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ON THE DISK-CYCLIC LINEAR RELATIONS

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Abstract. The study of linear dynamical systems for linear relations was initiated by C.-C. Chen et al. in (2017). Then E. Abakumov et al. extended hypercyclicty to linear relations in (2018). We extend the concept of disk-cyclicity studied in M. Amouch, O. Benchiheb (2020), Z. Z. Jamil, M. Helal (2013), Y.-X. Liang, Z.-H. Zhou (2015), Z. J. Zeana (2002) for linear operators to linear relations.

Keywords: hypercyclicity; linear relation; disk-cyclic linear relation; disk transitive linear relation

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1. Introduction

Let H and K be two complex infinite dimensional separable Hilbert spaces. We denote by $\mathcal{L}(H,K)$ (or $\mathcal{B}(H,K)$) the set of all linear operators (or bounded linear operators) acting from H into K. When $K=H$, we write $\mathcal{B}(H)=\mathcal{B}(H,H)$ and $\mathcal{L}(H)=\mathcal{L}(H,H)$. One of the most significative notions of linear dynamical properties is the hypercyclicity. An operator $T\in\mathcal{B}(H)$ is said to be *hypercyclic* if there exists a vector $x\in H$ such that the *orbit*

$$\operatorname{Orb}(T,x):=\{T^nx\colon\ n\geqslant 0\}$$

is dense in H.

If S is the unilateral backward shift on $l^2(\mathbb{N})$, then λS is hypercyclic if and only if $|\lambda|>1$, see [18]. This motives the following notion introduced in [21] and studied by [5], [6], [7], [13], [14], [15], [20], [21]. An operator $T\in\mathcal{B}(H)$ is said to be *disk-cyclic* if there exists a vector x in H such that the set

$$\mathbb{D}\operatorname{Orb}(T,x):=\{\alpha T^nx\colon\ \alpha\in\mathbb{D},\ n\geqslant 0\}$$

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is dense in H, where $\mathbb{D} := \{\alpha \in \mathbb{C} \colon |\alpha| \leqslant 1\}$. In this case, the vector x is called a *disk-cyclic vector* for T.

An equivalent concept of disk-cyclicity is the disk transitivity. A bounded operator T on H is said to be *disk transitive* [7] if for any pair (U, V) of nonempty open subsets of H there exist $\alpha \in \mathbb{D} \setminus \{0\}$ and $n \geqslant 0$ such that

$$\alpha T^n(U) \cap V \neq \emptyset.$$

A disk-cyclicity criterion that can be used to prove that an operator is disk-cyclic is one of the most important characterization of the disk-cyclicity. A bounded linear operator T satisfies the *disk-cyclicity criterion* if there exist two dense sets $\mathcal{D}_1, \mathcal{D}_2 \subset X$, an increasing sequence of positive integers $\{n_k\}$, a sequence $\{\alpha_{n_k}\}$ in $\mathbb{D} \setminus \{0\}$ and a sequence of maps $S_{n_k} \colon \mathcal{D}_2 \to H$ provided that:

(i) $\alpha_{n_k} T^{n_k} x \to 0$ for every $x \in \mathcal{D}_1$;
(ii) $\alpha_{n_k}^{-1} S_{n_k} y \to 0$ for every $y \in \mathcal{D}_2$;
(iii) $T^{n_k} S_{n_k} y \to y$ for every $y \in \mathcal{D}_2$.

Another disk-cyclic criterion which is equivalent to the above criterion was introduced in [13]. For $T \in \mathcal{B}(H)$ we say that T satisfies the *three open sets conditions for disk-cyclicity* if for any pair (U, V) of nonempty open sets in H and for any neighbourhood W of zero in H there exist $n \geqslant 0$ and $\alpha \in \mathbb{D}$ such that

$$\alpha T^n(U) \cap W \neq \emptyset \quad \text{and} \quad \alpha T^n(W) \cap V \neq \emptyset.$$

In [1] Abakumov et al. extended hypercyclicty to linear relation, and Chen et al. [10] studied some linear dynamical system notions for linear relation. Motivated by these generalizations, we extend, in this paper, the concept of disk-cyclicity and related concepts to linear relations. In Section 2, we recall some basic properties of linear relations that we will need in the sequel. Section 3 is devoted to introducing and to studying the disk-cyclicity of a linear relation. We show that this property is stable under quasi-conjugacy. We also show that if a linear relation T is disk-cyclic, then the range of $T - \lambda I$ is dense in H for every $\lambda \in \mathbb{D}$. As a consequence, the eigenvalues of the adjoint of a disk-cyclic linear relation are outside $\mathbb{D}$. In the last section, we introduce and we characterize the notion of disk transitive linear relation. Among other things, we show that a linear relation is disk transitive if and only if it is disk-cyclic.

From [1], [2], [10], [11] we recall some basic definitions and notations of linear relations. A *linear relation* or a *multivalued linear operator* T on H is a mapping from a subspace

$$\mathcal{D}(T) := \{x \in X\colon\ Tx \text{ is a nonempty subset of } H\}$$

called the domain of T into $2^H \setminus \emptyset$, the set of all non empty subsets of H, provided that

$$T(\lambda x + \mu y) = \lambda T(x) + \mu T(y)$$

for all $x, y \in \mathcal{D}(T)$ and all nonzero scalars λ and μ. We denoted by $\mathcal{LR}(H)$ the set of all linear relations on H. Let $T \in \mathcal{LR}(H)$. Then for $x \in \mathcal{D}(T)$, $y \in Tx$ if and only if $Tx = y + T(0)$. Notice that $T(0) = \{0\}$ if and only if T maps the points of its domain to singletons; in this case T is said to be a *single valued operator.*

A linear relation T on H is uniquely determined by its graph $G(T)$, which is defined by

$$G(T) := \{(x, y) \in H \times H\colon\ x \in \mathcal{D}(T) \text{ and } y \in T(x)\}.$$

The inverse of T is the linear relation T^{-1} defined by

$$G(T^{-1}) := \{(y, x) \in H \times H\colon\ (x, y) \in G(T)\}.$$

For T and $S \in \mathcal{LR}(H)$, the linear relations $T + S$ and TS are defined respectively by

$$G(T + S) := \{(x, y + z) \in H \times H\colon\ (x, y) \in G(T) \text{ and } (x, z) \in G(S)\}$$

and

$$G(TS) := \{(x, y) \in H \times H\colon\ \exists\, z \in H \text{ such that } (x, z) \in G(S) \text{ and } (z, y) \in G(T)\}.$$

For $T \in \mathcal{LR}(H)$, the image of a subset M of H by T and the inverse image of a subset N of H by T^{-1} are defined respectively by

$$T(M) := \bigcup_{x \in \mathcal{D}(T) \cap M} Tx \quad \text{and} \quad T^{-1}(N) := \{x \in D(T)\colon\ Tx \cap Y \neq \emptyset\}.$$

The subspace $\ker(T) := T^{-1}(0)$ is called the kernel of T and $R(T) := T(\mathcal{D}(T))$ is the range of T.

Lemma 2.1 ([2], Lemma 2.5)**.** Let A, B and $C \in \mathcal{LR}(H)$. Then:

(i) $(A + B)C \subset AC + BC$. If $C(0) \subset \ker(A) \cup \ker(B)$, then

$$(A + B)C = AC + BC.$$

(ii) If A is everywhere defined, then $A(B + C) = AB + AC$.

For a positive integer n, T^n is defined as follows: $T^0 = I$ (the identity operator in H), $T^1 = T$ and if T^{n-1} is defined, then

$$T^n x := TT^{n-1}x = \bigcup_{y \in \mathcal{D}(T) \cap T^{n-1}x} Ty,$$

where $\mathcal{D}(T^n) := \{x \in \mathcal{D}(T^{n-1})\colon\ \mathcal{D}(T) \cap T^{n-1}x \neq \emptyset\}$.

For $y \in D(T^{-1}) := R(T)$, the inverse image of y by T is defined by

$$T^{-1}y := \{x \in D(T)\colon\ y \in Tx\}.$$

By induction, we can show that $(T^n)^{-1} = (T^{-1})^n$ for all $n \in \mathbb{N}$.

We say that $T \in \mathcal{LR}(H)$ is *continuous* if for each neighbourhood V in $R(T)$, $T^{-1}(V)$ is a neighbourhood in $\mathcal{D}(T)$. If $\mathcal{D}(T) = H$ and T is continuous, then in this case, T is said to be *bounded*. T is *closed* if its graph $G(T)$ is closed. The set of all closed and bounded linear relations will be denoted by $\mathcal{BCR}(H)$. Notice that if T is closed, then $T(0)$ is closed. We say that $T \in \mathcal{BCR}(H)$ satisfies the *stabilization property* [8] if $T(0) = T^2(0)$.

The adjoint T^* of $T \in \mathcal{LR}(H)$ is defined by

$$G(T^*) := \{(y, y') \in H \times H\colon\ \langle x', y\rangle = \langle y', x\rangle\ \ \forall\,(x, x') \in G(T)\}$$

and we have (see [11], [19])

$$\ker(T^*) = R(T)^\perp \quad \text{and} \quad T^*(0) = \mathcal{D}(T)^\perp.$$

If $\overline{\mathcal{D}(T)} = H$, then T^* is a single valued operator.

A linear operator S is called a *selection* of T if $\mathcal{D}(S) = \mathcal{D}(T)$ and

$$Tx = Sx + T(0) \quad \forall\, x \in \mathcal{D}(T).$$

Moreover, if S is continuous, then T is continuous.

Linear relations are studied by numerous mathematicians, see for instance [2], [3], [4], [8], [9], [11], [16], [17], [19] and the reference therein. In the sequel, all linear relations are nonzero and satisfy $\overline{\bigcup_{n \geqslant 1} T^n(0)} \neq H$.

3. Disk-cyclic linear relations

In the same direction as in [1], [10], we introduce the notion of disk-cyclicity for linear relations.

Definition 3.1. Let $T \in \mathcal{BCR}(H)$. We say that T is a *disk-cyclic linear relation* if there exist a nonzero vector $x \in H$ such that

$$\mathbb{D}\operatorname{Orb}(T,x) := \bigcup_{n\geqslant 0}\bigcup_{\alpha\in\mathbb{D}} \alpha T^n x$$

is dense in H. In this case, the vector x is called a *disk-cyclic vector* of T and $\mathbb{D}\operatorname{Orb}(T,x)$ is the *disk-orbit* of T at x.

The set of all disk-cyclic linear relations on a separable Hilbert space H and the set of all disk-cyclic vectors for T are respectively denoted by $\mathbb{D}\mathcal{CR}(H)$ and $\mathbb{D}\mathcal{CR}(T)$, with $\mathbb{D}\mathcal{CR}(T) = \emptyset$ if $T \notin \mathbb{D}\mathcal{CR}(H)$.

Following [1], a relation $T \in \mathcal{BCR}(H)$ is *hypercyclic* if there exists a sequence $\{x_m, m \in \mathbb{N}\}$ provided that:

(i) $\{x_m, m \in \mathbb{N}\}$ is dense in H,

(ii) for each m, $\bigcup\limits_{n\in\mathbb{N}} T^n x_m$ is dense in H.

Remark 3.1. Let $T \in \mathcal{BCR}(H)$ be a bounded linear relation such that $\overline{T^n(0)} \neq H$ for each $n \geqslant 1$ and assume that T satisfies the stabilization property. If T is a hypercyclic linear relation, then T is a disk-cyclic linear relation. Indeed, suppose that T is a hypercyclic linear relation, then by [1], Corollary 2.1 there exists a vector x in H such that $\bigcup\limits_{n\in\mathbb{N}} T^n x$ is dense in H. We then have

$$H = \overline{\bigcup_{n\in\mathbb{N}} T^n x} \subset \overline{\bigcup_{\alpha\in\mathbb{D}}\bigcup_{n\geqslant 0} \alpha T^n x} = \overline{\mathbb{D}\operatorname{Orb}(T,x)} \subset H.$$

Therefore, T is a disk-cyclic linear relation.

In general, T being a disk-cyclic linear relation does not imply that T is a hypercyclic linear relation, see for instance [7], Example 2.20.

In the following example, we show that every linear relation which has a disk-cyclic selection is a disk-cyclic linear relation.

Example 3.1. Let $A \in \mathcal{B}(X)$ be a selection of a linear relation $T \in \mathcal{BCR}(H)$. If A is disk-cyclic, then T is a disk-cyclic linear relation. Indeed, if A is a selection of a linear relation $T \in \mathcal{BCR}(H)$, then $Tx = Ax + T(0)$ for all $x \in H$. By Lemma 2.1, we have

$$T^2x = T(Tx) = T(Ax + T(0)) = TAx + T^2(0) = A^2x + TA(0) + T^2(0)$$
$$= A^2x + T(0) + T^2(0) = A^2x + T^2(0).$$

By induction, we can prove that

$$T^n x = A^n x + T^n(0) \quad \forall\, n \in \mathbb{N} \cup \{0\}.$$

Since A is a disk-cyclic linear operator, then

$$H = \overline{\{\alpha A^n x\colon\ n \geqslant 0, \alpha \in \mathbb{D}\}} \subset \overline{\mathbb{D}\,\mathrm{Orb}(T, x)} \subset H.$$

Consequently, we obtain T is a disk-cyclic linear relation.

In the following example, we show that every noninjective disk-cyclic linear operator is a selection of a disk-cyclic linear relation.

Example 3.2. Let $S \in \mathcal{B}(H)$ be a disk-cyclic linear operator such that $\ker(S) \neq \{0\}$, we consider the bounded linear relation defined by

$$T\colon\ H \to 2^H \setminus \emptyset,$$
$$x \mapsto S^{-1}S^2(x).$$

Then S is a selection of T. Indeed, we have

$$Tx = S^{-1}S^2(x) = S^{-1}S(Sx) = Sx + \ker(S) = Sx + T(0)$$

for all $x \in \mathcal{D}(T) = H$, which means that S is a selection of T. Since S is disk-cyclic linear operator, then by Example 3.1, we deduce that T is a disk-cyclic linear relation.

Example 3.3. Let S be the bounded linear operator acting on $l_2(\mathbb{N})$ as follows:

$$S\colon\ l_2(\mathbb{N}) \to l_2(\mathbb{N}),$$
$$x = (x_1, x_2, \ldots) \mapsto 2(x_2, x_3, \ldots).$$

Then S is a disk-cyclic linear operator by Example 3.3 in [7]. Let T be the bounded linear relation defined by

$$T\colon\ l_2(\mathbb{N}) \to 2^{l_2(\mathbb{N})} \setminus \emptyset,$$
$$x \mapsto Sx + S^{-1}(0).$$

Then T is a disk-cyclic linear relation since S is a selection of T.

Proposition 3.1. *Let $T \in \mathcal{BCR}(H)$, $S \in \mathcal{BCR}(K)$ and $G \in \mathcal{B}(H,K)$ such that $SG = GT$ and $R(G)$ is dense in K. Then*

$$G(\mathbb{D}\mathcal{CR}(T)) \subset \mathbb{D}\mathcal{CR}(S).$$

In particular, if T is disk-cyclic, then S is disk-cyclic.

Proof. If T is not disk-cyclic, then $\mathbb{D}\mathcal{CR}(T) = \emptyset$ and hence, $G(\mathbb{D}\mathcal{CR}(T)) = \emptyset \subset \mathbb{D}\mathcal{CR}(S)$. Now suppose T is disk-cyclic. Let $x \in \mathbb{D}\mathcal{CR}(T)$, then $\mathbb{D}\operatorname{Orb}(T,x)$ is dense in H. We thus get

$$\begin{aligned}
\overline{\mathbb{D}\operatorname{Orb}(S,Gx)} &= \overline{\bigcup_{\alpha\in\mathbb{D}}\bigcup_{n\geqslant 0}\alpha S^n Gx} = \overline{\bigcup_{\alpha\in\mathbb{D}}\bigcup_{n\geqslant 0}\alpha GT^n x} \\
&= \overline{\bigcup_{\alpha\in\mathbb{D}}\bigcup_{n\geqslant 0}\alpha G(T^n x)} = \overline{\bigcup_{\alpha\in\mathbb{D}}\bigcup_{n\geqslant 0} G(\alpha T^n x)} \\
&= \overline{G\Big(\bigcup_{\alpha\in\mathbb{D}}\bigcup_{n\geqslant 0}\alpha T^n x\Big)} \supseteq G\Big(\overline{\bigcup_{\alpha\in\mathbb{D}}\bigcup_{n\geqslant 0}\alpha T^n x}\Big) \\
&= G(H) = R(G).
\end{aligned}$$

Since $R(G)$ is dense in K, then $\mathbb{D}\operatorname{Orb}(S,Gx)$ is also dense in K. Therefore Gx is an element of $\mathbb{D}\mathcal{CR}(S)$. □

Corollary 3.1. *Let $T \in \mathcal{BCR}(H)$ and $G \in \mathcal{B}(X)$. If $TG = GT$ and $R(G)$ is dense in H, then:*

(i) *$Gx \in \mathbb{D}\mathcal{CR}(T)$ for every $x \in \mathbb{D}\mathcal{CR}(T)$,*
(ii) *$\lambda\mathbb{D}\mathcal{CR}(T) = \mathbb{D}\mathcal{CR}(T)$ for all $\lambda \in \mathbb{C}\setminus\{0\}$.*

Lemma 3.1 ([1], Lemma 2.1)**.** *Let A and B be two subsets of a Banach space X with $\operatorname{int}(\overline{A}) = \emptyset$. Then*

$$\operatorname{int}(\overline{B}) = \operatorname{int}(\overline{A}\cup\overline{B}).$$

Proposition 3.2. *Let $T \in \mathcal{BCR}(H)$. If T is disk-cyclic, then the range of T is dense in H.*

Proof. Suppose that T is a disk-cyclic linear relation. Then there exists a nonzero vector $x \in H$ such that $\mathbb{D}\operatorname{Orb}(T,x)$ is dense in H. We set

$$\mathcal{D}_x := \{\alpha x\colon \alpha\in\mathbb{D}\} \quad \text{and} \quad A = \bigcup_{n\geqslant 1}\bigcup_{\alpha\in\mathbb{D}}\alpha T^n x.$$

Let $y \in \mathbb{D}\operatorname{Orb}(T,x) \setminus \mathcal{D}_x$, then there exist $n \geqslant 1$ and $\alpha \in \mathbb{D}$ such that $y \in \alpha T^n x$. If $\alpha = 0$, then $y = 0$, which is a contradiction with $0 \in \mathcal{D}_x$. So, assume that $\alpha \neq 0$, then

$$y \in \alpha T^n x = T^n(\alpha x) \subset R(T^n) \subset R(T).$$

Therefore

$$\mathbb{D}\operatorname{Orb}(T,x) \setminus \mathcal{D}_x \subset \overline{R(T)}. \tag{3.1}$$

Since $\overline{\operatorname{span}\{x\}} = \operatorname{span}\{x\} \neq H$, then $\operatorname{int}(\operatorname{span}\{x\}) = \emptyset$. Furthermore, as $\mathcal{D}_x$ is a subset of $\operatorname{span}\{x\}$, we obtain $\operatorname{int}(\overline{\mathcal{D}_x}) = \emptyset$. Using Lemma 3.1, we get

$$\operatorname{int}(\overline{\mathcal{D}_x} \cup \overline{A}) = \operatorname{int}(\overline{A}).$$

On the other hand, we have

$$H = \operatorname{int}(H) = \operatorname{int}(\overline{\mathbb{D}\operatorname{Orb}(T,x)}) = \operatorname{int}(\overline{A \cup \mathcal{D}_x}) = \operatorname{int}(\overline{A} \cup \overline{\mathcal{D}_x}) = \operatorname{int}(\overline{A}) \subset \overline{A} \subset H,$$

which implies that A is dense in H.

Now, we show that $\mathcal{D}_x \subset \overline{R(T)}$. Let $\alpha \in \mathbb{D} \setminus \{0\}\}$, then

$$\alpha x \in H = \overline{\mathbb{D}\operatorname{Orb}(T,x) \setminus \mathcal{D}_x}.$$

Hence, there exists a sequence $\{y_i\}$ in $\mathbb{D}\operatorname{Orb}(T,x) \setminus \{\alpha x \colon \alpha \in \mathbb{D}\}$ such that $\{y_i\}$ converges to αx, as $i \to \infty$. So, for all $i \geqslant 1$ there exist $n_i \geqslant 1$ and $\alpha_i \in \mathbb{D} \setminus \{0\}$ such that

$$y_i \in \alpha_i T^{n_i} x \subset R(T) \quad \text{and} \quad y_i \to \alpha x.$$

Then

$$\mathcal{D}_x \subset \overline{R(T)}. \tag{3.2}$$

Combining (3.1) and (3.2), we conclude that

$$\mathbb{D}\operatorname{Orb}(T,x) \subset \overline{R(T)} \subset H.$$

As $\mathbb{D}\operatorname{Orb}(T,x)$ is dense in H, then the range of T is dense in H. □

Remark 3.2. In general, the converse of Proposition 3.2 is not true. Indeed, let $A \in \mathcal{B}(l_2(\mathbb{N}))$ be the bounded operator defined by

$$A(x_1, x_2, \ldots) = \frac{1}{2}(x_2, x_3, \ldots).$$

Then the range of A is dense in $l_2(\mathbb{N})$ and by Example 2.22 in [12], A is not hypercyclic. Furthermore, according to [7], Corollary 3.6, A is not disk-cyclic.

The following result is [11], Exercise II.3.21, but for the convenience of the reader we give here a proof.

Lemma 3.2. *Let $T \in \mathcal{BCR}(H)$ and let M be a nonempty subset of H. Then*

$$T(\overline{M}) \subset \overline{T(M)}.$$

P r o o f. Since T is continuous and closed, then according to [11], Corollary II 4.6, T has a continuous selection A and $T(0)$ is closed. As A is continuous, then $A(\overline{M}) \subset \overline{A(M)}$. Therefore,

$$T(\overline{M}) = A(\overline{M}) + T(0) \subset \overline{A(M)} + T(0) \subset \overline{A(M) + T(0)} = \overline{T(M)}.$$

□

Proposition 3.3. *Let $T \in \mathbb{D}\mathcal{CR}(H)$ and $S \in \mathcal{BCR}(H)$ be such that $TS = ST$, $T(0) = TS(0)$ and the range of S is dense in H. Then*

$$Sx \subset \mathbb{D}\mathcal{CR}(T)$$

for all $x \in \mathbb{D}\mathcal{CR}(T)$.

P r o o f. Let x be a disk-cyclic vector for T. Then the set $\mathbb{D}\operatorname{Orb}(T,x)$ is dense in H. Now, let $y \in Sx$. Then

$$TSx = T(y + S(0)) = Ty + TS(0) = Ty + T(0) = Ty.$$

Since $TS = ST$, then

$$ST^n x = T^n Sx = T^n y$$

for all $n \geqslant 1$. Since $x \in \mathbb{D}\mathcal{CR}(T)$, then $\mathbb{D}\operatorname{Orb}(T,x) \setminus \mathcal{D}_x$ is also dense in H (see the proof of Proposition 3.2). By Lemma 3.2, we have

$$\begin{aligned} R(S) = S(H) &= S(\overline{\mathbb{D}\operatorname{Orb}(T,x) \setminus \mathcal{D}_x}) \subset \overline{S(\mathbb{D}\operatorname{Orb}(T,x) \setminus \mathcal{D}_x)} \\ &= \overline{S\Big(\bigcup_{\alpha\in\mathbb{D}} \bigcup_{n\geqslant 1} \alpha T^n x\Big)} = \overline{\bigcup_{\alpha\in\mathbb{D}} \bigcup_{n\geqslant 1} S(\alpha T^n x)} = \overline{\bigcup_{\alpha\in\mathbb{D}} \bigcup_{n\geqslant 1} \alpha S T^n x} \\ &= \overline{\bigcup_{\alpha\in\mathbb{D}} \bigcup_{n\geqslant 1} \alpha T^n Sx} = \overline{\bigcup_{\alpha\in\mathbb{D}} \bigcup_{n\geqslant 1} \alpha T^n y} \subset \overline{\mathbb{D}\operatorname{Orb}(T,y)} \subset H. \end{aligned}$$

Since the range of S is dense in H, we get that $\mathbb{D}\operatorname{Orb}(T,y)$ is dense in H. Therefore, y is a disk-cyclic vector for T and so Sx is a subset of $\mathbb{D}\mathcal{CR}(T)$. □

Theorem 3.1. *Let $T \in \mathcal{BCR}(H)$ satisfy the criterion of stabilization. Then T is a disk-cyclic linear relation if and only if T^p is a disk-cyclic linear relation for all $p \in \mathbb{N}$.*

P r o o f. Suppose that T is a disk-cyclic linear relation. Then by Proposition 3.2, the range of T is dense in H. Since $T(0) = T^2(0)$, then by virtue of Proposition 3.3,

$$T(\mathbb{D}\mathcal{CR}(T)) \subset \mathbb{D}\mathcal{CR}(T). \tag{3.3}$$

Hence, by induction we have

$$T^n(\mathbb{D}\mathcal{CR}(T)) \subset \mathbb{D}\mathcal{CR}(T) \quad \forall\, n \geqslant 1.$$

Now, we show that T^2 is a disk-cyclic linear relation. By assumption there exists $x \in H$ such that $\mathbb{D}\operatorname{Orb}(T, x)$ is dense in H. Let $y \in T^n x \subset \mathbb{D}\mathcal{CR}(T)$. Using the fact that $T(0) = T^2(0)$ and Lemma 2.1, we get

$$T^{2n}x = T^nT^nx = T^n(y + T^n(0)) = T^ny + T^{2n}(0) = T^ny + T^n(0) = T^ny$$

for all $n \geqslant 1$. Consequently,

$$\mathbb{D}\operatorname{Orb}(T^2, x) \setminus \mathcal{D}_x = \mathbb{D}\operatorname{Orb}(T, y) \setminus \mathcal{D}_y.$$

Since y is a disk-cyclic vector for T, it follows from the proof of Proposition 3.2 that $\mathbb{D}\operatorname{Orb}(T, y) \setminus \mathcal{D}_y$ is also dense in H. Therefore $\mathbb{D}\operatorname{Orb}(T^2, x)$ is dense in H, which implies that T^2 is a disk-cyclic linear relation. By induction, we show that for all $p \geqslant 1$, T^p is a disk-cyclic linear relation. $\square$

Let $T \in \mathcal{LR}(H)$ and M be a subspace of H. Then the restriction of T to M denoted by T_M is the linear relation defined by

$$G(T_M) := G(T) \cap (M \times H).$$

Lemma 3.3. *Let $T \in \mathcal{LR}(H)$ and M be a nontrivial closed subspace of H such that $T(M) \subset M$ and $T(M^\perp) \subset M^\perp$. If P is the orthogonal projection onto $M^\perp$,*

$$(TP)^n = T^nP = PT^n$$

for all $n \geqslant 1$.

P r o o f. Since H is a Hilbert space and M is a closed subspace of H, then $H = M \oplus M^{\perp}$. Let $x \in H$, then there exist $a \in M$ and $b \in M^{\perp}$ such that $x = a + b$. Since $T(M) \subset M$ and $T(M^{\perp}) \subset M^{\perp}$,

$$PTx = P(Ta + Tb) = PTb = Tb = TPx.$$

Hence $TP = PT$. By induction, we obtain $(TP)^n = T^n P = PT^n$, for all $n \geqslant 1$. □

Proposition 3.4. *Let $T \in \mathbb{D}\mathcal{CR}(H)$. Let M be a nontrivial closed subspace of H such that $T(M) \subset M$ and let P be the orthogonal projection onto $M^{\perp}$. Then*

$$Px \neq 0$$

for all $x \in \mathbb{D}\mathcal{CR}(T)$.

P r o o f. Let $x \in \mathbb{D}\mathcal{CR}(T) \subset H$. For the sake of contradiction assume that $Px = 0$. So, $x \in M$. As $T(M) \subset M$, then $\alpha T^n x \subset \alpha T^n M \subset \alpha M = M$ for all $\alpha \in \mathbb{D} \setminus \{0\}$ and all $n \geqslant 0$. This implies that

$$H = \overline{\bigcup_{\alpha \in \mathbb{D}} \bigcup_{n \geqslant 0} \alpha T^n x} \subset \overline{M} = M,$$

which is a contradiction. Therefore $Px \neq 0$. □

Proposition 3.5. *Let $T \in \mathbb{D}\mathcal{CR}(H)$ and M be a nontrivial subspace of H such that $T(M) \subset M$ and $T(M^{\perp}) \subset M^{\perp}$. Then T_M and $T_{M^{\perp}}$ are disk-cyclic linear relations.*

P r o o f. Let P be the bounded projection onto $M^{\perp}$. Since T is a disk-cyclic linear relation, there exists $x \in H$ such that the set $\mathbb{D}\operatorname{Orb}(T, x)$ is dense in H. It follows from the proof of Proposition 3.2 that $\mathbb{D}\operatorname{Orb}(T, x) \setminus \mathcal{D}_x$ is also dense in H. As $H = M \oplus M^{\perp}$, there exist $x_1 \in M$ and $x_2 \in M^{\perp}$ such that $x = x_1 + x_2$. Hence $Px = x_2$. By Lemma 3.3, we have $(TP)^n = T^n P = PT^n$ for all $n \geqslant 1$. Therefore we obtain

$$\begin{aligned} M^{\perp} = P(H) &= P(\overline{\mathbb{D}\operatorname{Orb}(T, x) \setminus \mathcal{D}_x}) \subset \overline{P(\mathbb{D}\operatorname{Orb}(T, x) \setminus \mathcal{D}_x)} = \overline{P\Big(\bigcup_{\alpha \in \mathbb{D}} \bigcup_{n \geqslant 1} \alpha T^n x\Big)} \\ &= \overline{\bigcup_{\alpha \in \mathbb{D}} \bigcup_{n \geqslant 1} \alpha P T^n x} = \overline{\bigcup_{\alpha \in \mathbb{D}} \bigcup_{n \geqslant 1} \alpha T^n P x} = \overline{\bigcup_{\alpha \in \mathbb{D}} \bigcup_{n \geqslant 1} \alpha (TP)^n x_2} \\ &= \overline{\bigcup_{\alpha \in \mathbb{D}} \bigcup_{n \geqslant 1} \alpha T^n_{M^{\perp}} x_2} \subset \overline{\mathbb{D}\operatorname{Orb}(T_{M^{\perp}}, x_2)} \subset \overline{M^{\perp}} = M^{\perp}. \end{aligned}$$

Finally, we conclude that $T_{M^{\perp}}$ is a disk-cyclic linear relation. With the same argument we show that T_M is also a disk-cyclic linear relation. □

Let $\{H_i\}_{i=1}^n$ be a family of separable Hilbert spaces and $T_i \in \mathcal{BCR}(H_i)$ for all $i \in \{1, \ldots, n\}$. We define (see [10])

$$\bigoplus_{i=1}^{n} H_i := \{(x_1, \ldots, x_n)\colon\ x_i \in H_i,\ 1 \leqslant i \leqslant n\}$$

and

$$\bigoplus_{i=1}^{n} T_i x := \{(y_1, \ldots, y_n)\colon\ y_i \in T_i x_i,\ 1 \leqslant i \leqslant n\},\ \text{where } x = (x_1, \ldots, x_n).$$

Let $k \in \mathbb{N}$, then

$$\Big(\bigoplus_{i=1}^{n} T_i\Big)^k x = \bigoplus_{i=1}^{n} T_i^k x.$$

Proposition 3.6. *Let $T_i \in \mathcal{BCR}(H_i)$ for all $i \in \{1, \ldots, m\}$. If $\bigoplus_{i=1}^{m} T_i$ is a disk-cyclic linear relation, then T_i is a disk-cyclic linear relation for each $i \in \{1, \ldots, m\}$.*

Proof. Let $y = (y_1, \ldots, y_m) \in \bigoplus_{i=1}^{m} H_i$. Since $\bigoplus_{i=1}^{m} T_i$ is a disk-cyclic linear relation, then there exists $x = (x_1, \ldots, x_m) \in \mathbb{D}\mathcal{CR}\Big(\bigoplus_{i=1}^{m} T_i\Big)$ such that $\mathbb{D}\,\mathrm{Orb}\Big(\bigoplus_{i=1}^{m} T_i, x\Big)$ is dense in $\bigoplus_{i=1}^{m} H_i$. Therefore there exists $\{y_k\}$ in $\mathbb{D}\,\mathrm{Orb}\Big(\bigoplus_{i=1}^{m} T_i, x\Big)$ such that $\{y_k\}$ converges to y as $k \to \infty$. Then for all $k \in \mathbb{N}$ there exists $\{\alpha_k\}$ in $\mathbb{D}$ and $\{n_k\}$ in $\mathbb{N}$ such that

$$y_k \to y \text{ with } y_k \in \alpha_k \Big(\bigoplus_{i=1}^{m} T_i\Big)^{n_k} x.$$

Let P_i be the bounded projection defined on $\bigoplus_{i=1}^{m} H_i$ such that $R(P_i) = H_i$. Then

$$P_i(y_k) \in \alpha_k T_i^{n_k} x_i \quad \text{and} \quad P_i(y_k) \to y_i.$$

Therefore $x_i \in \mathbb{D}\mathcal{CR}(T_i)$ for each $i \in \{1, \ldots, m\}$. $\square$

4. Disk transitive linear relation

Here we define and study the concept of disk transitive linear relation.

Definition 4.1. Let $T \in \mathcal{BCR}(H)$. We say that T is *disk transitive* if for any pair (U, V) of nonempty open subsets of H there exist $\alpha \in \mathbb{D} \setminus \{0\}$ and $n \geqslant 0$ such that $\alpha T^n(U) \cap V \neq 0$.

Let $S \in \mathcal{B}(H)$ be a disk transitive linear operator. Let U and V be two open nonempty sets of H, then there exist $n \geqslant 0$ and $\alpha \in \mathbb{D} \setminus \{0\}$ such that

$$\alpha S^n(U) \cap V \neq \emptyset.$$

Let $y \in \alpha S^n(U) \cap V$. Hence, there exists $x \in U$ such that $y = \alpha S^n x$.

If S is a selection of a linear relation $T \in \mathcal{BCR}(H)$, then by virtue of Example 3.1, we have $T^n x = S^n x + T^n(0)$ and hence, $y = \alpha S^n x \in \alpha T^n x \subset \alpha T^n U$. Consequently, $\alpha T^n(U) \cap V \neq \emptyset$ and therefore T is a disk transitive linear relation.

Proposition 4.1. *Let $T \in \mathcal{BCR}(H)$, $S \in \mathcal{BCR}(K)$ and $A \in \mathcal{B}(H, K)$ be such that $SA = AT$ and the range of A is dense in K. If T is a disk transitive linear relation, then S is a disk transitive linear relation.*

P r o o f. Let U and V be two nonempty open subsets of K. Since A is bounded and with dense range, then $A^{-1}(U)$ and $A^{-1}(V)$ are two nonempty open subsets of H. As T is a disk transitive linear relation, there exist $n \geqslant 0$ and $\alpha \in \mathbb{D} \setminus \{0\}$ such that

$$\alpha T^n A^{-1}(U) \cap A^{-1}(V) \neq \emptyset.$$

Let $y \in A^{-1}(V)$ and $x \in A^{-1}(U)$ such that $y \in \alpha T^n x$. Since $SA = AT$, we obtain

$$\alpha S^n Ax = \alpha AT^n x = A(\alpha T^n x) = A(y + \alpha T^n(0)) = Ay + \alpha AT^n(0) = Ay + \alpha S^n A(0).$$

So, $Ay \in \alpha S^n Ax \subset \alpha S^n(U)$ and $Ay \in V$. Thus,

$$\alpha S^n(U) \cap V \neq \emptyset.$$

Finally, S is a disk transitive linear relation. □

The following theorem gives a characterization of a disk transitive linear relation.

Theorem 4.1. *Let $T \in \mathcal{BCR}(H)$. Then the following assertions are equivalent:*

(i) *T is disk transitive.*

(ii) *For each pair* (U, V) *of nonempty open subsets of* H *there exist* $|\alpha| \geqslant 1$ *and* $n \geqslant 0$ *such that*

$$\alpha T^{-n}(U) \cap V \neq \emptyset.$$

(iii) *For any nonempty open subset* U *of* H,

$$\bigcup_{\alpha \in \mathbb{D} \setminus \{0\}} \bigcup_{n \geqslant 0} \alpha T^n(U)$$

is dense in H.

(iv) *For any nonempty open subset* V *of* H,

$$\bigcup_{\alpha \in \mathbb{C}, |\alpha| \geqslant 1} \bigcup_{n \geqslant 0} \alpha T^{-n}(V)$$

is dense in H.

Proof. (i) $\Longrightarrow$ (ii). Let (U, V) be a pair of nonempty open subsets of H. Since T is disk transitive, then there exist $\alpha \in \mathbb{D} \setminus \{0\}$ and $n \geqslant 0$ such that $\alpha T^n(U) \cap V \neq \emptyset$. Hence

$$(\alpha U + T^{-n}(0)) \cap T^{-n}(V) \neq \emptyset.$$

Let $x \in (\alpha U + T^{-n}(0)) \cap T^{-n}(V)$. Then there exist $u \in U$, $y \in T^{-n}(0)$ and $v \in V$ such that $x = \alpha u + y$ and $x \in T^{-n}(v)$. Hence

$$T^{-n}(v) = x + T^{-n}(0) = \alpha u + y + T^{-n}(0) = \alpha u + T^{-n}(0)$$

which means that $\alpha u \in T^{-n}(v)$. We thus get $u \in \beta T^{-n}(V) \cap U$ with $|\beta| = 1/|\alpha| \geqslant 1$. Therefore $\beta T^{-n}(V) \cap U \neq \emptyset$.

(ii) $\Longrightarrow$ (i). It is similar to (i) $\Longrightarrow$ (ii).

(i) $\Longleftrightarrow$ (iii). Assume that T is a disk transitive linear relation. Let U be a nonempty open subset of H and let $(O_i)_{i \geqslant 1}$ be a countable basis of open sets of H. For each $i \geqslant 1$ we can find $n_i \geqslant 0$ and $\alpha_i \in \mathbb{D} \setminus \{0\}$ such that $\alpha_i T^{n_i}(U) \cap O_i \neq \emptyset$. We then obtain that

$$\bigcup_{\alpha \in \mathbb{D} \setminus \{0\}} \bigcup_{n \geqslant 0} \alpha T^n(U)$$

is dense in H.

Conversely, let U and V be two open nonempty subsets of H. Since the set

$$\bigcup_{\alpha \in \mathbb{D} \setminus \{0\}} \bigcup_{n \geqslant 0} \alpha T^n(U)$$

is dense in H, then there exist $\alpha \in \mathbb{D} \setminus \{0\}$ and $n \geqslant 0$ such that

$$\alpha T^n(U) \cap V \neq 0,$$

which means that T is a disk transitive linear relation.

(ii) $\iff$ (iv). It is similar to (i) $\iff$ (iii). $\square$

In the sequel, we denote by $B(x, r)$ the open ball centered at $x \in H$ and with radius $r > 0$.

Theorem 4.2. *Let $T \in \mathcal{BCR}(H)$. Then the following assertions are equivalent:*

(1) *T is a disk transitive linear relation.*

(2) *For each $(x, y) \in H^2$ there exist sequences of positive integers $\{n_k\}$, $\{x_k\}$ in H, $\{\alpha_k\}$ in $\mathbb{D} \setminus \{0\}$ and $\{y_k\}$ in H such that*

$$x_k \to x, \quad y_k \to y \quad \text{and} \quad \alpha_k T^{n_k} x_k = y_k + T^{n_k}(0).$$

(3) *For each $(x, y) \in H^2$ and for each neighbourhood W of 0 there exist $z, t \in H$, $\alpha \in \mathbb{D} \setminus \{0\}$ and $n \in \mathbb{N}$ such that*

$$x - z \in W, \quad t - y \in W \quad \text{and} \quad \alpha T^n z = t + T^n(0).$$

Proof. (1) $\Longrightarrow$ (2): Suppose that T is disk transitive. Let $x, y \in H$ and let $B_k := B(x, 1/k)$ and $B'_k := B(y, 1/k)$ for all $k \geqslant 1$. Then B_k and B'_k are nonempty open subsets of H. As T is a disk transitive linear relation, then there exist two sequences $\{\alpha_k\} \subset \mathbb{D} \setminus \{0\}$ and $\{n_k\}$ in $\mathbb{N}$ such that $T^{n_k}(\alpha_k B_k) \cap B'_k \neq \emptyset$ for all $k \geqslant 1$. Hence, there exists a sequence $\{y_k\}$ in H such that

$$y_k \in T^{n_k}(\alpha_k B_k) \cap B'_k$$

for all $k \geqslant 1$. Consequently, for each $k \geqslant 1$ there exists $x_k \in B_k$ such that

$$y_k \in T^{n_k}(\alpha_k x_k) \cap B'_k.$$

Therefore we have

$$\alpha_k T^{n_k} x_k = y_k + \alpha_k T^{n_k}(0) = y_k + T^{n_k}(0).$$

Moreover,

$$x_k \to x \quad \text{and} \quad y_k \to y.$$

(2) $\Longrightarrow$ (3): Assume that for each $(x, y) \in H^2$ there exist sequences $\{n_k\}$ in $\mathbb{N}$, $\{x_k\}$ in H, $\{\alpha_k\}$ in $\mathbb{D} \setminus \{0\}$ and $\{y_k\}$ in H provided that

$$x_k - x \to 0, \quad y_k - y \to 0 \quad \text{and} \quad \alpha_k T^{n_k} x_k = y_k + T^{n_k}(0).$$

Let W be a neighbourhood of zero. Then there exists some $k_0 \geqslant 1$ such that $x - x_{k_0} \in W$ and $y_{k_0} - y \in W$. Set $z := x_{k_0}$ and $t := y_{k_0}$. We thus have

$$x - z \in W, \quad t - y \in W \quad \text{and} \quad \alpha_{k_0} T^{n_{k_0}} z = t + T^{n_{k_0}}(0).$$

(3) $\Longrightarrow$ (1): Let U and V be two nonempty open subsets of H. Let $(x, y) \in U \times V$. For each $k \geqslant 1$, $W_k := B(0, 1/k)$ is a neighbourhood of zero. By assumption there exist sequences $\{x_k\}$ in H, $\{\alpha_k\}$ in $\mathbb{D} \setminus \{0\}$, $\{n_k\}$ in $\mathbb{N}$ and $\{y_k\} \subset H$ such that

$$\|x_k - x\| < \frac{1}{k}, \quad \|y_k - y\| < \frac{1}{k} \quad \text{and} \quad y_k \in \alpha_k T^{n_k} x_k.$$

Then $\{x_k\}$ converges to x and $\{y_k\}$ converges to y as $k \to \infty$. Therefore for k large enough we have $x_k \in U$ and $y_k \in V$. Thus

$$\emptyset \neq \alpha_k T^{n_k} x_k \cap V \subset \alpha_k T^{n_k} U \cap V$$

and we conclude that T is a disk transitive linear relation. □

Lemma 4.1. *Let $T \in \mathcal{BCR}(H)$. If $x \in \mathbb{D}\mathcal{CR}(T)$, then for any nonempty open set U of H there exist $n \geqslant 0$ and $\gamma \in \mathbb{D} \setminus \{0\}$ such that*

$$\gamma T^n x \cap U \neq \emptyset.$$

P r o o f. Since x is a disk-cyclic vector for T, then the set

$$\mathbb{D}\,\mathrm{Orb}(T, x) = \bigcup_{n \geqslant 0} \bigcup_{\alpha \in \mathbb{D}} \alpha T^n x$$

is dense in H. Let U be a nonempty open subset of H. Then

$$\Big(\bigcup_{n \geqslant 0} \bigcup_{\alpha \in \mathbb{D}} \alpha T^n x\Big) \cap U \neq \emptyset.$$

Now, we distinguish two cases:

(i) If $0 \notin U$, then there exist $n \geqslant 0$ and $\alpha \in \mathbb{D} \setminus \{0\}$ such that

$$\alpha T^n x \cap U \neq \emptyset.$$

(ii) If $0 \in U$, then we can find an open set V of H such that

$$0 \notin V \quad \text{and} \quad V \subset U.$$

Using the above argument, we deduce that there exist $m \geqslant 0$ and $\beta \in \mathbb{D} \setminus \{0\}$ such that

$$\beta T^m x \cap V \neq \emptyset$$

and so

$$\beta T^m x \cap U \neq \emptyset.$$

Finally, in both cases there exist $n \geqslant 0$ and $\gamma \in \mathbb{D} \setminus \{0\}$ such that $\gamma T^n x \cap U \neq \emptyset$. $\square$

Proposition 4.2. *Let $T \in \mathcal{BCR}(H)$. Then T is a disk transitive linear relation if and only if*

$$\mathbb{D}\mathcal{CR}(T) = \bigcap_{k\in\mathbb{N}} \bigcup_{n\geqslant 0} \bigcup_{\alpha\in\mathbb{C}, |\alpha|\geqslant 1} \alpha T^{-n}(V_k)$$

is a dense G_δ-set in H, where $(V_k)_{k\in\mathbb{N}}$ is a countable basis of open subsets of H.

Proof. Let T be disk-cyclic. Let $(V_k)_{k\in\mathbb{N}}$ be a countable basis of open subsets of H. From Lemma 4.1 we have

$$\begin{aligned} x \in \mathbb{D}\mathcal{CR}(T) &\iff \forall\, k \geqslant 1,\ V_k \cap \Big(\bigcup_{n\geqslant 0} \bigcup_{\alpha\in\mathbb{D}} \alpha T^n x\Big) \neq \emptyset \\ &\iff \forall\, k \geqslant 1, \quad \exists\, \beta \in \mathbb{D} \setminus \{0\}, \quad \exists\, n \geqslant 0 \quad \text{such that} \quad V_k \cap \beta T^n x \neq \emptyset \\ &\iff \forall\, k \geqslant 1, \quad \exists\, \beta \in \mathbb{D} \setminus \{0\}, \quad \exists\, n \geqslant 0 \quad \text{such that} \quad \beta x \in T^{-n}(V_k) \\ &\iff \forall\, k \geqslant 1, \quad \exists\, \alpha \in \mathbb{C}, |\alpha| \geqslant 1, \quad \exists\, n \geqslant 0 \quad \text{such that} \quad x \in \alpha T^{-n}(V_k) \\ &\iff x \in \bigcap_{k\in\mathbb{N}} \bigcup_{n\geqslant 0} \bigcup_{\alpha\in\mathbb{C}, |\alpha|\geqslant 1} \alpha T^{-n}(V_k). \end{aligned}$$

Now, we show that $\mathbb{D}\mathcal{CR}(T)$ is dense in H. For each $k \geqslant 1$ we set

$$O_k := \bigcup_{n\geqslant 0} \bigcup_{\alpha\in\mathbb{C}, |\alpha|\geqslant 1} \alpha T^{-n}(V_k).$$

Since T is disk transitive, then by Theorem 4.1, O_k is dense in H. As O_k is an open set of H (see [1], Remark 2.2), by the Baire category theorem, we obtain $\bigcap_{k\in\mathbb{N}} O_k = \mathbb{D}\mathcal{CR}(T)$ is dense G_δ-set in H.

Conversely, let U and V be two nonempty open subsets of H. Since $(V_k)_{k\in\mathbb{N}}$ is a countable basis of open subsets of H and $\bigcap_{k\in\mathbb{N}} O_k = \mathbb{D}\mathcal{CR}(T)$ is dense in H, then for $U = \bigcup_{k\in I} V_k$ with $I \subset \mathbb{N}$, we have $\bigcap_{k\in\mathbb{N}} O_k \cap U \neq \emptyset$. Hence, $O_k \cap U \neq \emptyset$ for all $k \in \mathbb{N}$. For $k \in I$ we have

$$\begin{aligned}\emptyset \neq O_k \cap V &= \Big(\bigcup_{n\geqslant 0}\bigcup_{\alpha\in\mathbb{C},|\alpha|\geqslant 1} \alpha T^{-n}(V_k)\Big) \cap V \\ &\subset \Big(\bigcup_{n\geqslant 0}\bigcup_{\alpha\in\mathbb{C},|\alpha|\geqslant 1} \alpha T^{-n}\Big(\bigcup_{i\in I} V_i\Big)\Big) \cap V \\ &= \Big(\bigcup_{n\geqslant 0}\bigcup_{\alpha\in\mathbb{C},|\alpha|\geqslant 1} \alpha T^{-n}(U)\Big) \cap V.\end{aligned}$$

Thus, $\big(\bigcup_{n\geqslant 0}\bigcup_{\alpha\in\mathbb{C},|\alpha|\geqslant 1} \alpha T^{-n}(U)\big) \cap V \neq \emptyset$ for all nonempty open subset V of H, which means that $\bigcup_{n\geqslant 0}\bigcup_{\alpha\in\mathbb{C},|\alpha|\geqslant 1} \alpha T^{-n}(U)$ is dense in H. Finally, by virtue of Theorem 4.1, we conclude that T is a disk transitive linear relation. □

Theorem 4.3. *Let $T \in \mathcal{BCR}(H)$. Then the following assertions are equivalent:*

(i) *T is a disk transitive linear relation.*

(ii) *T is a disk-cyclic linear relation.*

Proof. Suppose that T is disk transitive, then by Proposition 4.2, $\mathbb{D}\mathcal{CR}(T)$ is dense in H. Hence, $\mathbb{D}\mathcal{CR}(T)$ is a nonempty set of H and so T is a disk-cyclic linear relation.

Conversely, assume that T is a disk-cyclic linear relation, then there exists a vector x in H such that the set $\mathbb{D}\,\mathrm{Orb}(T,x)$ is dense in H. Let (U,V) be a pair of nonempty open sets of H. Then

$$\mathbb{D}\,\mathrm{Orb}(T,x) \cap U \neq \emptyset \quad \text{and} \quad \mathbb{D}\,\mathrm{Orb}(T,x) \cap V \neq \emptyset.$$

According to Lemma 4.1, there exist $m, n \geqslant 0$ and $\alpha, \beta \in \mathbb{D} \setminus \{0\}$ such that

$$U \cap \alpha T^n x \neq \emptyset \quad \text{and} \quad V \cap \beta T^m x \neq \emptyset.$$

We choose $n \geqslant m$. Since $U \cap \alpha T^n x \neq \emptyset$ and $V \cap \beta T^m x \neq \emptyset$, there exist two elements z_1 and z_2 such that $z_1 \in U \cap \alpha T^n x$ and $z_2 \in V \cap \beta T^m x$. So, we distinguish two cases:

Case 1: $|\alpha| \leqslant |\beta|$. Since $z_2 \in \beta T^m x$ and $\beta \neq 0$,

$$\begin{aligned}z_2 \in \beta T^m x &\iff z_2 \in T^m(\beta x) \iff (\beta x, z_2) \in G(T^m) \iff (z_2, \beta x) \in G((T^m)^{-1}) \\ &\iff (z_2, \beta x) \in G((T^{-m})) \iff \beta x \in T^{-m} z_2 \iff x \in \frac{1}{\beta} T^{-m} z_2.\end{aligned}$$

Thus,

$$z_1 \in \alpha T^n x \subset \frac{\alpha}{\beta} T^{n-m} z_2 \subset \frac{\alpha}{\beta} T^{n-m}(V).$$

We set $p := n - m \geqslant 0$ and $\gamma := \alpha/\beta$. Therefore

$$\gamma T^p(V) \cap U \neq \emptyset \text{ with } p \geqslant 0 \text{ and } \gamma \in \mathbb{D} \setminus \{0\}.$$

Therefore T is a disk transitive linear relation.

Case 2: $|\beta| \leqslant |\alpha|$. As $(z_1, z_2) \in T^n x \times V$, then

$$z_1 \in \alpha T^n x \Longleftrightarrow x \in \frac{1}{\alpha} T^{-n} z_1,$$

which implies

$$z_2 \in \beta T^m x \subset \frac{\beta}{\alpha} T^{m-n} z_1 \subset \frac{\beta}{\alpha} T^{m-n}(U) \subset \gamma T^{-p}(U)$$

with $p := n - m$ and $\gamma := \beta/\alpha$. Since $n \geqslant m$ and $|\beta| \leqslant |\alpha|$,

$$\gamma T^{-p}(U) \cap V \neq \emptyset \text{ with } p \in \mathbb{N} \cup \{0\} \text{ and } \gamma \in \mathbb{D} \setminus \{0\}.$$

Hence,

$$\emptyset \neq \gamma T^{-p}(U) \cap V \subset \gamma \Big(\bigcup_{\alpha \in \mathbb{C}, |\alpha| \geqslant 1} \bigcup_{n \geqslant 0} \alpha T^{-n}(U) \Big) \cap V$$

and so

$$\gamma \Big(\bigcup_{\alpha \in \mathbb{C}, |\alpha| \geqslant 1} \bigcup_{n \geqslant 0} \alpha T^{-n}(U) \Big) \cap V \neq \emptyset$$

for any nonempty open subset V of H. Thus, we deduce that the set

$$G := \gamma \Big(\bigcup_{\alpha \in \mathbb{C}, |\alpha| \geqslant 1} \bigcup_{n \geqslant 0} \alpha T^{-n}(U) \Big)$$

is dense in H.

Now, we consider the map h_γ defined on H by $h_\gamma(x) = \gamma^{-1} x$. Clearly, h_γ is a homeomorphism. Since h_γ is closed,

$$\overline{\bigcup_{\alpha \in \mathbb{C}, |\alpha| \geqslant 1} \bigcup_{n \geqslant 0} \alpha T^{-n}(U)} = \overline{\frac{1}{\gamma} G} = \overline{h_\gamma(G)} = h_\gamma(\overline{G}) = h_\gamma(H) = \frac{1}{\gamma} H = H.$$

It follows from Theorem 4.1 that T is a disk transitive linear relation. □

Theorem 4.4. *Let $T \in \mathbb{D}\mathcal{CR}(H)$ and $\lambda \in \mathbb{D}$. Then the range of $T - \lambda I$ is dense in H.*

Proof. If $\lambda = 0$, then by Proposition 3.2, $R(T)$ is dense in H. Now, let $\lambda \in \mathbb{D} \setminus \{0\}$. Suppose that $R(T - \lambda I)$ is not dense in H. Since T is disk-cyclic, then by virtue of Proposition 4.2 and Theorem 4.3 there exists $x \in \mathbb{D}\mathcal{CR}(T)$ such that $x \notin \overline{(T - \lambda I)H}$. By the Hahn Banach theorem, there exists a continuous linear functional ψ on H such that $\psi(x) \neq 0$ and $\psi(\overline{R(T - \lambda I)}) = \{0\}$. This implies that

$$\psi(Ty) - \lambda\psi(y) = \psi((T - \lambda I)y) = 0$$

for all $y \in H$. Hence, $\psi(Ty) = \lambda\psi(y)$. Using [9], Lemma 4.2, we get for all $n \geqslant 1$, $R(T^n - \lambda^n I) \subset R(T - \lambda I)$. Thus

$$\psi(T^n y) = \lambda^n \psi(y) \tag{4.1}$$

for all $n \geqslant 1$ and all $y \in H$.

Now since $\mathbb{D}\,\mathrm{Orb}(T, x)$ is dense in H, there exists $\{x_k\}$ in $\mathbb{D}\,\mathrm{Orb}(T, x)$ such that $\{x_k\}$ converges to $3x$. Then $\psi(x_k) \to 3\psi(x)$ as $k \to \infty$. For each $k \geqslant 1$ there exists $n_k \geqslant 1$ and α_k in $\mathbb{D}$ such that $x_k \in \alpha_k T^{n_k} x$. Using equality (4.1) and $T^{n_k}\alpha_k x = x_k + T^{n_k}(0)$, we obtain

$$\psi(x_k) = \psi(\alpha_k T^{n_k} x) = \alpha_k \psi(T^{n_k} x) = \alpha_k \lambda^{n_k} \psi(x).$$

Thus, $\alpha_k \lambda^{n_k} \psi(x) \to 3\psi(x)$. Since $|\alpha_k \lambda^{n_k}| \leqslant 1$ and $\psi(x) \neq 0$, $|\alpha_k \lambda^{n_k}| \to 3 \leqslant 1$ as $k \to \infty$, which is a contradiction. Finally, we conclude that the range of $T - \lambda I$ is dense in H. □

As an immediate consequence of the previous results, we obtain the following corollary.

Corollary 4.1. *Let $T \in \mathbb{D}\mathcal{CR}(H)$. Then*

$$\sigma_p(T^*) \subset \mathbb{C} \setminus \mathbb{D}.$$

Proof. Suppose that $\sigma_p(T^*)$ is a nonempty subset of $\mathbb{C}$. Let $\lambda \in \mathbb{D}$, then, by Theorem 4.4, we deduce that $R(T - \lambda I)$ is dense in H. This implies that

$$\ker(T - \lambda I)^\star = R(T - \lambda I)^\perp = \overline{R(T - \lambda I)}^\perp = H^\perp = \{0\}.$$

Moreover, since λI is a bounded linear operator,

$$\ker(T - \lambda I)^* = \ker(T^* - \overline{\lambda} I) = \{0\},$$

which implies that $\overline{\lambda} \notin \sigma_p(T^*)$. Since $\lambda \in \mathbb{D}$ is equivalent to $\overline{\lambda} \in \mathbb{D}$, we obtain $\lambda \notin \sigma_p(T^*)$. Thus, $\sigma_p(T^*)$ is a subset of $\mathbb{C} \setminus \mathbb{D}$. □

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