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SUBCLASS OF ANALYTIC FUNCTIONS RELATED WITH MILLER-ROSS-TYPE POISSON DISTRIBUTION SERIES

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Abstract. The purpose of the present paper is to find a necessary and sufficient condition for the Miller-Ross-type Poisson distribution series to be in the class $\mathcal{P}^*(\alpha,\beta,\gamma)$ of analytic functions with negative coefficients. Also, we investigate several inclusion properties of the classes of Janowski type close-to-starlike functions, Janowski type close-to-convex functions and Janowski type quasi-convex functions associated with the operator $\mathbb{I}^s_{\theta,\varepsilon}$ defined by this distribution. Further, we consider an integral operator related to the Miller-Ross-type Poisson distribution series. Several corollaries and consequences of the main results are also considered.

Keywords: analytic function; starlike function; convex function; Hadamard product; Miller-Ross-type Poisson distribution series

MSC 2020: 30C45

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of analytic functions in the open unit disk $\mathbb{U}=\{\vartheta\in\mathbb{C}\colon |\vartheta|<1\}$ given by the series expansion

$$\hbar(\vartheta)=\vartheta+\sum_{\iota=2}^{\infty}a_\iota\vartheta^\iota, \tag{1.1}$$

and $\mathcal{S}$ denote the subclass of functions of $\mathcal{A}$ which are univalent in $\mathbb{U}$. Further, let $\mathcal{T}$ be the subclass of $\mathcal{A}$ consisting of functions of the form

$$\hbar(\vartheta)=\vartheta-\sum_{\iota=2}^{\infty}|a_\iota|\vartheta^\iota,\quad \vartheta\in\mathbb{U}. \tag{1.2}$$

Also, let $\mathcal{S}^*$ and $\mathcal{K}$ be the usual subclasses of functions whose members are univalent starlike and univalent convex in $\mathbb{U}$, respectively.

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For the analytic functions $\hbar$ and $\Im$ on $\mathbb{U}$ with $\hbar(0) = \Im(0)$, $\hbar(\vartheta)$ is subordinate to $\Im(\vartheta)$ in $\mathbb{U}$, and we write $\hbar(\vartheta) \prec \Im(\vartheta)$, if there exists an analytic function ϖ on $\mathbb{U}$ such that $\varpi(0) = 0$, $|\varpi(\vartheta)| < 1$, and $\hbar(\vartheta) = \Im(\varpi(\vartheta))$ for $\vartheta \in \mathbb{U}$. If $\Im$ is univalent in $\mathbb{U}$, then the subordination is equivalent to $\hbar(0) = \Im(0)$ and $\hbar(\mathbb{U}) \subset \Im(\mathbb{U})$.

For $-1 \leqslant B < A \leqslant 1$, a function p, which is analytic in $\mathbb{U}$ with $p(0) = 1$, is said to belong to the class $\mathcal{P}(A, B)$ if

$$p(\vartheta) \prec \frac{1 + A\vartheta}{1 + B\vartheta} \quad (\vartheta \in \mathbb{U}). \tag{1.3}$$

The class $\mathcal{P}(A, B)$ was introduced by Janowski [19]. A function $\hbar \in \mathcal{A}$ is said to be in $\mathcal{S}^*(A, B)$ if $\vartheta\hbar'/\hbar \in \mathcal{P}(A, B)$, and in $\mathcal{K}(A, B)$ if $\vartheta\hbar' \in \mathcal{S}^*(A, B)$. The class $\mathcal{S}^*(A, B)$ and related classes were studied by Janowski [19] and Silverman and Silvia [35]. We note that $\mathcal{S}^*(1-2\delta, -1) = \mathcal{S}^*(\delta)$ and $\mathcal{K}(1-2\delta, -1) = \mathcal{K}(\delta)$, where $\mathcal{S}^*(\delta)$ and $\mathcal{K}(\delta)$ are the class of starlike functions of order δ and the class of convex functions of order δ, respectively, $0 \leqslant \delta < 1$. Clearly, we have $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(0) = \mathcal{K}$.

A function $\hbar \in \mathcal{A}$ is said to be close-to-star, denoted by $\mathcal{CS}^*$, if and only if there exists $\Im \in \mathcal{S}^*$ such that

$$\Re\left(\frac{\hbar(\vartheta)}{\Im(\vartheta)}\right) > 0 \tag{1.4}$$

for all $\vartheta \in \mathbb{U}$. Also, a function $\hbar \in \mathcal{A}$ is said to be close-to-convex, denoted by $\mathcal{CK}$, if and only if there exists $\Im \in \mathcal{K}$ such that

$$\Re\left(\frac{\hbar'(\vartheta)}{\Im'(\vartheta)}\right) > 0 \tag{1.5}$$

for all $\vartheta \in \mathbb{U}$.

Furthermore, a function $\hbar \in \mathcal{A}$ is said to be quasi-convex, denoted by $\mathcal{QK}$, if and only if there exists $\Im \in \mathcal{K}$ such that

$$\Re\left(\frac{(\vartheta\hbar'(\vartheta))'}{\Im'(\vartheta)}\right) > 0 \tag{1.6}$$

for all $\vartheta \in \mathbb{U}$.

The class of close-to-star was introduced by Reade in [31] and the class of close-to-convex was introduced by Kaplan in [21], while the class of quasi-convex was introduced by Noor and Thomas in [29]. Similarly, we denote the class of close-to-star functions of order δ, close-to-convex functions of order δ, and quasi-convex functions of order δ by $\mathcal{CS}^*(\delta)$, $\mathcal{CK}(\delta)$, and $\mathcal{QK}(\delta)$, respectively, where $0 \leqslant \delta < 1$.

Clearly, we have $\mathcal{CS}^*(0) = \mathcal{CS}^*$, $\mathcal{CK}(0) = \mathcal{CK}$, and $\mathcal{QK}(0) = \mathcal{QK}$.

Several researchers obtained coefficient bounds of some classes of convex and starlike functions, see [3], [4], [2]. Altıntaş and Kiliç [5] obtained coefficient bounds of the subclasses of analytic functions defined as follows.

Definition 1.1. A function $\hbar(\vartheta)$ in the form (1.1) is Janowski type close-to-starlike in $\mathbb{U}$, denoted by $\mathcal{CS}^*(A,B)$, if there is a starlike function $\Im(\vartheta)$ such that

$$\frac{\hbar(\vartheta)}{\Im(\vartheta)} \prec \frac{1+A\vartheta}{1+B\vartheta},$$

where $\vartheta \in \mathbb{U}$ and $-1 \leqslant B < A \leqslant 1$.

Definition 1.2. A function $\hbar(\vartheta)$ in the form (1.1) is Janowski type close-to-convex in $\mathbb{U}$, denoted by $\mathcal{CK}(A,B)$, if there is a function $\Im \in \mathcal{K}$ such that

$$\frac{\hbar'(\vartheta)}{\Im'(\vartheta)} \prec \frac{1+A\vartheta}{1+B\vartheta},$$

where $\vartheta \in \mathbb{U}$ and $-1 \leqslant B < A \leqslant 1$.

Definition 1.3. A function $\hbar(\vartheta)$ in the form (1.1) is Janowski type quasi-convex in $\mathbb{U}$, denoted by $\mathcal{QK}(A,B)$, if there is a function $\Im \in \mathcal{K}$ such that

$$\frac{(\vartheta\hbar'(\vartheta))'}{\Im'(\vartheta)} \prec \frac{1+A\vartheta}{1+B\vartheta},$$

where $\vartheta \in \mathbb{U}$ and $-1 \leqslant B < A \leqslant 1$.

If we let $\Im(\vartheta) = \hbar(\vartheta)$ in Definitions 1.1, 1.2 and 1.3, we have

$$\mathcal{K}(A,B) \subset \mathcal{QK}(A,B) \subset \mathcal{CK}(A,B) \quad \text{and} \quad \mathcal{S}^*(A,B) \subset \mathcal{CS}^*(A,B).$$

Very recently, Joshi et al. [20] obtained sufficient conditions for the normalized Wright functions to be in the class $\mathcal{D}(\alpha,\beta,\gamma)$ defined as follows.

Definition 1.4. A function $\hbar \in \mathcal{A}$ is said to be in the class $\mathcal{D}(\alpha,\beta,\gamma)$, $0 < \beta \leqslant 1$, $0 \leqslant \alpha < 1/(2\gamma)$, and $1/2 \leqslant \gamma \leqslant 1$, if it satisfies the inequality

$$\left|\frac{\hbar'(\vartheta)-1}{2\gamma(\hbar'(\vartheta)-\alpha)+(\hbar'(\vartheta)-1)]}\right| < \beta, \quad \vartheta \in \mathbb{U}.$$

Let $\mathcal{P}^*(\alpha,\beta,\gamma) = \mathcal{D}(\alpha,\beta,\gamma) \cap \mathcal{T}$.

The class $\mathcal{P}^*(\alpha,\beta,\gamma)$ was introduced by Kulkarni [23]. In particular, the class $\mathcal{P}^*(\alpha,\beta,1) = \mathcal{P}^*(\alpha,\beta)$ was studied by Gupta and Jain [18] and the class $\mathcal{P}^*(0,\beta,1) = \mathcal{D}^*(\beta)$ was introduced and studied by Kim and Lee [22].

Let $\mathbb{E}_{\theta,\varepsilon}(\vartheta)$ be the Miller-Ross function [25] defined by

$$\mathbb{E}_{\theta,\varepsilon}(\vartheta) = \vartheta^{\theta} \sum_{\iota=0}^{\infty} \frac{(\varepsilon\vartheta)^{\iota}}{\Gamma(\iota+\theta+1)} \quad (\theta, \varepsilon, \vartheta \in \mathbb{C}). \tag{1.7}$$

Also, let $E_{\varsigma,\mu}(\vartheta)$ be the two parameter Mittag-Leffler function [37] defined by

$$E_{\varsigma,\mu}(\vartheta) = \sum_{\iota=0}^{\infty} \frac{\vartheta^{\iota}}{\Gamma(\varsigma\iota+\mu)} \quad (\vartheta, \varsigma, \mu \in \mathbb{C},\ \mathrm{Re}(\varsigma) > 0,\ \mathrm{Re}(\mu) > 0). \tag{1.8}$$

If $\mu = 1$, from (1.8) we obtain the one parameter Mittag-Leffler function [26]

$$E_{\varsigma}(\vartheta) = \sum_{\iota=0}^{\infty} \frac{\vartheta^{\iota}}{\Gamma(\varsigma\iota+1)} \quad (\vartheta, \varsigma \in \mathbb{C},\ \mathrm{Re}(\varsigma) > 0). \tag{1.9}$$

Several properties of the Mittag-Leffler function and generalized Mittag-Leffler function can be found in [7], [8], [13], [17].

From (1.7) and (1.8), the Miller-Ross function may be written as

$$\mathbb{E}_{\theta,\varepsilon}(\vartheta) = \vartheta^{\theta} E_{1,1+\theta}(\varepsilon\vartheta).$$

Very recently, Şeker et al. [32] introduced a power series whose coefficients are a Miller-Ross-type Poisson distribution

$$\hbar^{s}_{\theta,\varepsilon}(\vartheta) := \vartheta + \sum_{\iota=2}^{\infty} \frac{s^{\theta}(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)\mathbb{E}_{\theta,\varepsilon}(s)} \vartheta^{\iota}, \quad \vartheta \in \mathbb{U}, \tag{1.10}$$

where $\theta > -1$, $\varepsilon > 0$.

We note that if we put $\theta = 0$ and $\varepsilon = 1$ in (1.10), we get the Poisson distribution series introduced by Porwal [30].

Also, Şeker et al. [32] defined the series

$$\mathbb{K}^{s}_{\theta,\varepsilon}(\vartheta) := 2\vartheta - \hbar^{s}_{\theta,\varepsilon}(\vartheta) = \vartheta - \sum_{\iota=2}^{\infty} \frac{s^{\theta}(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)\mathbb{E}_{\theta,\varepsilon}(s)} \vartheta^{\iota}, \quad \vartheta \in \mathbb{U}. \tag{1.11}$$

Now we consider the linear operator $\mathbb{I}^{s}_{\theta,\varepsilon} \colon \mathcal{A} \to \mathcal{A}$ defined by the convolution or Hadamard product

$$\mathbb{I}^{s}_{\theta,\varepsilon}\hbar(\vartheta) := \hbar^{s}_{\theta,\varepsilon}(\vartheta) * \hbar(\vartheta) = \vartheta + \sum_{\iota=2}^{\infty} \frac{s^{\theta}(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)\mathbb{E}_{\theta,\varepsilon}(s)} a_{\iota}\vartheta^{\iota}, \quad \vartheta \in \mathbb{U}, \tag{1.12}$$

where $\theta > -1$ and $\varepsilon > 0$.

In recent years, several researchers have used distribution series such as the Poisson distribution series [10], [15], [28], [30], the Pascal distribution series [11], [6], [14], [33], [27], the hypergeometric-type distribution series, the confluent hypergeometric distribution series [9], [12], [24], [34], [36], and the Mittag-Leffler-type Poisson distribution [1], [16] to obtain some necessary and sufficient conditions for these distributions to belong to certain classes of univalent functions. Motivated with the works mentioned, in the present paper we determine a necessary and sufficient condition for $\mathbb{K}^s_{\theta,\varepsilon}$ to be in our class $\mathcal{P}^*(\alpha,\beta,\gamma)$. Also, we investigate several inclusion properties of the classes $\mathcal{QK}(A,B)$, $\mathcal{CS}^*(A,B)$, and $\mathcal{CK}(A,B)$ associated to the operator $\mathbb{I}^s_{\theta,\varepsilon}$ defined by (1.12). Finally, we give sufficient conditions for the function $\hbar$ such that its image by the integral operator

$$\Im^s_{\theta,\varepsilon}(\vartheta) = \int_0^{\vartheta} \frac{\mathbb{K}^s_{\theta,\varepsilon}(t)}{t}\,\mathrm{d}t$$

belongs to the class $\mathcal{P}^*(\alpha,\beta,\gamma)$.

2. Preliminary lemmas

To establish our main results, we need the following lemmas.

Lemma 2.1 ([23]). *A function $\hbar \in \mathcal{T}$ of the form (1.2) is in the class $\mathcal{P}^*(\alpha,\beta,\gamma)$ if and only if*

$$\sum_{\iota=2}^{\infty}[1+\beta(1-2\gamma)]\iota|a_\iota| \leqslant 2\beta\gamma(1-\alpha), \tag{2.1}$$

where $0<\beta\leqslant 1$, $0\leqslant\alpha<1/(2\gamma)$, and $1/2\leqslant\gamma\leqslant 1$. The result (2.1) is sharp.

Lemma 2.2 ([5]). *If $\hbar \in \mathcal{QK}(A,B)$ is of the form (1.1), then*

$$|a_\iota| \leqslant \frac{1}{\iota}\left[1+\frac{(\iota-1)(A-B)}{1-B}\right] \quad (\iota\geqslant 2). \tag{2.2}$$

The result is sharp.

Lemma 2.3 ([5]). *If $\hbar \in \mathcal{CS}^*(A,B)$ is of the form (1.1), then*

$$|a_\iota| \leqslant \iota\left[1+\frac{(\iota-1)(A-B)}{1-B}\right] \quad (\iota\geqslant 2). \tag{2.3}$$

The result is sharp.

Lemma 2.4 ([5]). *If $\hbar \in C\mathcal{K}(A,B)$ is of the form* (1.1), *then*

$$|a_\iota| \leqslant \left[1+\frac{(\iota-1)(A-B)}{1-B}\right] \quad (\iota \geqslant 2). \tag{2.4}$$

The result is sharp.

3. Necessary and sufficient condition for $\mathbb{K}^s_{\theta,\varepsilon} \in \mathcal{P}^*(\alpha,\beta,\gamma)$

Firstly, we obtain a necessary and sufficient condition for $\mathbb{K}^s_{\theta,\varepsilon}$ to be in the class $\mathcal{P}^*(\alpha,\beta,\gamma)$.

Theorem 3.1. *Let $\theta > -1$ and $\varepsilon > 0$, then $\mathbb{K}^s_{\theta,\varepsilon} \in \mathcal{P}^*(\alpha,\beta,\gamma)$ if and only if*

$$\frac{(1+\beta(1-2\gamma))\varepsilon}{\mathbb{E}_{\theta,\varepsilon}(s)}[s\mathbb{E}_{\theta,\varepsilon}(s)+(1-\theta)\mathbb{E}_{\theta+1,\varepsilon}(s)] \leqslant 2\beta\gamma(1-\alpha). \tag{3.1}$$

Proof. Since $\mathbb{K}^s_{\theta,\varepsilon}$ is defined by (1.11), in view of Lemma 2.1 it is sufficient to show that

$$\sum_{\iota=2}^{\infty}[1+\beta(1-2\gamma)]\iota\frac{s^\theta(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)}\frac{1}{\mathbb{E}_{\theta,\varepsilon}(s)} \leqslant 2\beta\gamma(1-\alpha). \tag{3.2}$$

Writing

$$\iota = (\theta+\iota-1)+(1-\theta)$$

in (3.2), we have

$$\begin{aligned}
&\sum_{\iota=2}^{\infty}[1+\beta(1-2\gamma)]\iota\frac{s^\theta(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)}\frac{1}{\mathbb{E}_{\theta,\varepsilon}(s)}\\
&\quad=\frac{1}{\mathbb{E}_{\theta,\varepsilon}(s)}\Bigg[(1+\beta(1-2\gamma))\sum_{\iota=2}^{\infty}(\theta+\iota-1)\frac{s^\theta(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)}\\
&\qquad+(1+\beta(1-2\gamma))(1-\theta)\sum_{\iota=2}^{\infty}\frac{s^\theta(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)}\Bigg]\\
&\quad=\frac{1}{\mathbb{E}_{\theta,\varepsilon}(s)}\Bigg[(1+\beta(1-2\gamma))\sum_{\iota=2}^{\infty}\frac{s^\theta(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta-1)}\\
&\qquad+(1+\beta(1-2\gamma))(1-\theta)\sum_{\iota=2}^{\infty}\frac{s^\theta(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)}\Bigg]\\
&\quad=\frac{\varepsilon}{\mathbb{E}_{\theta,\varepsilon}(s)}\Bigg[(1+\beta(1-2\gamma))s\sum_{\iota=0}^{\infty}\frac{s^\theta(\varepsilon s)^{\iota}}{\Gamma(\iota+\theta+1)}
\end{aligned}$$

$$\times (1+\beta(1-2\gamma))(1-\theta)s\sum_{\iota=0}^{\infty}\frac{s^{\theta}(\varepsilon s)^{\iota}}{\Gamma(\iota+\theta+2)}\Bigg]$$

$$= \frac{\varepsilon}{\mathbb{E}_{\theta,\varepsilon}(s)}[(1+\beta(1-2\gamma))s\mathbb{E}_{\theta,\varepsilon}(s)+(1+\beta(1-2\gamma))(1-\theta)\mathbb{E}_{\theta+1,\varepsilon}(s)],$$

but this last expression is upper bounded by $2\beta\gamma(1-\alpha)$ if and only if (3.1) holds. □

4. Inclusion relations

In this section we prove the inclusion relations of the classes $\mathcal{QK}(A,B)$, $\mathcal{CS}^*(A,B)$, and $\mathcal{CK}(A,B)$ associated to the operator $\mathbb{I}^s_{\theta,\varepsilon}$ defined by (1.12).

Theorem 4.1. *Let $\theta > -1$ and $\varepsilon > 0$. If $\hbar \in \mathcal{QK}(A,B)$ and the inequality*

$$\text{(4.1)}\qquad \frac{1+\beta(1-2\gamma)\varepsilon}{(1-B)\mathbb{E}_{\theta,\varepsilon}(s)}[(A-B)s\mathbb{E}_{\theta,\varepsilon}(s)+((A-B)(1-\theta)+(1-A))\mathbb{E}_{\theta+1,\varepsilon}(s)] \leqslant 2\beta\gamma(1-\alpha)$$

is satisfied then $\mathbb{I}^s_{\theta,\varepsilon}\hbar \in \mathcal{P}^*(\alpha,\beta,\gamma)$.

Proof. According to Lemma 2.1 it is sufficient to show that

$$\text{(4.2)}\qquad \sum_{\iota=2}^{\infty}[1+\beta(1-2\gamma)]\iota\frac{s^{\theta}(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)\mathbb{E}_{\theta,\varepsilon}(s)}|a_{\iota}| \leqslant 2\beta\gamma(1-\alpha).$$

Since $\hbar \in \mathcal{QK}(A,B)$, using the inequality (2.2) and writing

$$\iota = (\theta+\iota-1)+(1-\theta)$$

in (4.2), we have

$$\sum_{\iota=2}^{\infty}[1+\beta(1-2\gamma)]\iota\frac{s^{\theta}(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)\mathbb{E}_{\theta,\varepsilon}(s)}|a_{\iota}|$$

$$\leqslant \frac{1+\beta(1-2\gamma)}{1-B}\left[\sum_{\iota=2}^{\infty}[\iota(A-B)+1-A]\frac{s^{\theta}(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)\mathbb{E}_{\theta,\varepsilon}(s)}\right]$$

$$= \frac{1+\beta(1-2\gamma)}{(1-B)\mathbb{E}_{\theta,\varepsilon}(s)}\Bigg[(A-B)\sum_{\iota=2}^{\infty}(\theta+\iota-1)\frac{s^{\theta}(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)}$$

$$+((A-B)(1-\theta)+(1-A))\sum_{\iota=2}^{\infty}\frac{s^{\theta}(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)}\Bigg]$$

$$= \frac{1+\beta(1-2\gamma)\varepsilon}{(1-B)\mathbb{E}_{\theta,\varepsilon}(s)}[(A-B)s\mathbb{E}_{\theta,\varepsilon}(s)+((A-B)(1-\theta)+(1-A))\mathbb{E}_{\theta+1,\varepsilon}(s)].$$

The last expression is upper bounded by $2\beta\gamma(1-\alpha)$ if (4.1) holds, which completes our proof. □

Theorem 4.2. *Let $\theta > -1$ and $\varepsilon > 0$. If $\hbar \in \mathcal{KS}^*(A,B)$ and the inequality*

$$\begin{aligned}(4.3)\qquad \frac{(1+\beta(1-2\gamma))\varepsilon}{(1-B)\mathbb{E}_{\theta,\varepsilon}(s)}&[(A-B)s^3\mathbb{E}_{\theta-2,\varepsilon}(s)\\ &+[(A-B)(6-3\theta)+(1-A)]s^2\mathbb{E}_{\theta-1,\varepsilon}(s)\\ &+[(A-B)(3\theta^2-9\theta+7)+(1-A)(3-2\theta)]s\mathbb{E}_{\theta,\varepsilon}(s)\\ &+[(A-B)(1-\theta)^3+(1-A)(1-\theta)^2]\mathbb{E}_{\theta+1,\varepsilon}(s)]\\ &\leqslant 2\beta\gamma(1-\alpha)\end{aligned}$$

is satisfied then $\mathbb{I}^s_{\theta,\varepsilon}\hbar \in \mathcal{P}^*(\alpha,\beta,\gamma)$.

Proof. According to Lemma 2.1 it is sufficient to show that

$$\sum_{\iota=2}^{\infty}[1+\beta(1-2\gamma)]\iota\frac{s^\theta(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)\mathbb{E}_{\theta,\varepsilon}(s)}|a_\iota| \leqslant 2\beta\gamma(1-\alpha).$$

Since $\hbar \in \mathcal{CS}^*(A,B)$, using the inequality (2.3), we have

$$\begin{aligned}(4.4)\qquad &\sum_{\iota=2}^{\infty}[1+\beta(1-2\gamma)]\iota\frac{s^\theta(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)\mathbb{E}_{\theta,\varepsilon}(s)}|a_\iota|\\ &\leqslant \frac{1+\beta(1-2\gamma)}{1-B}\left[\sum_{\iota=2}^{\infty}[(A-B)\iota^3+(1-A)\iota^2]\frac{s^\theta(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)\mathbb{E}_{\theta,\varepsilon}(s)}\right].\end{aligned}$$

Writing

$$\begin{aligned}\iota^3 &= (\theta+\iota-1)(\theta+\iota-2)(\theta+\iota-3)+(6-3\theta)(\theta+\iota-1)(\theta+\iota-2)\\ &\quad\times(3\theta^2-9\theta+7)(\theta+\iota-1)+(1-\theta)^3\end{aligned}$$

and

$$\iota^2 = (\theta+\iota-1)(\theta+\iota-2)+(3-2\theta)(\theta+\iota-1)+(1-\theta)^2$$

in (4.4), we have

$$\begin{aligned}&\sum_{\iota=2}^{\infty}[1+\beta(1-2\gamma)]\iota\frac{s^\theta(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)\mathbb{E}_{\theta,\varepsilon}(s)}|a_\iota|\\ &\leqslant \frac{1+\beta(1-2\gamma)}{(1-B)\mathbb{E}_{\theta,\varepsilon}(s)}\Bigg[(A-B)\sum_{\iota=2}^{\infty}(\theta+\iota-1)(\theta+\iota-2)(\theta+\iota-3)\frac{s^\theta(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)}\\ &\quad+[(A-B)(6-3\theta)+(1-A)]\sum_{\iota=2}^{\infty}(\theta+\iota-1)(\theta+\iota-2)\frac{s^\theta(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)}\\ &\quad+[(A-B)(3\theta^2-9\theta+7)+(1-A)(3-2\theta)]\sum_{\iota=2}^{\infty}(\theta+\iota-1)\frac{s^\theta(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)}\end{aligned}$$

$$+ [(A-B)(1-\theta)^3 + (1-A)(1-\theta)^2] \sum_{\iota=2}^{\infty} \frac{s^{\theta}(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)} \Bigg]$$

$$= \frac{1+\beta(1-2\gamma)}{(1-B)\mathbb{E}_{\theta,\varepsilon}(s)} \Bigg[(A-B) \sum_{\iota=2}^{\infty} \frac{s^{\theta}(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta-3)}$$

$$+ [(A-B)(6-3\theta) + (1-A)] \sum_{\iota=2}^{\infty} \frac{s^{\theta}(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta-2)}$$

$$+ [(A-B)(3\theta^2-9\theta+7) + (1-A)(3-2\theta)] \sum_{\iota=2}^{\infty} \frac{s^{\theta}(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta-1)}$$

$$+ [(A-B)(1-\theta)^3 + (1-A)(1-\theta)^2] \sum_{\iota=2}^{\infty} \frac{s^{\theta}(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)} \Bigg]$$

$$= \frac{(1+\beta(1-2\gamma))\varepsilon}{(1-B)\mathbb{E}_{\theta,\varepsilon}(s)} \Bigg[(A-B)s \sum_{\iota=0}^{\infty} \frac{s^{\theta}(\varepsilon s)^{\iota}}{\Gamma(\iota+\theta-1)}$$

$$+ [(A-B)(6-3\theta) + (1-A)]s \sum_{\iota=0}^{\infty} \frac{s^{\theta}(\varepsilon s)^{\iota}}{\Gamma(\iota+\theta)}$$

$$+ [(A-B)(3\theta^2-9\theta+7) + (1-A)(3-2\theta)]s \sum_{\iota=0}^{\infty} \frac{s^{\theta}(\varepsilon s)^{\iota}}{\Gamma(\iota+\theta+1)}$$

$$+ [(A-B)(1-\theta)^3 + (1-A)(1-\theta)^2]s \sum_{\iota=0}^{\infty} \frac{s^{\theta}(\varepsilon s)^{\iota}}{\Gamma(\iota+\theta+2)} \Bigg]$$

$$= \frac{(1+\beta(1-2\gamma))\varepsilon}{(1-B)\mathbb{E}_{\theta,\varepsilon}(s)} \Big[(A-B)s^3\mathbb{E}_{\theta-2,\varepsilon}(s) + [(A-B)(6-3\theta)+(1-A)]s^2\mathbb{E}_{\theta-1,\varepsilon}(s).$$
$$+ [(A-B)(3\theta^2-9\theta+7) + (1-A)(3-2\theta)]s\mathbb{E}_{\theta,\varepsilon}(s)$$
$$+ [(A-B)(1-\theta)^3 + (1-A)(1-\theta)^2]\mathbb{E}_{\theta+1,\varepsilon}(s) \Big],$$

but this last expression is upper bounded by $2\beta\gamma(1-\alpha)$ if and only if (4.3) holds. □

Theorem 4.3. *Let $\theta > -1$ and $\varepsilon > 0$. If $\hbar \in C\mathcal{K}(A,B)$ and the inequality*

$$\frac{(1+\beta(1-2\gamma))\varepsilon}{(1-B)\mathbb{E}_{\theta,\varepsilon}(s)} [(A-B)s^2\mathbb{E}_{\theta-1,\varepsilon}(s) + [(A-B)(6-3\theta)+(1-A)]s\mathbb{E}_{\theta,\varepsilon}(s) + [(A-B)(1-\theta)^2 + (1-A)(1-\theta)]\mathbb{E}_{\theta+1,\varepsilon}(s)] \leqslant 2\beta\gamma(1-\alpha) \tag{4.5}$$

is satisfied then $\mathbb{I}^{s}_{\theta,\varepsilon}\hbar \in \mathcal{P}^*(\alpha,\beta,\gamma)$.

P r o o f. According to Lemma 2.1 it is sufficient to show that

$$\sum_{\iota=2}^{\infty}[1+\beta(1-2\gamma)]\iota\frac{s^{\theta}(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)\mathbb{E}_{\theta,\varepsilon}(s)}|a_{\iota}|\leqslant 2\beta\gamma(1-\alpha).$$

Since $\hbar\in\mathcal{CK}(A,B)$, using the inequality (2.4), we have

$$\sum_{\iota=2}^{\infty}[1+\beta(1-2\gamma)]\iota\frac{s^{\theta}(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)\mathbb{E}_{\theta,\varepsilon}(s)}|a_{\iota}| \leqslant\frac{1+\beta(1-2\gamma)}{1-B}\biggl[\sum_{\iota=2}^{\infty}[(A-B)\iota^{2}+(1-A)\iota]\frac{s^{\theta}(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)\mathbb{E}_{\theta,\varepsilon}(s)}\biggr]. \tag{4.6}$$

Writing

$$\iota^{2}=(\theta+\iota-1)(\theta+\iota-2)+(3-2\theta)(\theta+\iota-1)+(1-\theta)^{2}$$

and

$$\iota=(\theta+\iota-1)+(1-\theta)$$

in (4.6), we have

$$\begin{aligned}
&\sum_{\iota=2}^{\infty}[1+\beta(1-2\gamma)]\iota\frac{s^{\theta}(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)\mathbb{E}_{\theta,\varepsilon}(s)}|a_{\iota}|\\
&\quad\leqslant\frac{1+\beta(1-2\gamma)}{(1-B)\mathbb{E}_{\theta,\varepsilon}(s)}\biggl[(A-B)\sum_{\iota=2}^{\infty}(\theta+\iota-1)(\theta+\iota-2)\frac{s^{\theta}(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)}\\
&\qquad+[(A-B)(3-2\theta)+(1-A)]\sum_{\iota=2}^{\infty}(\theta+\iota-1)\frac{s^{\theta}(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)}\\
&\qquad+[(A-B)(1-\theta)^{2}+(1-A)(1-\theta)]\sum_{\iota=2}^{\infty}\frac{s^{\theta}(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)}\biggr]\\
&\quad=\frac{1+\beta(1-2\gamma)}{(1-B)\mathbb{E}_{\theta,\varepsilon}(s)}\biggl[(A-B)\sum_{\iota=2}^{\infty}\frac{s^{\theta}(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta-2)}\\
&\qquad+[(A-B)(3-2\theta)+(1-A)]\sum_{\iota=2}^{\infty}\frac{s^{\theta}(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta-1)}\\
&\qquad+[(A-B)(1-\theta)^{3}+(1-A)(1-\theta)^{2}]\sum_{\iota=2}^{\infty}\frac{s^{\theta}(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)}\biggr]\\
&\quad=\frac{(1+\beta(1-2\gamma))\varepsilon}{(1-B)\mathbb{E}_{\theta,\varepsilon}(s)}\biggl[(A-B)s\sum_{\iota=0}^{\infty}\frac{s^{\theta}(\varepsilon s)^{\iota}}{\Gamma(\iota+\theta-1)}\\
&\qquad+[(A-B)(6-3\theta)+(1-A)]s\sum_{\iota=0}^{\infty}\frac{s^{\theta}(\varepsilon s)^{\iota}}{\Gamma(\iota+\theta)}
\end{aligned}$$

$$+ [(A-B)(3\theta^2-9\theta+7)+(1-A)(3-2\theta)]s\sum_{\iota=0}^{\infty}\frac{s^{\theta}(\varepsilon s)^{\iota}}{\Gamma(\iota+\theta+1)}$$
$$+ [(A-B)(1-\theta)^2+(1-A)(1-\theta)]s\sum_{\iota=0}^{\infty}\frac{s^{\theta}(\varepsilon s)^{\iota}}{\Gamma(\iota+\theta+2)}\Bigg]$$
$$= \frac{(1+\beta(1-2\gamma))\varepsilon}{(1-B)\mathbb{E}_{\theta,\varepsilon}(s)}[(A-B)s^2\mathbb{E}_{\theta-1,\varepsilon}(s)+[(A-B)(6-3\theta)+(1-A)]s\mathbb{E}_{\theta,\varepsilon}(s)$$
$$+ [(A-B)(1-\theta)^2+(1-A)(1-\theta)]\mathbb{E}_{\theta+1,\varepsilon}(s)],$$

but this last expression is upper bounded by $2\beta\gamma(1-\alpha)$ if and only if (4.5) holds. □

5. An integral operator

Theorem 5.1. *Let $\theta > -1$ and $\varepsilon > 0$. If the integral operator $\Im^s_{\theta,\varepsilon}$ is given by*

$$\Im^s_{\theta,\varepsilon}(\vartheta) := \int_0^{\vartheta}\frac{\mathbb{K}^s_{\theta,\varepsilon}(t)}{t}\,\mathrm{d}t, \quad \vartheta\in\mathbb{U}, \tag{5.1}$$

then $\Im^s_{\theta,\varepsilon}\in\mathcal{P}^*(\alpha,\beta,\gamma)$, if and only if

$$[1+\beta(1-2\gamma)]\varepsilon\frac{\mathbb{E}_{\theta+1,\varepsilon}(s)}{\mathbb{E}_{\theta,\varepsilon}(s)} \leqslant 2\beta\gamma(1-\alpha). \tag{5.2}$$

P r o o f. According to (1.11) it follows that

$$\Im^s_{\theta,\varepsilon}(\vartheta) = \vartheta - \sum_{\iota=2}^{\infty}\frac{s^{\theta}(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)\mathbb{E}_{\theta,\varepsilon}(s)}\frac{\vartheta^{\iota}}{\iota}, \quad \vartheta\in\mathbb{U}.$$

Using Lemma 2.1, the function $\Im^s_{\theta,\varepsilon}(\vartheta)$ belongs to $\mathcal{P}^*(\alpha,\beta,\gamma)$ if and only if

$$\sum_{\iota=2}^{\infty}[1+\beta(1-2\gamma)]\frac{s^{\theta}(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)\mathbb{E}_{\theta,\varepsilon}(s)}$$
$$\leqslant 2\beta\gamma(1-\alpha)\sum_{\iota=2}^{\infty}[1+\beta(1-2\gamma)]\frac{s^{\theta}(\varepsilon s)^{\iota-1}}{\Gamma(\iota+\theta)\mathbb{E}_{\theta,\varepsilon}(s)}$$
$$= \frac{[1+\beta(1-2\gamma)]\varepsilon}{\mathbb{E}_{\theta,\varepsilon}(s)}s\sum_{\iota=0}^{\infty}\frac{s^{\theta}(\varepsilon s)^{\iota}}{\Gamma(\iota+\theta+2)}$$
$$= [1+\beta(1-2\gamma)]\varepsilon\frac{\mathbb{E}_{\theta+1,\varepsilon}(s)}{\mathbb{E}_{\theta,\varepsilon}(s)},$$

but this last expression is upper bounded by $2\beta\gamma(1-\alpha)$ if and only if (5.2) holds. □

6. Corollaries and consequences

By specializing the parameters α and γ in Theorem 3.1 and Theorem 5.1, we obtain the following special cases for the subclasses $\mathcal{P}^*(\alpha,\beta)$ and $\mathcal{D}^*(\beta)$.

Corollary 6.1. *Let $\theta > -1$ and $\varepsilon > 0$, then $\mathbb{K}^s_{\theta,\varepsilon} \in \mathcal{P}^*(\alpha,\beta)$ if and only if*

$$\frac{(1-\beta)\varepsilon}{\mathbb{E}_{\theta,\varepsilon}(s)}[s\mathbb{E}_{\theta,\varepsilon}(s) + (1-\theta)\mathbb{E}_{\theta+1,\varepsilon}(s)] \leqslant 2(1-\alpha).$$

Corollary 6.2. *Let $\theta > -1$ and $\varepsilon > 0$, then $\mathbb{K}^s_{\theta,\varepsilon} \in \mathcal{D}^*(\beta)$ if and only if*

$$\frac{(1-\beta)\varepsilon}{\mathbb{E}_{\theta,\varepsilon}(s)}[s\mathbb{E}_{\theta,\varepsilon}(s) + (1-\theta)\mathbb{E}_{\theta+1,\varepsilon}(s)] \leqslant 2.$$

Corollary 6.3. *Let $\theta > -1$ and $\varepsilon > 0$. If the integral operator $\Im^s_{\theta,\varepsilon}$ is given by* (5.1)*, then $\Im^s_{\theta,\varepsilon} \in \mathcal{P}^*(\alpha,\beta)$, if and only if*

$$\frac{\mathbb{E}_{\theta+1,\varepsilon}(s)}{\mathbb{E}_{\theta,\varepsilon}(s)} \leqslant \frac{2\beta(1-\alpha)}{(1-\beta)\varepsilon}.$$

Corollary 6.4. *Let $\theta > -1$ and $\varepsilon > 0$. If the integral operator $\Im^s_{\theta,\varepsilon}$ is given by* (5.1)*, then $\Im^s_{\theta,\varepsilon} \in \mathcal{D}^*(\beta)$, if and only if*

$$\frac{\mathbb{E}_{\theta+1,\varepsilon}(s)}{\mathbb{E}_{\theta,\varepsilon}(s)} \leqslant \frac{2\beta}{(1-\beta)\varepsilon}.$$

By setting $A = 1$ and $B = -1$ in Theorems 4.1, 4.2 and 4.3, we obtain the following corollaries.

Corollary 6.5. *Let $\theta > -1$ and $\varepsilon > 0$. If $\hbar \in \mathcal{QCV}$ and the inequality*

$$\frac{1+\beta(1-2\gamma)\varepsilon}{\mathbb{E}_{\theta,\varepsilon}(s)}[s\mathbb{E}_{\theta,\varepsilon}(s) + (1-\theta)\mathbb{E}_{\theta+1,\varepsilon}(s)] \leqslant 2\beta\gamma(1-\alpha)$$

is satisfied then $\mathbb{I}^s_{\theta,\varepsilon}\hbar \in \mathcal{P}^*(\alpha,\beta,\gamma)$.

Corollary 6.6. *Let $\theta > -1$ and $\varepsilon > 0$. If $\hbar \in \mathcal{CS}^*$ and the inequality*

$$\begin{aligned}\frac{(1+\beta(1-2\gamma))\varepsilon}{\mathbb{E}_{\theta,\varepsilon}(s)}&[s^3\mathbb{E}_{\theta-2,\varepsilon}(s)+(6-3\theta)s^2\mathbb{E}_{\theta-1,\varepsilon}(s)\\&\qquad+(3\theta^2-9\theta+7)s\mathbb{E}_{\theta,\varepsilon}(s)+(1-\theta)^3\mathbb{E}_{\theta+1,\varepsilon}(s)]\\&\leqslant 2\beta\gamma(1-\alpha).\end{aligned}$$

is satisfied then $\mathbb{I}^s_{\theta,\varepsilon}\hbar \in \mathcal{P}^*(\alpha,\beta,\gamma)$.

Corollary 6.7. *Let $\theta > -1$ and $\varepsilon > 0$. If $\hbar \in \mathcal{CK}$ and the inequality*

$$\frac{(1+\beta(1-2\gamma))\varepsilon}{\mathbb{E}_{\theta,\varepsilon}(s)}[s^2\mathbb{E}_{\theta-1,\varepsilon}(s)+(6-3\theta)s\mathbb{E}_{\theta,\varepsilon}(s)+(1-\theta)^2\mathbb{E}_{\theta+1,\varepsilon}(s)]\leqslant 2\beta\gamma(1-\alpha)$$

is satisfied then $\mathbb{I}^s_{\theta,\varepsilon}\hbar \in \mathcal{P}^*(\alpha,\beta,\gamma)$.

7. Conclusions

Making use of the Miller-Ross-type Poisson distribution series $\mathbb{K}^s_{\theta,\varepsilon}$,we find a necessary and sufficient condition for this series to be in a class of analytic functions with negative coefficients. Also, we investigate several inclusion properties of the classes Janowski type close-to-starlike functions, Janowski type close-to-convex functions and Janowski type quasi-convex functions associated with the operator $\mathbb{I}^s_{\theta,\varepsilon}$ defined by the series $\mathbb{K}^s_{\theta,\varepsilon}$. Several corollaries and consequences of the main results are also considered. Following this study, one can obtain new conditions and inclusion relations for analytic functions in different subclasses defined in the disk $\mathbb{U}$ using the series $\mathbb{K}^s_{\theta,\varepsilon}$.

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