A,B). For each $m \in \mathbb{N}$, let $a_m \in \wp(A_m)$ be such that $\mathcal{V}_{1/m}(a_m) \cap B = \emptyset$. Since X is a compact metric space, there exists a subsequence $\{a_{m_l}\}_{l=1}^{\infty}$ of $\{a_m\}_{m=1}^{\infty}$ converging to an element a_0 of X. Since $\wp$ is upper semicontinuous (Theorem 5.1), $\limsup \wp(A_{m_l}) \subset \wp(A)$. Hence, $a_0 \in \wp(A)$. We claim that $\mathcal{V}_{1/(2n)}(a_0) \cap B = \emptyset$. Suppose that $\mathcal{V}_{1/(2n)}(a_0) \cap B \neq \emptyset$. Since $\{a_{m_l}\}_{l=1}^{\infty}$ converges to a_0 and $\{B_{m_l}\}_{l=1}^{\infty}$ converges to B, there exists $k \in \mathbb{N}$ such that a_{m_k} belongs to $\mathcal{V}_{1/(2n)}(a_0)$, and $\mathcal{V}_{1/(2n)}(a_0) \cap B_{m_k} \neq \emptyset$. Let z be a point of $\mathcal{V}_{1/(2n)}(a_0) \cap B_{m_k}$. Note that $d(a_{m_k}, z) \leq d(a_{m_k}, a_0) + d(a_0, z) < 1/(2n) + 1/(2n) = 1/n$. Thus,

$\mathcal{V}_{1/n}(a_{m_k}) \cap B_{m_k} \neq \emptyset$, a contradiction. Hence, $\mathcal{V}_{1/(2n)}(a_0) \cap B = \emptyset$, and $a_0 \in \wp(A) \setminus B$. Therefore, $(A, B) \in (2^X \times 2^X) \setminus \mathcal{N}$, and $\mathcal{N}$ is a G_δ-set.

Observe that $\mathcal{F} = \mathcal{M} \cap \mathcal{N}$. Thus, $\mathcal{F}$ is a G_δ-set. By [11, Theorem 3.11], $\mathcal{F}$ is a polish space. Also, note that $\wp(2^X) = \pi_2(\mathcal{F})$, where $\pi_2\colon 2^X \times 2^X \twoheadrightarrow 2^X$ is given by $\pi_2((A, B)) = B$ for every element (A, B) of $2^X \times 2^X$. Therefore, $\wp(2^X)$ is an analytic set. $\square$

9. Effros continua

We prove that every continuum with the uniform property of Effros is an Effros continuum (see definition below).

Notation 9.1. Let X be a homogeneous continuum, let $\mathcal{H}(X)$ be its group of homeomorphisms with the compact-open topology. If $x \in X$, then define the *evaluation map* $\gamma_x\colon \mathcal{H}(X) \to X$ by $\gamma_x(h) = h(x)$.

Remark 9.2. It is known that if X is a compactum and $\mathcal{H}(X)$ is the group of homeomorphisms of X, then $\mathcal{H}(X)$ is a topological group with the compact-open topology [4, page 441].

Let X be a continuum and let $\mathcal{H}(X)$ be the family of homeomorphisms of X with the compact-open topology. In [1], D. P. Bellamy and K. F. Porter defined an *Effros continuum* as a homogeneous continuum X such that for each element x of X, the evaluation map $\gamma_x\colon \mathcal{H}(X) \twoheadrightarrow X$ is open (Notation 9.1). Observe that they constructed a homogeneous continuum that is not an Effros continuum, see [1].

Notation 9.3. Let X be a compactum. For every $U \in \mathfrak{U}_X$ (Remark 2.2), let $\widehat{U}$ be the entourage of the diagonal $\Delta_{\mathcal{H}(X)} \subset \mathcal{H}(X) \times \mathcal{H}(X)$ defined by:

$$\widehat{U} = \{(g_1, g_2) \in \mathcal{H}(X) \times \mathcal{H}(X)\colon\ \varrho_X(g_1(x), g_2(x)) < U \ \text{ for all } \ x \in X\}.$$

It is known that the family $\widehat{\mathcal{U}} = \{\widehat{U}\colon\ U \in \mathfrak{U}_X\}$ is a base for a uniformity on $\mathcal{H}(X)$; the uniformity generated by this family is called the *uniformity of uniform convergence* induced by $\mathfrak{U}_X$ and is denoted by $\widehat{\mathfrak{U}}_X$, see [4, page 440].

We need a special case of [4, 8.2.7]:

Theorem 9.4. *Let X be a compactum. Then the topology on $\mathcal{H}(X)$ induced by the uniformity $\widehat{\mathfrak{U}}_X$ of uniform convergence coincides with the compact-open topology on $\mathcal{H}(X)$, and depends only on the topology induced on X by the uniformity $\mathfrak{U}_X$ (Remark 2.2).*

Theorem 9.5. *Let Z be a Tychonoff space and let $\mathfrak{U}$ be a uniformity that induces the topology of Z. If Z has the uniform property of Effros with respect to $\mathfrak{U}$, then for each element z of Z and every $U \in \mathfrak{U}$, there exists an open subset $O_{U,z}$ with $z \in O_{U,z}$ such that if $z' \in O_{U,z}$, there exists a U-homeomorphism $h\colon Z \twoheadrightarrow Z$ such that $h(z) = z'$.*

Proof: Let z be a point of Z and let $U \in \mathfrak{U}$. Since Z has the uniform property of Effros with respect to $\mathfrak{U}$, there exists an Effros entourage V for U. Note that $\mathrm{Int}_Z(B(z,V))$ is an open subset of Z containing z. Let $z' \in \mathrm{Int}_Z(B(z,V))$. Then $\varrho_Z(z,z') < V$. Thus, there exists a U-homeomorphism $h\colon Z \twoheadrightarrow Z$ such that $h(z) = z'$. □

Theorem 9.6. *Let X be a continuum. Then X is an Effros continuum if and only if X has the uniform property of Effros.*

Proof: If X is an Effros continuum, by [13, Theorem 5.9] (or [15, Theorem 1.4.60]), then X has the uniform property of Effros.

Suppose X is a continuum with the uniform property of Effros. By Theorem 6.8, X is a homogeneous continuum. Let x be a point of X, let $\mathcal{O}$ be an open subset of $\mathcal{H}(X)$ and let $x' \in \gamma_x(\mathcal{O})$. Then there exists $h \in \mathcal{O}$ such that $\gamma_x(h) = h(x) = x'$. Since $\mathcal{O}$ is an open subset of $\mathcal{H}(X)$, there exists $U \in \mathfrak{U}_X$ such that $h \in B(h,\widehat{U}) \subset \mathcal{O}$ (Notation 9.3). By Theorem 9.5, there exists an open subset W of X such that $x \in W$ and if $z \in W$, then there exists a U-homeomorphism of X with $h(x') = z$. Thus, $\varrho_{\mathcal{H}(X)}(e,h) < \widehat{U}$. We show that $W \subset \gamma_x(\mathcal{O})$. To this end, let $w \in W$. Then there exists a U-homeomorphism f such that $f(x') = w$. Since f is a U-homeomorphism, we have that $\varrho_{\mathcal{H}(X)}(e,f) < \widehat{U}$. Hence, $\varrho_{\mathcal{H}(X)}(h, f\circ h) < \widehat{U}$. Thus, $f \circ h \in B(h,\widehat{U})$ and $f\circ h(x) = f(h(x)) = f(x') = w$. Hence, $w \in \gamma_x(B(h,\widehat{U})) \subset \gamma_x(\mathcal{O})$. Therefore, $\gamma_x(\mathcal{O})$ is an open subset of X and γ_x is an open map. □

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S. Macías:
Current address:
Escuela de Matemáticas, Facultad de Ciencias,
Universidad Industrial de Santander, Ciudad Universitaria,
Carrera 27 Calle 9, Bucaramanga, Santander, A. A. 678, Colombia
Permanent address:
Instituto de Matemáticas, Universidad Nacional Autónoma de México,
Circuito Exterior, Ciudad Universitaria, CDMX, C. P. 04510, México

E-mail: sergiom@matem.unam.mx

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