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SOME LOWER BOUNDS FOR THE QUOTIENTS OF NORMALIZED ERROR FUNCTION AND THEIR PARTIAL SUMS

Basem Aref Frasin

Abstract. The purpose of the present paper is to determine lower bounds for $\Re\left\{\frac{\mathcal{E}_k f(z)}{(\mathcal{E}_k f)_m(z)}\right\}, \Re\left\{\frac{(\mathcal{E}_k f)_m(z)}{\mathcal{E}_k f(z)}\right\}, \Re\left\{\frac{\mathcal{E}'_k f(z)}{(\mathcal{E}_k f)'_m(z)}\right\}$ and $\Re\left\{\frac{(\mathcal{E}_k f)'_m(z)}{\mathcal{E}'_k f(z)}\right\}$, where $\mathcal{E}_k f$ is the generalized normalized error function of the form $\mathcal{E}_k f(z) = z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{((n-1)k+1)(n-1)!} z^n$ and $(\mathcal{E}_k f)_m$ its partial sum. Furthermore, we give lower bounds for $\Re\left\{\frac{\mathbb{I}[\mathcal{E}_k f](z)}{(\mathbb{I}[\mathcal{E}_k f])_m(z)}\right\}$ and $\Re\left\{\frac{(\mathbb{I}[\mathcal{E}_k f])_m(z)}{\mathbb{I}[\mathcal{E}_k f](z)}\right\}$, where $\mathbb{I}[\mathcal{E}_k f]$ is the Alexander transform of $\mathcal{E}_k f$. Several examples of the main results are also considered.

1. Introduction and preliminaries

The partial sum and certain ratios of partial sums to their derivatives is one of the important problems associated with the series of analytic functions. Silvia [22] was the first one who introduced the concept of finding the lower bound of the real part of the ratio of the partial sum of analytic functions to its infinite series sum. In [21], Silverman investigated partial sums for different subclasses of analytic functions. For more work on partial sums, the interested readers are referred to [10, 11, 13, 16, 18, 19, 20].

Recently, some researchers have studied on partial sums of special functions. For example, Orhan and Yagmur in [23] determined lower bounds for the normalized Struve functions to its sequence of partial sums. Some lower bounds for the quotients of normalized Dini functions and their partial sum, as well as for the quotients of the derivative of normalized Dini functions and their partial sums were obtained by Aktaş and Orhan in [3]. Din et al. [8] found the partial sums of two kinds normalized Wright functions and the partial sums of Alexander transform of these normalized Wright functions. While Kazımoğlu in [15] studied the partial sums of the normalized Miller-Ross Function. Very recently, Frasin and Cotîrlă [12] determined lower bounds for the normalized Le Roy-type Mittag-Leffler function.

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Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\mathfrak{U} = \{z : |z| < 1\}$ and hold the normalization condition $f(0) = f'(0) - 1 = 0$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathfrak{U}$.

Also let the transform $\mathbb{I}[f]\colon \mathfrak{U} \to \mathbb{C}$ of f be defined by

$$\mathbb{I}[f] = \int_0^z f(t)\,dt = z + \sum_{n=2}^{\infty} \frac{a_n}{n} z^n ,$$

where $\mathbb{I}[f]$ is called the Alexander transform of f [4].

The error function, which is denoted by erf is defined by [1]

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)\, n!} , \qquad (z \in \mathbb{C}) , \tag{1.2}$$

whereas the imaginary error function, which is denoted by erf i, is defined by

$$\operatorname{erf} i\,(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)\, n!} , \qquad (z \in \mathbb{C}) . \tag{1.3}$$

The error function defined has many applications in partial differential equations physics as well as in probability science, statistics and applied mathematics. In quantum mechanics, error function is important in estimate the probability of observing a particle in a specified region. Various properties and inequalities of error function were presented by Alzer [5] and Coman [7], whereas Elbert et al. [9] studied the properties of complementary error function.

A generalization of the error function (1.2) is given by [1]

$$\begin{aligned} \operatorname{erf}_k(z) &= \frac{k!}{\sqrt{\pi}} \int_0^z e^{-t^k} dt \\ &= \frac{k!}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{kn+1}}{(kn+1)\, n!} , \qquad (k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ z \in \mathbb{C}) . \end{aligned} \tag{1.4}$$

Also, a generalization of the imaginary error function (1.3) is given by

$$\begin{aligned} \operatorname{erf} i_k(z) &= \frac{k!}{\sqrt{\pi}} \int_0^z e^{t^k} dt \\ &= \frac{k!}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{z^{kn+1}}{(kn+1)\, n!} , \qquad (k \in \mathbb{N}_0,\ z \in \mathbb{C}) . \end{aligned} \tag{1.5}$$

From (1.4) and (1.5), we immediately have

$$\operatorname{erf}_0(z) = \frac{z}{e\sqrt{\pi}}, \quad \operatorname{erf}_1(z) = \frac{1 - e^z}{\sqrt{\pi}} = -\operatorname{erf} i_1(z) ,$$
$$\operatorname{erf}_2(z) = \operatorname{erf}(z) \quad \text{and} \quad \operatorname{erf} i_2(z) = \operatorname{erf} i\,(z) .$$

It is clear that the generalized error function $\operatorname{erf}_k(z)$ and the generalized imaginary error function $\operatorname{erf} i_k(z)$ are not members of the the family $\mathcal{A}$. Thus, it is natural to consider the following normalizations of these functions:

$$\begin{aligned}\mathcal{E}_k f(z) &= \frac{\sqrt{\pi}}{k!} z^{\left(1-\frac{1}{k}\right)} \operatorname{erf}_k\left(z^{1/k}\right) \\ &= z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{((n-1)k+1)(n-1)!} z^n, \qquad (k \in \mathbb{N}), \end{aligned} \tag{1.6}$$

and

$$\begin{aligned}\mathcal{E}i_k f(z) &= \frac{\sqrt{\pi}}{k!} z^{\left(1-\frac{1}{k}\right)} \operatorname{erf} i_k\left(z^{1/k}\right) \\ &= z + \sum_{n=2}^{\infty} \frac{1}{((n-1)k+1)(n-1)!} z^n, \qquad (k \in \mathbb{N}). \end{aligned} \tag{1.7}$$

We notice that if we put $k = 2$ in (1.6), we get the normalization $\mathcal{E}_2 f = Erf$ given by Ramachandran et al. [6] and if we put $k = 2$ in (1.7), we get the the normalization $\mathcal{E}i_2 f = Erfi$ given by Mohammed et al. [17].

From (1.6) and (1.7), we have

$$\mathcal{E}_1 f(z) = \sqrt{\pi} \operatorname{erf}_1(z) = 1 - e^z, \quad \mathcal{E}i_1 f(z) = \sqrt{\pi} \operatorname{erf} i_1(z) = e^z - 1,$$
$$\mathcal{E}_2 f(z) = \frac{\sqrt{\pi z}}{2} \operatorname{erf}\left(\sqrt{z}\right) \quad \text{and} \quad \mathcal{E}i_2 f(z) = \frac{\sqrt{\pi z}}{2} \operatorname{erf} i\left(\sqrt{z}\right).$$

In this paper, we determine the lower bounds for

$$\Re\left\{\frac{\mathcal{E}_k f(z)}{(\mathcal{E}_k f)_m(z)}\right\}, \Re\left\{\frac{(\mathcal{E}_k f)_m(z)}{\mathcal{E}_k f(z)}\right\}, \Re\left\{\frac{\mathcal{E}_k' f(z)}{(\mathcal{E}_k f)_m'(z)}\right\} \quad \text{and} \quad \Re\left\{\frac{(\mathcal{E}_k f)_m'(z)}{\mathcal{E}_k' f(z)}\right\},$$

where $\mathcal{E}_k f$ is the generalized normalized error function of the form (1.6) and $(\mathcal{E}_k f)_m$ its partial sum given by

$$(\mathcal{E}_k f)_m(z) = z + \sum_{n=1}^{m} \frac{(-1)^{n-1}}{((n-1)k+1)(n-1)!} z^{n+1}, \quad m \in \mathbb{N}, \tag{1.8}$$

and for $m = 0$, we have $(\mathcal{E}_k f(z))_0 = z$. Furthermore, we give lower bounds for

$$\Re\left\{\frac{\mathbb{I}[\mathcal{E}_k f](z)}{(\mathbb{I}[\mathcal{E}_k f])_m(z)}\right\} \quad \text{and} \quad \Re\left\{\frac{(\mathbb{I}[\mathcal{E}_k f])_m(z)}{\mathbb{I}[\mathcal{E}_k f](z)}\right\},$$

where $\mathbb{I}[\mathcal{E}_k f]$ is the Alexander transform of $\mathcal{E}_k f$.

2. Some lemmas

In this section, we provide some useful lemmas which will be useful to complete the proof of the main results.

Lemma 2.1. *Let $k \in \mathbb{N}$. Then the function $\mathcal{E}_k f : \mathfrak{U} \to \mathbb{C}$ defined by* (1.6) *satisfies the following inequalities:*

(i)

$$|\mathcal{E}_k f(z)| \leq \frac{k+\ln 4}{k} \qquad (z \in \mathfrak{U}) ,$$

(ii)

$$\left|(\mathcal{E}_k f(z))'\right| \leq \frac{k-1+e+\ln 4}{k} \qquad (z \in \mathfrak{U}) ,$$

(iii)

$$||\mathbb{I}[\mathcal{E}_k f](z)]|| \leq \frac{k+\ln 2}{k} \qquad (z \in \mathfrak{U}) .$$

Proof. (i) By using the following well-known series sums

$$\sum_{n=1}^{\infty} \frac{1}{n2^n} = \ln 2 , \tag{2.1}$$

and the following inequalities

$$n! \geq 2^{n-1} \ (n \in \mathbb{N} = \{1,2,3,\ldots\}) , \tag{2.2}$$

$$nk < nk+1 \qquad (k,\ n \in \mathbb{N}) , \tag{2.3}$$

we have

$$\begin{aligned} |\mathcal{E}_k f(z)| &= \left| z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{((n-1)k+1)(n-1)!} z^n \right| \\ &= \left| z + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(nk+1)} z^{n+1} \right| \\ &\leq 1 + \sum_{n=1}^{\infty} \frac{1}{n!(nk+1)} \\ &\leq 1 + \sum_{n=1}^{\infty} \frac{1}{n!kn} \\ &\leq 1 + \frac{2}{k} \sum_{n=1}^{\infty} \frac{1}{n2^n} = 1 + \frac{\ln 4}{k} . \end{aligned}$$

(ii) To prove (ii), using (2.1) with the following well-known series sums

$$\sum_{n=1}^{\infty} \frac{1}{n!} = e-1 \tag{2.4}$$

and the inequalities (2.2), (2.3) we have

$$\begin{aligned}\left|\mathcal{E}_k f\,(z)'\right| &= \left|1+\sum_{n=1}^{\infty}\frac{(n+1)\,(-1)^n}{n!\,(nk+1)}z^n\right|\\ &\leq 1+\sum_{n=1}^{\infty}\frac{(n+1)}{n!\,(kn)}\\ &= 1+\sum_{n=1}^{\infty}\frac{1}{kn!}+\sum_{n=1}^{\infty}\frac{1}{n!\,(kn)}\\ &= 1+\sum_{n=1}^{\infty}\frac{1}{kn!}+\frac{2}{k}\sum_{n=1}^{\infty}\frac{1}{n2^n}\\ &= 1+\frac{(e-1)}{k}+\frac{\ln 4}{k}.\end{aligned}$$

(iii) Making the use of (2.1) and the inequality

$$(n+1)! \geq 2^n \quad (n\in\mathbb{N})\ ,$$

we thus find

$$\begin{aligned}\left|\mathbb{I}[\mathcal{E}_k f](z)\right| &= \left|z+\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)\,n!\,(nk+1)}z^{n+1}\right|\\ &\leq 1+\sum_{n=1}^{\infty}\frac{1}{(n+1)!\,(nk+1)}\\ &\leq 1+\frac{1}{k}\sum_{n=1}^{\infty}\frac{1}{n2^n}\\ &= 1+\frac{\ln 2}{k}.\end{aligned}$$

□

3. Partial sums

In this section, we will use the following well-known result:
Let $w(z)$ be an analytic function in $\mathfrak{U}$. Then

$$\Re\left\{\frac{1+w(z)}{1-w(z)}\right\}>0\,,\ z\in\mathfrak{U}\quad\text{if and only if}\quad |w(z)|<1\,,\ z\in\mathfrak{U}\,.$$

The lower bounds for $\Re\left\{\mathcal{E}_k f(z)/(\mathcal{E}_k f)_m(z)\right\}$ and $\Re\left\{(\mathcal{E}_k f)_m(z)/\mathcal{E}_k f(z)\right\}$, will be determined in Theorem 3.1 below.

Theorem 3.1. *Let $k\geq 2$; $k\in\mathbb{N}$. Then*

$$\Re\left\{\frac{\mathcal{E}_k f\,(z)}{(\mathcal{E}_k f)_m(z)}\right\}\geq\frac{k-\ln 4}{k}\,,\qquad z\in\mathfrak{U}\,, \tag{3.1}$$

and

$$\Re\left\{\frac{(\mathcal{E}_k f)_m(z)}{\mathcal{E}_k f\,(z)}\right\}\geq\frac{k}{k+\ln 4}\,,\qquad z\in\mathfrak{U}\,. \tag{3.2}$$

Proof. From inequality (i) of Lemma 2.1, we get

$$1+\sum_{n=1}^{\infty}|A_n|\leq 1+\frac{\ln 4}{k}\,,$$

or equivalently

$$\Big(\frac{k}{\ln 4}\Big)\sum_{n=1}^{\infty}|A_n|\leq 1\,,$$

where

$$A_n=\frac{(-1)^n}{n!\,(nk+1)}\,. \tag{3.3}$$

Consider the function $w(z)$ defined by

$$\begin{aligned}\frac{1+w(z)}{1-w(z)}&=\Big(\frac{k}{\ln 4}\Big)\Big[\frac{\mathcal{E}_k f\,(z)}{(\mathcal{E}_k f)_m(z)}-\frac{k-\ln 4}{k}\Big]\\ &=\frac{1+\sum_{n=1}^{m}A_nz^n+\big(\frac{k}{\ln 4}\big)\sum_{n=m+1}^{\infty}A_nz^n}{1+\sum_{n=1}^{m}A_nz^n}\,. \end{aligned}\tag{3.4}$$

Now, from (3.4) we can write

$$w(z)=\frac{\big(\frac{k}{\ln 4}\big)\sum_{n=m+1}^{\infty}A_nz^n}{2+2\sum_{n=1}^{m}A_nz^n+\big(\frac{k}{\ln 4}\big)\sum_{n=m+1}^{\infty}A_nz^n}$$

and

$$|w(z)|\leq\frac{\big(\frac{k}{\ln 4}\big)\sum_{n=m+1}^{\infty}|A_n|}{2-2\sum_{n=1}^{m}|A_n|-\big(\frac{k}{\ln 4}\big)\sum_{n=m+1}^{\infty}|A_n|}\,.$$

This implies that $|w(z)|\leq 1$ if and only if

$$2\Big(\frac{k}{\ln 4}\Big)\sum_{n=m+1}^{\infty}|A_n|\leq 2-2\sum_{n=1}^{m}|A_n|\,.$$

Which further implies that

$$\sum_{n=1}^{m}|A_n|+\Big(\frac{k}{\ln 4}\Big)\sum_{n=m+1}^{\infty}|A_n|\leq 1\,. \tag{3.5}$$

It suffices to show that the left hand side of (3.5) is bounded above by

$$\Big(\frac{k}{\ln 4}\Big)\sum_{n=1}^{\infty}|A_n|\,,$$

which is equivalent to

$$\Big(\frac{k-\ln 4}{\ln 4}\Big)\sum_{n=1}^{m}|A_n|\geq 0\,.$$

To prove (3.2), we write

$$\frac{1+w(z)}{1-w(z)} = \Big(\frac{k+\ln 4}{\ln 4}\Big)\Big[\frac{(\mathcal{E}_k f)_m(z))}{\mathcal{E}_k f\,(z)} - \frac{k}{k+\ln 4}\Big]$$
$$= \frac{1+\sum_{n=1}^{m} A_n z^n - \big(\frac{k}{\ln 4}\big)\sum_{n=m+1}^{\infty} A_n z^n}{1+\sum_{n=1}^{m} A_n z^n}\,.$$

Therefore

$$|w(z)| \leq \frac{\big(\frac{k+\ln 4}{\ln 4}\big)\sum_{n=m+1}^{\infty}|A_n|}{2-2\sum_{n=1}^{m}|A_n| - \big(\frac{k-\ln 4}{\ln 4}\big)\sum_{n=m+1}^{\infty}|A_n|} \leq 1\,.$$

The last inequality is equivalent to

$$\sum_{n=1}^{m}|A_n| + \Big(\frac{k}{\ln 4}\Big)\sum_{n=m+1}^{\infty}|A_n| \leq 1\,. \tag{3.6}$$

Since the left hand side of (3.6) is bounded above by $\big(\frac{k}{\ln 4}\big)\sum_{n=1}^{\infty}|A_n|$, this completes the proof. □

We next turn to ratios involving derivatives.

Theorem 3.2. *Let $k \geq 4$, $k \in \mathbb{N}$. Then*

$$\Re\left\{\frac{\mathcal{E}_k' f(z)}{(\mathcal{E}_k f)_m'(z)}\right\} \geq \frac{k+1-e-\ln 4}{k}\,, \qquad z \in \mathfrak{U}, \tag{3.7}$$

and

$$\Re\left\{\frac{(\mathcal{E}_k f)_m'(z)}{\mathcal{E}_k' f(z)}\right\} \geq \frac{k}{k-1+e+\ln 4}\,, \qquad z \in \mathfrak{U}. \tag{3.8}$$

Proof. From part (ii) of Lemma 2.1, we observe that

$$1+\sum_{n=1}^{\infty}(n+1)\,|A_n| \leq \frac{k-1+e+\ln 4}{k}\,,$$

where A_n as given in (3.3). This implies that

$$\Big(\frac{k}{e-1+\ln 4}\Big)\sum_{n=1}^{\infty}(n+1)\,|A_n| \leq 1\,.$$

Consider

$$\frac{1+w(z)}{1-w(z)} = \Big(\frac{k}{e-1+\ln 4}\Big)\Big[\frac{\mathcal{E}_k' f(z)}{(\mathcal{E}_k f)_m'(z)} - \frac{k+1-e-\ln 4}{k}\Big]$$
$$= \frac{1+\sum_{n=1}^{m}(n+1)A_n z^n + \Big(\frac{k}{e-1+\ln 4}\Big)\sum_{n=m+1}^{\infty}(n+1)A_n z^n}{1+\sum_{n=1}^{m}(n+1)A_n z^n}\,.$$

Therefore

$$|w(z)| \leq \frac{\Big(\frac{k}{e-1+\ln 4}\Big)\sum_{n=m+1}^{\infty}(n+1)\,|A_n|}{2-2\sum_{n=1}^{m}(n+1)\,|A_n| - \Big(\frac{k}{e-1+\ln 4}\Big)\sum_{n=m+1}^{\infty}(n+1)\,|A_n|} \leq 1\,.$$

The last inequality is equivalent to

$$\sum_{n=1}^{m}(n+1)\,|A_n| + \left(\frac{k}{e-1+\ln 4}\right)\sum_{n=m+1}^{\infty}(n+1)\,|A_n| \le 1\,. \tag{3.9}$$

It suffices to show that the left hand side of (3.9) is bounded above by

$$\left(\frac{k}{e-1+\ln 4}\right)\sum_{n=1}^{\infty}(n+1)\,|A_n|$$

which is equivalent to

$$\left(\frac{k+1-e-\ln 4}{e-1+\ln 4}\right)\sum_{n=1}^{m}(n+1)\,|A_n| \ge 0\,.$$

To prove the result (3.8), we write

$$\frac{1+w(z)}{1-w(z)} = \left(\frac{k-1+e+\ln 4}{e-1+\ln 4}\right)\left[\frac{(\mathcal{E}_k f)'_m(z)}{\mathcal{E}'_k f(z)} - \frac{k}{k-1+e+\ln 4}\right]$$

where

(3.10)

$$|w(z)| \le \frac{\left(\frac{k-1+e+\ln 4}{e-1+\ln 4}\right)\sum_{n=m+1}^{\infty}(n+1)\,|A_n|}{2-2\sum_{n=1}^{m}(n+1)\,|A_n| - \left(\frac{k+1-e-\ln 4}{e-1+\ln 4}\right)\sum_{n=m+1}^{\infty}(n+1)\,|A_n|} \le 1\,.$$

But the last inequality (3.10) is equivalent to

$$\sum_{n=1}^{m}(n+1)\,|A_n| + \left(\frac{k}{e-1+\ln 4}\right)\sum_{n=m+1}^{\infty}(n+1)\,|A_n| \le 1\,. \tag{3.11}$$

It is observed that the left hand side of (3.11) is bounded above by

$$\left(\frac{k}{e-1+\ln 4}\right)\sum_{n=1}^{\infty}(n+1)\,|A_n|\ .$$

□

Finally, we prove the following theorem.

Theorem 3.3. *Let $k \in \mathbb{N}$. Then*

$$\Re\left\{\frac{\mathbb{I}[\mathcal{E}_k f](z)}{(\mathbb{I}[\mathcal{E}_k f])_m(z)}\right\} \ge \frac{k-\ln 2}{k}\,, \qquad z \in \mathfrak{U}\,, \tag{3.12}$$

and

$$\Re\left\{\frac{(\mathbb{I}[\mathcal{E}_k f])_m(z)}{\mathbb{I}[\mathcal{E}_k f](z)}\right\} \ge \frac{k}{k+\ln 2}\,, \qquad z \in \mathfrak{U}\,. \tag{3.13}$$

where $\mathbb{I}[\mathcal{E}_k f]$ is the Alexander transform of $\mathcal{E}_k f$.

Proof. To prove (3.12), we consider from part (iii) of Lemma 2.1 so that

$$1+\sum_{n=1}^{\infty}\frac{|A_n|}{n+1}\leq\frac{k+\ln 2}{k}\,,$$

or equivalently

$$\left(\frac{k}{\ln 2}\right)\sum_{n=1}^{\infty}\frac{|A_n|}{n+1}\leq 1$$

where A_n as given in (3.3). Now, we write

$$\begin{aligned}\frac{1+w(z)}{1-w(z)}&=\left(\frac{k}{\ln 2}\right)\left[\frac{\mathbb{I}[\mathcal{E}_k f](z)}{(\mathbb{I}[\mathcal{E}_k f])_m(z)}-\frac{k-\ln 2}{k}\right]\\ &=\frac{1+\sum_{n=1}^{m}\frac{|A_n|}{n+1}z^n+\left(\frac{k}{\ln 2}\right)\sum_{n=m+1}^{\infty}\frac{|A_n|}{n+1}z^n}{1+\sum_{n=1}^{m}\frac{|A_n|}{n+1}z^n}\,. \qquad (3.14)\end{aligned}$$

Now, from (3.14) we can write

$$w(z)=\frac{\left(\frac{k}{\ln 2}\right)\sum_{n=m+1}^{\infty}\frac{A_n}{n+1}z^n}{2+2\sum_{n=1}^{m}\frac{A_n}{n+1}z^n+\left(\frac{k}{\ln 2}\right)\sum_{n=m+1}^{\infty}\frac{A_n}{n+1}z^n}\,.$$

Using the fact that $|w(z)|\leq 1$, we get

$$\frac{\left(\frac{k}{\ln 2}\right)\sum_{n=m+1}^{\infty}\frac{|A_n|}{n+1}}{2-2\sum_{n=1}^{m}\frac{|A_n|}{n+1}-\left(\frac{k}{\ln 2}\right)\sum_{n=m+1}^{\infty}\frac{|A_n|}{n+1}}\leq 1\,.$$

The last inequality is equivalent to

$$\sum_{n=1}^{m}\frac{|A_n|}{n+1}+\left(\frac{k}{\ln 2}\right)\sum_{n=m+1}^{\infty}\frac{|A_n|}{n+1}\leq 1\,. \qquad (3.15)$$

It suffices to show that the left hand side of (3.15) is bounded above by

$$\left(\frac{k}{\ln 2}\right)\sum_{n=1}^{\infty}\frac{|A_n|}{n+1}\,,$$

which is equivalent to

$$\left(\frac{k-\ln 2}{k}\right)\sum_{n=1}^{m}\frac{|A_n|}{n+1}\geq 0\,.$$

The proof of (3.13) is similar to the proof of Theorem 3.1. □

4. Special cases

If we take $k=2$ in Theorem 3.1, we obtain the following corollary.

Corollary 4.1. *The following inequalities hold true:*

$$\Re\left\{\frac{Erf\,(z)}{(Erf)_m(z)}\right\} \geq 1-\ln 2 \approx 0.306\,85\,, \qquad z\in\mathfrak{U}, \tag{4.1}$$

and

$$\Re\left\{\frac{(Erf)_m(z)}{Erf\,(z)}\right\} \geq \frac{1}{1+\ln 2} \approx 0.590\,62\,, \qquad z\in\mathfrak{U}. \tag{4.2}$$

In particular for $m=0$, we get

$$\Re\left\{\sqrt{\frac{\pi}{z}}\,\mathrm{erf}\left(\sqrt{z}\right)\right\} \geq 2-2\ln 2 \approx 0.613\,71\,, \qquad z\in\mathfrak{U}, \tag{4.3}$$

and

$$\Re\left\{\frac{\sqrt{\frac{z}{\pi}}}{\mathrm{erf}\left(\sqrt{z}\right)}\right\} \geq \frac{1}{2\,(1+\ln 2)} \approx 0.295\,31\,, \qquad z\in\mathfrak{U}. \tag{4.4}$$

If we take $k=2$ in Theorem 3.3, we obtain the following corollary.

Corollary 4.2. *The following inequalities hold true:*

$$\Re\left\{\frac{\mathbb{I}[Erf](z)}{(\mathbb{I}[Erf])_m(z)}\right\} \geq \frac{2-\ln 2}{2} \approx 0.653\,43\,, \qquad z\in\mathfrak{U}, \tag{4.5}$$

and

$$\Re\left\{\frac{(\mathbb{I}[Erf])_m(z)}{\mathbb{I}[Erf](z)}\right\} \geq \frac{2}{2+\ln 2} \approx 0.742\,63\,, \qquad z\in\mathfrak{U}. \tag{4.6}$$

Remark 4.3. The results (4.1) and (4.2) of Corollary 4.1 were obtained in [2].

Remark 4.4. Putting $m=0$ in inequality (3.7), we obtain $\Re\left\{\mathcal{E}_k' f(z)\right\} > 0$. In view of Noshiro-Warschawski Theorem (see [14]), we have that the normalized Le Roy-type Mittag-Leffler function is univalent in $\mathfrak{U}$ if $k\geq 4$, $k\in\mathbb{N}$.

5. Conclusions

In the present article, we have considered the generalized and normalized error function $\mathcal{E}_k f$ and determined lower bounds for the following real parts

$$\Re\left\{\frac{\mathcal{E}_k f(z)}{(\mathcal{E}_k f)_m(z)}\right\}, \Re\left\{\frac{(\mathcal{E}_k f)_m(z)}{\mathcal{E}_k f(z)}\right\}, \Re\left\{\frac{\mathcal{E}_k' f(z)}{(\mathcal{E}_k f)_m'(z)}\right\} \quad \text{and} \quad \Re\left\{\frac{(\mathcal{E}_k f)_m'(z)}{\mathcal{E}_k' f(z)}\right\},$$

$$\Re\left\{\frac{\mathbb{I}\left[\mathcal{E}_k f\right](z)}{(\mathbb{I}\left[\mathcal{E}_k f\right])_m(z)}\right\} \quad \text{and} \quad \Re\left\{\frac{(\mathbb{I}\left[\mathcal{E}_k f\right])_m(z)}{\mathbb{I}\left[\mathcal{E}_k f\right](z)}\right\},$$

where $(\mathcal{E}_k f)_m$ is the partial sum and $\mathbb{I}\left[\mathcal{E}_k f\right]$ is the Alexander transform of $\mathcal{E}_k f$.

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