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ON A CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY q-SĂLĂGEAN OPERATOR ASSOCIATED WITH OPERATOR ON HILBERT SPACE

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Abstract. Using a Hilbert space operator, we define a new subclass of analytic functions defined by p-valent q-Sălăgean operator and determine coefficient estimates, distortion bounds, radii of close-to-convexity, starlikeness, and convexity for the functions in this class. We also investigate extreme points and the modified Hadamard product.

Keywords: analytic function; coefficient estimates; Hadamard product; Hilbert space operator

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1. Introduction

The sets of real numbers, complex numbers and positive integers are denoted by

$$\mathbb{R} = (-\infty, \infty), \quad \mathbb{C} = \mathbb{C}^* \cup \{0\} \quad \text{and} \quad \mathbb{N} = \{1, 2, 3, \ldots\} = \mathbb{N}_0 \setminus \{0\},$$

respectively.

Let $\mathcal{H}$ be the class of analytic functions in the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}.$$

For two functions $\mathfrak{f}, \mathfrak{g} \in \mathcal{H}$, we say that the function $\mathfrak{f}$ is subordinate to $\mathfrak{g}$ in $\mathbb{D}$, and write

$$\mathfrak{f}(z) \prec \mathfrak{g}(z), \quad z \in \mathbb{D},$$

if there exists a Schwarz function $\Theta \in \mathcal{H}$ with

$$\Theta(0) = 0 \quad \text{and} \quad |\Theta(z)| < 1, \quad z \in \mathbb{D}$$

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such that

$$\mathfrak{f}(z) = \mathfrak{g}(\Theta(z)), \quad z \in \mathbb{D}.$$

It is known that

$$\mathfrak{f}(z) \prec \mathfrak{g}(z), \quad z \in \mathbb{D} \Rightarrow \mathfrak{f}(0) = \mathfrak{g}(0) \quad \text{and} \quad \mathfrak{f}(\mathbb{D}) \subset \mathfrak{g}(\mathbb{D}).$$

Also, we need the following basic definitions of the q-calculus which are used in this paper (see, for details, [5], [6]).

For $0 < q < 1$, the q-number and the q-factorial are defined by

$$[n]_q = \begin{cases} \dfrac{1-q^n}{1-q}, & n \in \mathbb{C}, \\ 1 + q + q^2 + \dots q^{n-1}, & n \in \mathbb{N}, \end{cases}$$

and

$$[n]_q! = \begin{cases} 1, & n = 0, \\ \displaystyle\prod_{r=1}^{n} [r]_q, & n \in \mathbb{N}, \end{cases}$$

respectively. As $q \to 1^-$, $[n]_q \to n$ and $[n]_q! \to n!$.

For a function f defined on a subset of $\mathbb{C}$, Jackson's q-derivative $\partial_q f$ is defined by (see [5], [6])

$$\partial_q f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z}, & z \neq 0, \\ f'(0), & z = 0, \end{cases} \tag{1.1}$$

provided that $f'(0)$ exists. Then for a function $g(z) = z^k$, we have

$$\partial_q(z^k) = [k]_q z^{k-1},$$
$$\lim_{q \to 1^-} (\partial_q(z^k)) = kz^{k-1} = g'(z),$$

where g' is the ordinary derivative.

Let $\mathcal{A}_p(n)$ denote the class of all functions of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \quad p, n \in \mathbb{N}, \ z \in \mathbb{D}, \tag{1.2}$$

which are analytic and p-valent in the open unit disk $\mathbb{D}$. In particular, we set

$$\mathcal{A}_p(1) := \mathcal{A}_p, \quad \mathcal{A}_1(1) = \mathcal{A}_1 := \mathcal{A}.$$

For a function $f \in \mathcal{A}_p(n)$ given by (1.2), we deduce from (1.1) that

$$\tag{1.3} \begin{cases} \partial_q^{(1)} f(z) = [p]_q z^{p-1} + \sum\limits_{k=p+n}^{\infty} [k]_q a_k z^{k-1} =: \partial_q f(z), \\ \partial_q^{(2)} f(z) = [p]_q [p-1]_q z^{p-2} + \sum\limits_{k=p+n}^{\infty} [k]_q [k-1]_q a_k z^{k-2}, \\ \vdots \\ \partial_q^{(p)} f(z) = [p]_q! + \sum\limits_{k=p+n}^{\infty} \dfrac{[k]_q!}{[k-p]_q!} a_k z^{k-p}, \end{cases}$$

where $\partial_q^{(p)} f(z)$ is the pth q-derivative of $f(z)$. It is clear that the jth q-derivative $\partial_q^{(j)} f$ of f is

$$\tag{1.4} \partial_q^{(j)} f(z) = \left(\prod_{r=1}^{j} [p-r+1]_q\right) z^{p-j} + \sum_{k=p+n}^{\infty} \left(\prod_{r=1}^{j} [k-r+1]_q\right) a_k z^{k-j}, \quad j \in \mathbb{N}.$$

For a function $f \in \mathcal{A}_p(n)$ given by (1.2), using Jackson's q-derivative operator $\partial_q f$, El-Qadeem and Mamon [2] introduced the p-valent q-Sălăgean operator $\mathfrak{D}_{p,q}^m \colon \mathcal{A}_p(n) \to \mathcal{A}_p(n)$, as follows:

$$\tag{1.5} \begin{cases} \mathfrak{D}_{p,q}^0 f(z) = f(z), \\ \mathfrak{D}_{p,q}^1 f(z) = \dfrac{z \partial_q f(z)}{[p]_q} =: \mathfrak{D}_{p,q} f(z), \\ \mathfrak{D}_{p,q}^m f(z) = \mathfrak{D}_{p,q}(\mathfrak{D}_{p,q}^{m-1} f(z)), \quad m \in \mathbb{N}. \end{cases}$$

If f is given by (1.2), then by (1.1) and (1.5), we have

$$\tag{1.6} \mathfrak{D}_{p,q}^m f(z) = z^p + \sum_{k=p+n}^{\infty} \left(\frac{[k]_q}{[p]_q}\right)^m a_k z^k, \quad m \in \mathbb{N}_0.$$

Let $\mathcal{T}_p(n)$ denote the subclass of $\mathcal{A}_p(n)$ consisting of functions of the form

$$\tag{1.7} f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k, \quad a_k \geqslant 0.$$

El-Qadeem and Mamon [2] introduced the following subclass of p-valent functions with negative coefficients:

For $-1 \leqslant A < B \leqslant 1$ and $0 \leqslant \alpha < [p]_q$, let $\mathcal{T}_{p,q}(m,n,\alpha;A,B)$ be the subclass of functions $f \in \mathcal{T}_p(n)$ given by (1.7) which satisfy

$$\frac{z\partial_q(\mathfrak{D}^m_{p,q}f(z))}{\mathfrak{D}^m_{p,q}f(z)} \prec ([p]_q-\alpha)\frac{1+Az}{1+Bz}+\alpha, \quad z\in\mathbb{D}. \tag{1.8}$$

We note that the subordination relation (1.8) is equivalent to

$$\left|\frac{z\partial_q(\mathfrak{D}^m_{p,q}f(z))/\mathfrak{D}^m_{p,q}f(z)-[p]_q}{B(z\partial_q(\mathfrak{D}^m_{p,q}f(z))/\mathfrak{D}^m_{p,q}f(z))-[A[p]_q+\alpha(B-A)]}\right| < 1, \quad z\in\mathbb{D}.$$

Let H be a complex Hilbert space and $L(H)$ denote the algebra of all bounded linear operators on H. For a complex-valued function f analytic in a domain E of the complex plain containing the spectrum $\sigma(T)$ of the bounded linear operator T, let $f(T)$ denote the operator on H defined by the Riesz-Dunford integral [1]

$$f(T)=\frac{1}{2\pi \mathrm{i}}\int_C (zI-T)^{-1}f(z)\,\mathrm{d}z,$$

where I is the identity operator on H and C is a positively oriented simple closed rectifiable closed contour containing the spectrum $\sigma(T)$ in the interior domain [3]. The operator $f(T)$ can also be defined by the series

$$f(T)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}T^k$$

which converges in the norm topology.

Such type of work was earlier considered by Ghanim and Darus [4], Joshi et al. [7], Kim et al. [8] and Yu [9].

Definition 1.1. For $0<q<1$, $-1\leqslant A<B\leqslant 1$, $0\leqslant\alpha<[p]_q$, $p,n\in\mathbb{N}$, $m\in\mathbb{N}_0$, a function $f\in\mathcal{T}_p(n)$ given by (1.7) is in the class $\mathcal{S}_{p,q}(m,n;A,B,\alpha;T)$ if

$$\|T\partial_q(\mathfrak{D}^m_{p,q}f(T))-[p]_q\mathfrak{D}^m_{p,q}f(T)\| < \|BT\partial_q(\mathfrak{D}^m_{p,q}f(T))-[A[p]_q+\alpha(B-A)]\mathfrak{D}^m_{p,q}f(T)\|$$

for all operators T with $\|T\|<1$ and $T\neq\Theta$ (Θ is the zero operator on H).

2. Coefficient bounds

Theorem 2.1. *A function* $f \in \mathcal{T}_p(n)$ *given by* (1.7) *is in the class* $\mathcal{S}_{p,q}(m,n;A,B,\alpha;T)$ *for all proper contractions* T *with* $T \neq \Theta$ *if and only if*

$$\sum_{k=p+n}^{\infty}\left(\frac{[k]_q}{[p]_q}\right)^m\big\{[k]_q(1+B)-[p]_q(1+A)-\alpha(B-A)\big\}a_k \leqslant ([p]_q-\alpha)(B-A). \tag{2.1}$$

The result is sharp for the function

$$f(z)=z^p-\frac{([p]_q-\alpha)(B-A)}{([k]_q/[p]_q)^m\{[k]_q(1+B)-[p]_q(1+A)-\alpha(B-A)\}}z^k, \quad k\geqslant p+n. \tag{2.2}$$

P r o o f. Assume that (2.1) holds. Then

$$\begin{aligned}
&\|T\partial_q(\mathfrak{D}^m_{p,q}f(T))-[p]_q\mathfrak{D}^m_{p,q}f(T)\| \\
&\quad -\|BT\partial_q(\mathfrak{D}^m_{p,q}f(T))-[A[p]_q+\alpha(B-A)]\mathfrak{D}^m_{p,q}f(T)\| \\
&=\left\|\sum_{k=p+n}^{\infty}([k]_q-[p]_q)\left(\frac{[k]_q}{[p]_q}\right)^m a_kT^k\right\| \\
&\quad -\left\|([p]_q-\alpha)(B-A)T^p-\sum_{k=p+n}^{\infty}(B[k]_q-A[p]_q-\alpha(B-A))\left(\frac{[k]_q}{[p]_q}\right)^m a_kT^k\right\| \\
&\leqslant \sum_{k=p+n}^{\infty}\big\{[k]_q(1+B)-[p]_q(1+A)-\alpha(B-A)\big\}\left(\frac{[k]_q}{[p]_q}\right)^m a_k-([p]_q-\alpha)(B-A) \\
&\leqslant 0
\end{aligned}$$

and hence $f \in \mathcal{T}_p(n)$ is in the class $\mathcal{S}_{p,q}(m,n;A,B,\alpha;T)$.

Conversely, let $f \in \mathcal{S}_{p,q}(m,n;A,B,\alpha;T)$, that is,

$$\|T\partial_q(\mathfrak{D}^m_{p,q}f(T))-[p]_q\mathfrak{D}^m_{p,q}f(T)\| < \|BT\partial_q(\mathfrak{D}^m_{p,q}f(T))-[A[p]_q+\alpha(B-A)]\mathfrak{D}^m_{p,q}f(T)\|.$$

From this inequality, it is obtained that

$$\begin{aligned}
&\left\|\sum_{k=p+n}^{\infty}([k]_q-[p]_q)\left(\frac{[k]_q}{[p]_q}\right)^m a_kT^k\right\| \\
&\quad <\left\|([p]_q-\alpha)(B-A)T^p-\sum_{k=p+n}^{\infty}(B[k]_q-A[p]_q-\alpha(B-A))\left(\frac{[k]_q}{[p]_q}\right)^m a_kT^k\right\|.
\end{aligned}$$

By choosing $T=rI$ $(0<r<1)$ in above inequality, we get

$$\frac{\sum_{k=p+n}^{\infty}([k]_q-[p]_q)([k]_q/[p]_q)^m a_kr^k}{([p]_q-\alpha)(B-A)r^p-\sum_{k=p+n}^{\infty}(B[k]_q-A[p]_q-\alpha(B-A))([k]_q/[p]_q)^m a_kr^k}<1.$$

As $r\to 1^-$, (2.1) is obtained. □

Corollary 2.1. *If a function $f \in \mathcal{T}_p(n)$ given by* (1.7) *is in the class* $\mathcal{S}_{p,q}(m,n;A,B,\alpha;T)$, *then*

$$a_k \leqslant \frac{([p]_q-\alpha)(B-A)}{([k]_q/[p]_q)^m\{[k]_q(1+B)-[p]_q(1+A)-\alpha(B-A)\}}, \quad k \geqslant p+n.$$

The result is sharp for the function f of the form (2.2).

Theorem 2.2. *The class $\mathcal{S}_{p,q}(m,n;A,B,\alpha;T)$ is closed under convex combination.*

P r o o f. Let the functions

$$f(z)=z^p-\sum_{k=p+n}^{\infty} a_k z^k \quad \text{and} \quad g(z)=z^p-\sum_{k=p+n}^{\infty} b_k z^k, \quad a_k, b_k \geqslant 0 \tag{2.3}$$

be in the class $\mathcal{S}_{p,q}(m,n;A,B,\alpha;T)$. Then, by Theorem 2.1, we have

$$\sum_{k=p+n}^{\infty}\left(\frac{[k]_q}{[p]_q}\right)^m\{[k]_q(1+B)-[p]_q(1+A)-\alpha(B-A)\}a_k \leqslant ([p]_q-\alpha)(B-A),$$
$$\sum_{k=p+n}^{\infty}\left(\frac{[k]_q}{[p]_q}\right)^m\{[k]_q(1+B)-[p]_q(1+A)-\alpha(B-A)\}b_k \leqslant ([p]_q-\alpha)(B-A).$$

For $0 \leqslant \tau \leqslant 1$, define the function φ as

$$\varphi(z)=\tau f(z)+(1-\tau)g(z).$$

Then, we get

$$\varphi(z)=z^p-\sum_{k=p+n}^{\infty}[\tau a_k+(1-\tau)b_k]z^k, \quad \tau a_k+(1-\tau)b_k \geqslant 0.$$

Now, we obtain

$$\begin{aligned}
&\sum_{k=p+n}^{\infty}\left(\frac{[k]_q}{[p]_q}\right)^m\{[k]_q(1+B)-[p]_q(1+A)-\alpha(B-A)\}[\tau a_k+(1-\tau)b_k]\\
&\quad=\tau\sum_{k=p+n}^{\infty}\left(\frac{[k]_q}{[p]_q}\right)^m\{[k]_q(1+B)-[p]_q(1+A)-\alpha(B-A)\}a_k\\
&\qquad+(1-\tau)\sum_{k=p+n}^{\infty}\left(\frac{[k]_q}{[p]_q}\right)^m\{[k]_q(1+B)-[p]_q(1+A)-\alpha(B-A)\}b_k\\
&\quad\leqslant \tau([p]_q-\alpha)(B-A)+(1-\tau)([p]_q-\alpha)(B-A)\\
&\quad=([p]_q-\alpha)(B-A).
\end{aligned}$$

So, $\varphi \in \mathcal{S}_{p,q}(m,n;A,B,\alpha;T)$. □

3. Hadamard product

Next for functions $f, g \in \mathcal{T}_p(n)$ defined by (2.3), we introduce the modified Hadamard product

$$f * g(z) = z^p - \sum_{k=p+n}^{\infty} a_k b_k z^k, \quad a_k, b_k \geqslant 0.$$

Theorem 3.1. *For functions $f, g \in \mathcal{T}_p(n)$ defined by* (2.3)*, let $f, g \in \mathcal{S}_{p,q}(m, n; A, B, \alpha; T)$. Then the Hadamard product $f * g \in \mathcal{S}_{p,q}(m, n; A, B, \eta; T)$, where*

$$\eta \leqslant [p]_q - \frac{([k]_q - [p]_q)([p]_q - \alpha)^2 (B-A)(1+B)}{([k]_q/[p]_q)^m \{[k]_q(1+B) - [p]_q(1+A) - \alpha(B-A)\}^2 - ([p]_q - \alpha)^2 (B-A)^2}.$$

Proof. Let the functions f and g belong to the class $\mathcal{S}_{p,q}(m, n; A, B, \alpha; T)$. From Theorem 2.1, we have

$$\sum_{k=p+n}^{\infty} \left(\frac{[k]_q}{[p]_q}\right)^m \frac{\{[k]_q(1+B) - [p]_q(1+A) - \alpha(B-A)\}}{([p]_q - \alpha)(B-A)} a_k \leqslant 1, \tag{3.1}$$

and

$$\sum_{k=p+n}^{\infty} \left(\frac{[k]_q}{[p]_q}\right)^m \frac{\{[k]_q(1+B) - [p]_q(1+A) - \alpha(B-A)\}}{([p]_q - \alpha)(B-A)} b_k \leqslant 1. \tag{3.2}$$

We need to find the largest η such that

$$\sum_{k=p+n}^{\infty} \left(\frac{[k]_q}{[p]_q}\right)^m \frac{\{[k]_q(1+B) - [p]_q(1+A) - \eta(B-A)\}}{([p]_q - \eta)(B-A)} a_k b_k \leqslant 1.$$

From (3.1) and (3.2) we find, by means of the Cauchy-Schwarz inequality, that

$$\sum_{k=p+n}^{\infty} \left(\frac{[k]_q}{[p]_q}\right)^m \frac{\{[k]_q(1+B) - [p]_q(1+A) - \alpha(B-A)\}}{([p]_q - \alpha)(B-A)} \sqrt{a_k b_k} \leqslant 1. \tag{3.3}$$

Thus it is enough to show that

$$\left(\frac{[k]_q}{[p]_q}\right)^m \frac{\{[k]_q(1+B) - [p]_q(1+A) - \eta(B-A)\}}{([p]_q - \eta)(B-A)} a_k b_k \leqslant \left(\frac{[k]_q}{[p]_q}\right)^m \frac{\{[k]_q(1+B) - [p]_q(1+A) - \alpha(B-A)\}}{([p]_q - \alpha)(B-A)} \sqrt{a_k b_k},$$

or, equivalently, that

$$\sqrt{a_k b_k} \leqslant \frac{([p]_q - \eta)\{[k]_q(1+B) - [p]_q(1+A) - \alpha(B-A)\}}{([p]_q - \alpha)\{[k]_q(1+B) - [p]_q(1+A) - \eta(B-A)\}}. \tag{3.4}$$

On the other hand, from (3.3) we have

$$\sqrt{a_k b_k} \leqslant \frac{([p]_q - \alpha)(B-A)}{([k]_q/[p]_q)^m\{[k]_q(1+B) - [p]_q(1+A) - \alpha(B-A)\}}. \tag{3.5}$$

Therefore in view of (3.4) and (3.5) it is enough to show that

$$\frac{([p]_q - \alpha)(B-A)}{([k]_q/[p]_q)^m\{[k]_q(1+B) - [p]_q(1+A) - \alpha(B-A)\}} \leqslant \frac{([p]_q - \eta)\{[k]_q(1+B) - [p]_q(1+A) - \alpha(B-A)\}}{([p]_q - \alpha)\{[k]_q(1+B) - [p]_q(1+A) - \eta(B-A)\}}$$

which simplifies to

$$\eta \leqslant [p]_q - \frac{([k]_q - [p]_q)([p]_q - \alpha)^2(B-A)(1+B)}{([k]_q/[p]_q)^m\{[k]_q(1+B) - [p]_q(1+A) - \alpha(B-A)\}^2 - ([p]_q - \alpha)^2(B-A)^2}.$$

□

4. Distortion theorem

Theorem 4.1. *If a function $f \in \mathcal{T}_p(n)$ given by (1.7) is in the class $\mathcal{S}_{p,q}(m,n;A,B,\alpha;T)$ for $\|T\| < 1$ with $T \neq \Theta$, then*

$$\begin{aligned}
&\left\{\prod_{r=1}^{j}[p-r+1]_q - \frac{[p]_q^m([p]_q-\alpha)(B-A)\prod_{r=1}^{j}[p+n-r+1]_q}{[p+n]_q^m\{[p+n]_q(1+B)-[p]_q(1+A)-\alpha(B-A)\}}\|T\|^n\right\}\|T\|^{p-j} \\
&\leqslant \|\partial_q^{(j)} f(T)\| \\
&\leqslant \left\{\prod_{r=1}^{j}[p-r+1]_q + \frac{[p]_q^m([p]_q-\alpha)(B-A)\prod_{r=1}^{j}[p+n-r+1]_q}{[p+n]_q^m\{[p+n]_q(1+B)-[p]_q(1+A)-\alpha(B-A)\}}\|T\|^n\right\}\|T\|^{p-j},
\end{aligned}$$

where $\partial_q^{(j)}$ is given by (1.4). The result is sharp for the function

$$f(z) = z^p - \frac{([p]_q - \alpha)(B-A)}{([p+n]_q/[p]_q)^m\{[p+n]_q(1+B) - [p]_q(1+A) - \alpha(B-A)\}} z^{p+n}.$$

P r o o f. According to Theorem 2.1, we obtain

$$\sum_{k=p+n}^{\infty} a_k \leqslant \frac{([p]_q-\alpha)(B-A)}{([p+n]_q/[p]_q)^m\{[p+n]_q(1+B)-[p]_q(1+A)-\alpha(B-A)\}}. \tag{4.1}$$

Thus the desired results follow from (1.4) and (4.1). □

5. Extreme points

Theorem 5.1. *Let*

$$f_{p+n-1}(z)=z^p$$

and

(5.1)

$$f_k(z)=z^p-\frac{([p]_q-\alpha)(B-A)}{([k]_q/[p]_q)^m\{[k]_q(1+B)-[p]_q(1+A)-\alpha(B-A)\}}z^k, \quad k\geqslant p+n.$$

Then $f\in\mathcal{S}_{p,q}(m,n;A,B,\alpha;T)$ if and only if it can be represented in the form

$$f(z)=\sum_{k=p+n-1}^{\infty}\gamma_k f_k(z) \quad \Bigl(\gamma_k\geqslant 0, \ \sum_{k=p+n-1}^{\infty}\gamma_k=1\Bigr).$$

P r o o f. Assume that $f(z)=\sum_{k=p+n-1}^{\infty}\gamma_k f_k(z)$. Then, we have

$$\begin{aligned}
&\sum_{k=p+n-1}^{\infty}\gamma_k f_k(z)\\
&=\gamma_{p+n-1}f_{p+n-1}(z)+\sum_{k=p+n}^{\infty}\gamma_k f_k(z)\\
&=\Bigl(1-\sum_{k=p+n}^{\infty}\gamma_k\Bigr)z^p\\
&\quad+\sum_{k=p+n}^{\infty}\gamma_k\Bigl(z^p-\frac{([p]_q-\alpha)(B-A)}{([k]_q/[p]_q)^m\{[k]_q(1+B)-[p]_q(1+A)-\alpha(B-A)\}}z^k\Bigr)\\
&=z^p-\sum_{k=p+n}^{\infty}\gamma_k\frac{([p]_q-\alpha)(B-A)}{([k]_q/[p]_q)^m\{[k]_q(1+B)-[p]_q(1+A)-\alpha(B-A)\}}z^k.
\end{aligned}$$

Therefore,

$$\sum_{k=p+n}^{\infty} \frac{([k]_q/[p]_q)^m\{[k]_q(1+B)-[p]_q(1+A)-\alpha(B-A)\}\gamma_k([p]_q-\alpha)(B-A)}{([k]_q/[p]_q)^m\{[k]_q(1+B)-[p]_q(1+A)-\alpha(B-A)\}}$$

$$\begin{aligned}&= ([p]_q-\alpha)(B-A)\sum_{k=p+n}^{\infty}\gamma_k\\ &= ([p]_q-\alpha)(B-A)(1-\gamma_{p+n-1})\\ &\leqslant ([p]_q-\alpha)(B-A).\end{aligned}$$

Hence, by Theorem 2.1, $f \in \mathcal{S}_{p,q}(m,n;A,B,\alpha;T)$.

Conversely, suppose that $f \in \mathcal{S}_{p,q}(m,n;A,B,\alpha;T)$. Since, by Corollary 2.1,

$$a_k \leqslant \frac{([p]_q-\alpha)(B-A)}{([k]_q/[p]_q)^m\{[k]_q(1+B)-[p]_q(1+A)-\alpha(B-A)\}}, \quad k \geqslant p+n,$$

setting

$$\gamma_k = \frac{([k]_q/[p]_q)^m\{[k]_q(1+B)-[p]_q(1+A)-\alpha(B-A)\}}{([p]_q-\alpha)(B-A)}a_k, \quad k \geqslant p+n$$

and $\gamma_{p+n-1} = 1 - \sum_{k=p+n}^{\infty}\gamma_k$, we obtain

$$f(z) = \gamma_{p+n-1}f_{p+n-1}(z) + \sum_{k=p+n}^{\infty}\gamma_k f_k(z).$$

This completes the proof of the theorem. □

6. Radii of close-to-convexity, starlikeness, and convexity

We now find the radii of close-to-convexity, starlikeness, and convexity for functions belonging to the class $\mathcal{S}_{p,q}(m,n;A,B,\alpha;T)$.

Theorem 6.1. *Let the function $f \in \mathcal{T}_p(n)$ defined by (1.7) belongs to the class $\mathcal{S}_{p,q}(m,n;A,B,\alpha;T)$. Then f is p-valently close-to-convex of order ϱ $(0 \leqslant \varrho < [p]_q)$ in the disk $\|T\| < r_1$, where*

$$r_1 = \inf_k \left\{ \frac{([p]_q-\varrho)[k]_q^{m-1}\{[k]_q(1+B)-[p]_q(1+A)-\alpha(B-A)\}}{[p]_q^m([p]_q-\alpha)(B-A)} \right\}^{1/(k-p)}, \ k \geqslant p+n.$$

The result is sharp for the extremal function f given by (2.2).

P r o o f. It is sufficient to show that

$$\|T^{1-p}\partial_q f(T) - [p]_q\| < [p]_q - \varrho. \tag{6.1}$$

Since $f \in \mathcal{S}_{p,q}(m,n;A,B,\alpha;T)$, by Theorem 2.1 we have

$$\sum_{k=p+n}^{\infty}\left(\frac{[k]_q}{[p]_q}\right)^m \frac{\{[k]_q(1+B)-[p]_q(1+A)-\alpha(B-A)\}}{([p]_q-\alpha)(B-A)}a_k \leqslant 1.$$

So the inequality

$$\|T^{1-p}\partial_q f(T) - [p]_q\| = \|\sum_{k=p+n}^{\infty}[k]_q a_k T^{k-p}\| \leqslant \sum_{k=p+n}^{\infty}[k]_q a_k \|T\|^{k-p} < [p]_q - \varrho$$

holds true if

$$\frac{[k]_q\|T\|^{k-p}}{[p]_q-\varrho} \leqslant \left(\frac{[k]_q}{[p]_q}\right)^m \frac{\{[k]_q(1+B)-[p]_q(1+A)-\alpha(B-A)\}}{([p]_q-\alpha)(B-A)}.$$

Then, (6.1) holds true if

$$\|T\|^{k-p} \leqslant \frac{([p]_q-\varrho)[k]_q^{m-1}\{[k]_q(1+B)-[p]_q(1+A)-\alpha(B-A)\}}{[p]_q^m([p]_q-\alpha)(B-A)}, \quad k \geqslant p+n,$$

which yields the close-to-convexity of the family and completes the proof. □

Theorem 6.2. *Let the function $f \in \mathcal{T}_p(n)$ defined by (1.7) belongs to the class $\mathcal{S}_{p,q}(m,n;A,B,\alpha;T)$. Then f is p-valently starlike of order ϱ $(0 \leqslant \varrho < [p]_q)$ in the disk $\|T\| < r_2$, where*

$$r_2 = \inf_k\left\{\frac{([p]_q-\varrho)[k]_q^m\{[k]_q(1+B)-[p]_q(1+A)-\alpha(B-A)\}}{([k]_q-\varrho)[p]_q^m([p]_q-\alpha)(B-A)}\right\}^{1/(k-p)}, \quad k \geqslant p+n.$$

The result is sharp for the extremal function f given by (2.2).

P r o o f. By using the technique employed in the proof of Theorem 6.1, we can show that

$$\left\|\frac{T\partial_q f(T)}{f(T)} - [p]_q\right\| < [p]_q - \varrho$$

for $\|T\| < r_2$, and prove that the assertion of the theorem is true. □

Theorem 6.3. *Let the function $f \in \mathcal{T}_p(n)$ defined by* (1.7) *belongs to the class $\mathcal{S}_{p,q}(m,n;A,B,\alpha;T)$. Then f is p-valently convex of order ϱ $(0 \leqslant \varrho < [p]_q)$ in the disk $\|T\| < r_3$, where*

$$r_3 = \inf_k \left\{ \frac{([p]_q - \varrho)[k]_q^{m-1}\{[k]_q(1+B) - [p]_q(1+A) - \alpha(B-A)\}}{([k]_q - \varrho)[p]_q^{m-1}([p]_q - \alpha)(B-A)} \right\}^{1/(k-p)}, \quad k \geqslant p+n$$

The result is sharp for the extremal function f given by (2.2).

P r o o f. By using the technique employed in the proof of Theorem 6.1, we can show that

$$\left\| \frac{\partial_q(T\partial_q f(T))}{\partial_q f(T)} - [p]_q \right\| < [p]_q - \varrho$$

for $\|T\| < r_3$, and prove that the assertion of the theorem is true. □

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