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# A NOTE ON KURZWEIL-HENSTOCK'S ANTICIPATING NON-STOCHASTIC INTEGRAL

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*Abstract.* Motivated by the study of anticipating stochastic integrals using Kurzweil-Henstock approach, we use anticipating interval-point pairs (with the tag as the right-end point of the interval) in studying non-stochastic integral, which we call the Kurzweil-Henstock anticipating non-stochastic integral. We prove the integration-by-parts and integration-by-substitution results, the convergence theorems using our new setting. Using the convergence theorems, we show that the Kurzweil-Henstock's anticipating non-stochastic integral is equivalent to the Lebesgue integral.

*Keywords*: Kurzweil-Henstock integral; anticipative integral; non-stochastic

*MSC 2020*: 60H05

## 1. Introduction

The Riemann integral was perhaps the first rigorous definition of the integral [8]. The theory of Riemann integration is well-known. It is inadequate for many theoretical purposes as it cannot be used to study highly oscillatory integrands and integrators.

The Lebesgue integral, which remedied the technical deficits in Riemann integration, was introduced by Henri Lebesgue in 1906 (see [4]). This integral was able to handle more irregular functions and provide more careful approximation techniques. However, Lebesgue integral is technically involved and a considerable amount of measure theory is required even to define the integral.

In the 1950s, Jaroslav Kurzweil and Ralph Henstock independently introduced another integral by a slight modification of the classical Riemann integral (see [1]). They used non-uniform meshes instead of uniform meshes as in the usual Riemann approach. This modification of the classical Riemann approach led to integrals which

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are more general than both the Riemann and Lebesgue integral. It preserves the intuitive approach of the Riemann integral.

The use of non-uniform mesh in Kurzweil-Henstock integral has allowed for highly oscillating points to be encompassed. In the study of stochastic integral using Kurzweil-Henstock approach, the tag in the interval-point pair cannot be any point in the interval. In the case of non-anticipating Itô-stochastic integrals the tag must be the left point of the interval, [3], [9]–[15]. Lim and Toh (see [7]) study the theory of non-stochastic integral by restricting the tag to be the left point of the interval, and show that this is equivalent to the Lesbesgue integral.

In this paper, we study non-stochastic integrals using the Kurzweil-Henstock approach by considering the tag to be the right-hand point of the interval. This is motivated by the study of anticipating stochastic integrals using the Kurzweil-Henstock approach used by Chew et al. (see [2]).

## 2. Some definitions

We consider throughout this paper functions and integrals which are defined on a closed and bounded interval $[a, b]$.

**Definition 2.1.** (1) A collection of sub-intervals $[x_{i-1}, x_i]$, for all $i = 1, 2, \dots, n$, where $a = x_0 < x_1 < x_2 < \dots < x_n = b$, is a (full) partition of $[a, b]$.

(2) A collection of nonoverlapping subintervals $\{([u, v], \xi)\}$ of $[a, b]$, that is, nonoverlapping subintervals such that $\cup[u, v] \subset [a, b]$ is said to be a partial division of $[a, b]$.

**Definition 2.2.** Let $\delta$ be a positive function on $[a, b]$. A full division $D = \{([u, v], \xi)\}$ of $[a, b]$ is said to be $\delta$-fine Henstock division if for all interval point pairs $([u, v], \xi)$, $\xi \in [u, v] \subset [\xi - \delta(\xi), \xi + \delta(\xi)]$.

Note that Definition 2.2 is used in Henstock's integration theory. Such a full division exists by continuous bisection (see [6]) or a consequence of Heine-Borel Theorem (see [5]).

**Definition 2.3.** A collection of interval-point pairs $D = \{([u, \xi], \xi)\}$ is said to be a partial anticipating (or $\mathcal{PA}$) $\delta$-fine division if for all interval point pairs $([u, \xi], \xi)$, $\xi \in [u, \xi] \subset [\xi - \delta(\xi), \xi]$.

In the study of stochastic integrals, Toh and Chew in [3], [9]–[15] used a belated partial division in which the tag $\xi$ was the left-hand point of the interval. In this study, we adopt the case when the tag is the right-hand point, aligned to the approach used in [2].

We noted that a full anticipating $\delta$-fine division of $[a,b]$ may not exist. In considering the interval $[a,b]$, let $\delta(\xi) = (\xi - a)/2 > 0$. Then any finite collection of intervals $(\xi_i - \delta(\xi_i), \xi_i] = ((\xi_i + a)/2, \xi_i]$ will leave some parts near to $x = a$ uncovered. Hence, for this chosen $\delta > 0$, we can only have a partial division. From Vitali Covering Theorem, a $\mathcal{PA}$ $\delta$-fine division of $[a,b]$ that misses out the interval $[a,b]$ by any arbitrary small part exists. We have:

**Definition 2.4.** Let $\delta$ be a positive function of $[a,b]$ and $\eta > 0$ be a positive number. A collection of interval point pairs $D = \{([u,\xi],\xi)\}$ is said to be a $\mathcal{PA}$ $(\delta,\eta)$-fine division of $[a,b]$ if:

(1) the set of intervals $\{([u,\xi],\xi)\}$ is non-overlapping and $[u,\xi] \subset [a,b]$,

(2) for each interval point pair $([u,\xi],\xi)$, $[u,\xi] \subseteq (\xi - \delta(\xi), \xi]$, and

(3) $\Big|[a,b] \setminus \bigcup_D [u,\xi]\Big| < \eta$.

## 3. Kurzweil-Henstock anticipating non-stochastic Stieltjes integral

**Definition 3.1** (KHANS integral)**.** A real-valued function $f$ is said to be Kurzweil-Henstock anticipative non-stochastic Stieltjes (KHANS) integrable to $A$ on $[a,b]$ with respect to the function $g$ if for every $\varepsilon > 0$ there exists a positive function $\delta > 0$ and a positive number $\eta > 0$ such that for any $\mathcal{PA}$ $(\delta,\eta)$-fine division $D = \{([u,\xi],\xi)\}$ of $[a,b]$, we have

$$\left|(D)\sum f(\xi)[g(\xi) - g(u)] - A\right| < \varepsilon,$$

where $(D)\sum f(\xi)[g(\xi) - g(u)]$ denotes the Riemann sum of $f$ with respect to the function $g$ over the division $D$ on the interval $[a,b]$.

Here, the function $f$ is known as the integrand and $g$ the integrator. In the event that the integrator $g$ can be understood from the context, we may simply say $f$ is KHANS integrable on $[a,b]$ (with respect to $g$). Note that the part of the interval $[a,b]$ that is not covered by $D$ has total measure of less than $\eta$, that is, $\Big|[a,b] \setminus \bigcup_D [u,\xi]\Big| < \eta$, for a given $\eta > 0$.

Note 3.1. For the special case when the integrator $g(x) \equiv x$, we name the integral Kurzweil-Henstock anticipative non-stochastic or KHAN integral.

**Definition 3.2** (KHAN integral)**.** A real-valued function $f$ is said to be Kurzweil-Henstock anticipative non-stochastic (KHAN) integrable to $A$ on $[a,b]$ if for every $\varepsilon > 0$ there exists a positive function $\delta > 0$ and a positive number $\eta > 0$

such that for any $\mathcal{PA}$ $(\delta,\eta)$-fine division $D = \{([u,\xi],\xi)\}$ of $[a,b]$ we have

$$\Bigl|(D)\sum f(\xi)(\xi-u) - A\Bigr| < \varepsilon.$$

**Theorem 3.1** (Uniqueness of KHANS integral). *If $f$ is a* KHANS *integrable function on $[a,b]$ (with respect to $g$), then the integral on $[a,b]$ is unique. That is, if both $A$ and $B$ are two values of the* KHANS *integrals of $f$ on $[a,b]$, then $A = B$.*

Since the KHANS integral of $f$ on $[a,b]$ is unique, we can use the notion KHANS $\int_a^b f(t)\,\mathrm{d}g(t)$ or simply $\int_a^b f\,\mathrm{d}g$ to denote the KHANS integral of $f$ on $[a,b]$ with respect to $g$ if the integral exists and if there is no ambiguity on the type of integral being referred to.

**Theorem 3.2** (Integrability of sum and difference). *Suppose $f$ and $h$ are* KHANS *integrable functions defined on $[a,b]$. Then $f\pm h$ is* KHANS *integrable on $[a,b]$. Furthermore,*

$$\int_a^b (f(t)\pm h(t))\,\mathrm{d}g(t) = \int_a^b f(t)\,\mathrm{d}g(t) \pm \int_a^b h(t)\,\mathrm{d}g(t).$$

**Theorem 3.3** (Integrability of scalar multiple). *Suppose $f$ is a* KHANS *integrable function defined on $[a,b]$ and $k\in\mathbb{R}$. Then $kf$ is* KHANS *integrable on $[a,b]$. Furthermore,*

$$\int_a^b kf(t)\,\mathrm{d}g(t) = k\int_a^b f(t)\,\mathrm{d}g(t).$$

Note that the proofs of the above theorems are typical results of integration theory. It is instructive for readers to go through the proof following classical results.

**Theorem 3.4** (Cauchy criterion). *A function $f$ is* KHANS *integrable on $[a,b]$ if and only if for every $\varepsilon > 0$ there exists a positive function $\delta > 0$ and a positive number $\eta > 0$ such that for any two $\mathcal{PA}$ $(\delta,\eta)$-fine divisions $D_1$ and $D_2$ we have*

$$\Bigl|(D_1)\sum f(\xi)[g(\xi)-g(u)] - (D_2)\sum f(\xi)[g(\xi)-g(u)]\Bigr| < \varepsilon.$$

P r o o f. Necessity follows from triangle inequality. We only need to prove the sufficiency. For any $\varepsilon > 0$, there exists a positive function $\delta > 0$ and a positive number $\eta > 0$ such that for any two $\mathcal{PA}$ $(\delta,\eta)$-fine divisions $D_1$ and $D_2$ we have

$$\Bigl|(D_1)\sum f(\xi)[g(\xi)-g(u)] - (D_2)\sum f(\xi)[g(\xi)-g(u)]\Bigr| < \frac{\varepsilon}{2}.$$

Take $\varepsilon/2 = 1/k$ for a positive integer $k$. Define a sequence of positive functions $\delta_1 > \delta_2 > \delta_3 > \ldots$ and a sequence of positive numbers $\eta_1 > \eta_2 > \eta_3 > \ldots$ For each

$k = 1, 2, 3, \ldots$, let $S_k$ denote $(D_k)\sum f(\xi)[g(\xi) - g(u)]$, where $D_k$ is a $(\delta_k, \eta_k)$-fine division. If $p > q$, then the $(\delta_p, \eta_p)$-fine division $S_p$ is also $(\delta_q, \eta_q)$-fine. Since both $S_p$ and $S_q$ are $(\delta_q, \eta_q)$-fine, $|S_p - S_q| < \varepsilon/2$. Hence, $\{S_p\}$ is a Cauchy sequence in $\mathbb{R}$. Let $S$ be the limit of this sequence. So there exists $N > 0$ such that for all $k \geqslant N$ we have $|S - S_k| < \varepsilon/2$. Let $D$ be any $\mathcal{PA}$ $(\delta_n, \eta_n)$-fine division with $n < k$. Then

$$\begin{aligned}\Big|S - (D)\sum f(\xi)[g(\xi) - g(u)]\Big| &= \Big|S - S_k + S_k - (D)\sum f(\xi)[g(\xi) - g(u)]\Big| \\ &\leqslant |S - S_k| + \Big|S_k - (D)\sum f(\xi)[g(\xi) - g(u)]\Big| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

showing that $f$ is KHANS integrable. □

**Theorem 3.5** (Integrability on a subinterval). *If $f$ is* KHANS *integrable on $[a, b]$, then it is* KHANS *integrable on any subinterval $[c, d]$ of $[a, b]$.*

P r o o f. For $\varepsilon > 0$, let $D_1$ and $D_2$ be any two $\mathcal{PA}$ $(\delta, \eta)$-fine divisions of $[c, d]$ and denote the Riemann sums of $f$ over $D_1$ and $D_2$ by $S_1$ and $S_2$, respectively, corresponding to Theorem 3.4 for $\varepsilon/2$. Similarly, take another $\mathcal{PA}$ $(\delta, \eta)$-fine division $D_3$ of $[a, c] \cup [d, b]$ and denote the corresponding Riemann sum by $S_3$. Then $D_1 \cup D_3$ is a $\mathcal{PA}$ $(\delta, \eta)$-fine division of $[a, b]$. By the Cauchy criterion (Theorem 3.4) on $[a, b]$,

$$|S_1 - S_2| = |(S_1 + S_3) - (S_2 + S_3)| < \varepsilon.$$

Hence, the proof is completed by applying the Cauchy Criterion with the interval $[a, b]$ being replaced by $[c, d]$. □

**Theorem 3.6** (Henstock's Lemma). *Suppose $f$ is* KHANS *integrable on $[a, b]$. Then, given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $\mathcal{PA}$ $\delta$-fine division $D = \{([u, \xi], \xi)\}$ on $[a, b]$, we have*

$$\left|(D)\sum\left\{f(\xi)[g(\xi) - g(u)] - \int_u^{\xi} f \,\mathrm{d}g\right\}\right| < \varepsilon.$$

P r o o f. Since $f$ is KHANS integrable on $[a, b]$, given $\varepsilon > 0$, there exists a positive function $\delta > 0$ and a positive number $\eta > 0$ such that for any $\mathcal{PA}$ $(\delta, \eta)$-fine division $D$,

$$\left|(D)\sum f(\xi)[g(\xi) - g(u)] - \int_a^b f \,\mathrm{d}g\right| < \frac{\varepsilon}{2}.$$

Let $D = \{([u, \xi], \xi)\}$ be a collection of $\delta$-fine partial divisions. Let the closure of $[a, b] \setminus \bigcup_D [u, \xi]$ be $\bigcup_{i=1}^{m} [a_i, b_i]$. By Theorem 3.5, $f$ is KHANS integrable on each $[a_i, b_i]$.

Given $\varepsilon > 0$, choose a positive function $\delta_i(\xi) > 0$ and $\eta_i > 0$ such that for any $\mathcal{PA}$ $(\delta_i, \eta_i)$-fine division $D$, we have

$$\left|(D_i)\sum f(\xi)[g(\xi_i) - g(u_i)] - \int_{a_i}^{b_i} f\,\mathrm{d}g\right| < \frac{\varepsilon}{2m}$$

for each $i = 1, 2, 3, \ldots, m$.

Impose the condition that $\delta_i \leqslant \delta$ for all $i = 1, 2, 3, \ldots, m$, and that $\sum \eta_i \leqslant \eta$. Then $D \bigcup_{i=1}^{m} D_i$ is a $\mathcal{PA}$ $(\delta, \eta)$-fine division of $[a, b]$. Hence,

$$\begin{aligned}
&\left|(D)\sum\left\{f(\xi)[g(\xi) - g(u)] - \int_u^{\xi} f\,\mathrm{d}g\right\}\right| \\
&\quad= \left|\left(D\bigcup_{i=1}^{m} D_i\right)\sum f(\xi)[g(\xi) - g(u)] - \sum_{i=1}^{m}(D_i)\sum f(\xi)[g(\xi) - g(u)]\right. \\
&\qquad\left. - \left(\int_a^b f\,\mathrm{d}g - \sum_{i=1}^{m}\int_{a_i}^{b_i} f\,\mathrm{d}g\right)\right| \\
&\quad\leqslant \left|\left(D\bigcup_{i=1}^{m} D_i\right)\sum f(\xi)[g(\xi) - g(u)] - \int_a^b f\,\mathrm{d}g\right| \\
&\qquad+ \left|\left(\sum_{i=1}^{m}(D_i)\sum f(\xi)[g(\xi) - g(u)] - \sum_{i=1}^{m}\int_{a_i}^{b_i} f\,\mathrm{d}g\right)\right| \\
&\quad= \left|\left(D\bigcup_{i=1}^{m} D_i\right)\sum f(\xi)[g(\xi) - g(u)] - \int_a^b f\,\mathrm{d}g\right| \\
&\qquad+ \sum_{i=1}^{m}\left|\left((D_i)\sum f(\xi)[g(\xi) - g(u)] - \int_{a_i}^{b_i} f\,\mathrm{d}g\right)\right| \\
&\quad\leqslant \frac{\varepsilon}{2} + m\left(\frac{\varepsilon}{2m}\right) = \varepsilon,
\end{aligned}$$

thereby completing the proof. □

**Theorem 3.7** (Strong Henstock's Lemma)**.** *Suppose $f$ is* KHANS *integrable on $[a, b]$. Then, given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $\mathcal{PA}$ $\delta$-fine partial division $D = \{([u, \xi], \xi)\}$ on $[a, b]$,*

$$(D)\sum\left|f(\xi)[g(\xi) - g(u)] - \int_u^{\xi} f\,\mathrm{d}g\right| < \varepsilon.$$

Proof. Suppose $f$ is KHANS integrable on $[a,b]$. We want to show that the above condition is equivalent to that of Henstock's Lemma in Theorem 3.6. We only need to prove the necessity. By Henstock's Lemma, given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $\mathcal{PA}$ $\delta$-fine division $D = \{([u,\xi],\xi)\}$ on $[a,b]$, we have

$$\left|(D)\sum\left\{f(\xi)[g(\xi)-g(u)] - \int_u^\xi f\,\mathrm{d}g\right\}\right| < \varepsilon.$$

Let $D_1$ be those parts $\{([u,\xi],\xi)\}$ in $D$ such that $f(\xi)[g(\xi)-g(u)] - \int_u^\xi f\,\mathrm{d}g > 0$ and $D_2$ be those parts $\{([u,\xi],\xi)\}$ in $D$ such that $f(\xi)[g(\xi)-g(u)] - \int_u^\xi f\,\mathrm{d}g < 0$. Then $D_1$ and $D_2$ are $\mathcal{PA}$ $\delta$-fine divisions such that

$$\left|(D_1)\sum\left\{f(\xi)[g(\xi)-g(u)] - \int_u^\xi f\,\mathrm{d}g\right\}\right| < \varepsilon$$

and

$$\left|(D_2)\sum\left\{f(\xi)[g(\xi)-g(u)] - \int_u^\xi f\,\mathrm{d}g\right\}\right| < \varepsilon.$$

Hence,

$$\begin{aligned}
&(D)\sum\left|f(\xi)[g(\xi)-g(u)] - \int_u^\xi f\,\mathrm{d}g\right| \\
&\quad= (D_1)\sum\left(f(\xi)[g(\xi)-g(u)] - \int_u^\xi f\,\mathrm{d}g\right) \\
&\qquad - (D_2)\sum\left(f(\xi)[g(\xi)-g(u)] - \int_u^\xi f\,\mathrm{d}g\right) \\
&\quad= \left|(D_1)\sum\left(f(\xi)[g(\xi)-g(u)] - \int_u^\xi f\,\mathrm{d}g\right)\right. \\
&\qquad \left. - (D_2)\sum\left(f(\xi)[g(\xi)-g(u)] - \int_u^\xi f\,\mathrm{d}g\right)\right| \\
&\quad\leqslant \left|(D_1)\sum\left(f(\xi)[g(\xi)-g(u)] - \int_u^\xi f\,\mathrm{d}g\right)\right| \\
&\qquad + \left|(D_2)\sum\left(f(\xi)[g(\xi)-g(u)] - \int_u^\xi f\,\mathrm{d}g\right)\right| \\
&\quad< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

thereby completing the proof. □

**Definition 3.3** (Absolute continuity). A function $F\colon [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\sum |F(v) - F(u)| < \varepsilon$ whenever $\{[u,v]\}$ is a collection of subintervals of $[a,b]$ with $\sum |v-u| < \delta$.

**Theorem 3.8** (Small Riemann sum). *Let $f$ be* KHANS *integrable with respect to $g$ on $[a,b]$. Then given $\varepsilon > 0$, there exists a positive function $\delta > 0$ and a positive number $\mu > 0$ such that for any given $\mathcal{PA}$ $\delta$-fine partial division $D = \{([u,\xi],\xi)\}$ of $[a,b]$ with $(D)\sum|\xi - u| < \mu$,*

$$\left|(D)\sum f(\xi)[g(\xi) - g(u)]\right| < \varepsilon.$$

Proof. Given $\varepsilon > 0$, choose a positive function $\delta > 0$ and a positive number $\eta > 0$ such that for any $\mathcal{PA}$ $(\delta,\eta)$-fine division $D_1 = \{([u,\xi],\xi)\}$ we have

$$\left|(D_1)\sum f(\xi)[g(\xi) - g(u)] - \int_a^b f\,\mathrm{d}g\right| < \frac{\varepsilon}{2}.$$

Take a $\mathcal{PA}$ $\delta$-fine partial division $D = \{([u,\xi],\xi)\}$ from $\overline{[a,b] \setminus \bigcup_{D_1}[u,\xi]}$. Since $D_1$ is $(\delta,\eta)$-fine, we have $\left|[a,b] \setminus \bigcup_{D_1}[u,\xi]\right| < \eta$, and hence $(D)\sum|\xi - u| < \eta$. Then $D \cup D_1$ is $(\delta,\eta)$-fine. Hence,

$$\left|(D \cup D_1)\sum f(\xi)[g(\xi) - g(u)] - \int_a^b f\,\mathrm{d}g\right| < \frac{\varepsilon}{2}.$$

By the triangle inequality,

$$\begin{aligned}\left|(D \cup D_1)\sum f(\xi)[g(\xi) - g(u)]\right| &= \left|(D)\sum f(\xi)[g(\xi) - g(u)] - \int_a^b f\,\mathrm{d}g\right.\\ &\quad\left. + \int_a^b f\,\mathrm{d}g - (D_1)\sum f(\xi)[g(\xi) - g(u)]\right|\\ &\quad + \left|\left((D_1)\sum f(\xi)[g(\xi) - g(u)] - \int_a^b f\,\mathrm{d}g\right)\right|\\ &\leqslant \left|(D \cup D_1)\sum f(\xi)[g(\xi) - g(u)] - \int_a^b f\,\mathrm{d}g\right|\\ &\quad + \left|(D_1)\sum f(\xi)[g(\xi) - g(u)] - \int_a^b f\,\mathrm{d}g\right|\\ &\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

thereby completing the proof. □

**Lemma 3.1.** *If $f$ is* KHANS *integrable on the interval $[a,b]$ with respect to $g$, then $F(t) = \int_a^t f\,\mathrm{d}g$ is absolutely continuous on $[a,b]$.*

P r o o f. Given $\varepsilon > 0$, by Theorem 3.8 there exists a positive function $\delta > 0$ and a positive number $\eta > 0$ such that whenever $D_k = \{([u,\xi],\xi)\}$ is a $\mathcal{PA}$ $\delta$-fine division of $[a,b]$ with $(D_k)\sum|\xi-u| < \eta$, we have

$$\left|(D_k)\sum f(\xi)[g(\xi)-g(u)]\right| < \frac{\varepsilon}{2}.$$

Let $\{[a_i,b_i]\}_{i=1}^m$ be a finite collection of disjoint subintervals from $[a,b]$ such that $\sum|b_i-a_i| < \eta$, where $\eta$ is chosen as above. By Theorem 3.5, $f$ is KHANS integrable on each $[a_i,b_i]$, where $i = 1,2,3,\ldots,m$. By Theorem 3.6, on each $[a_i,b_i]$ there exists $\delta_i > 0$ such that whenever $D_i = \{([u,\xi],\xi)\}$ is a $\mathcal{PA}$ $\delta_i$-fine division of $[a_i,b_i]$, we have

$$\left|(D_i)\sum f(\xi)[g(\xi)-g(u)] - \int_{a_i}^{b_i} f\,\mathrm{d}g\right| < \frac{\varepsilon}{2^{i+2}}.$$

Assume that $\delta_i \leqslant \delta$ for each $i = 1,2,3,\ldots,m$. So $D = \bigcup\limits_{i=1}^m D_i$ is a $\delta$-fine partial division with $\left(\bigcup D_i\right)\sum|\xi-u| \leqslant \sum|b_i-a_i| < \eta$. By Theorem 3.8,

$$\left|\left(\bigcup_{i=1}^m D_i\right)\sum f(\xi)[g(\xi)-g(u)]\right| < \frac{\varepsilon}{2}.$$

Consequently,

$$\begin{aligned}\sum_{i=1}^m |[F(b_i)-F(a_i)]| &\leqslant \left|\sum_{i=1}^m\left\{\int_{a_i}^{b_i} f\,\mathrm{d}g - (D_i)\sum f(\xi)[g(\xi)-g(u)]\right\}\right| \\ &\quad + \left|\sum_{i=1}^m\left\{(D_i)\sum f(\xi)[g(\xi)-g(u)]\right\}\right| \\ &\leqslant \sum_{i=1}^m\left|\left\{(D_i)\sum f(\xi)[g(\xi)-g(u)] - \int_{a_i}^{b_i} f\,\mathrm{d}g\right\}\right| \\ &\quad + \left|\left(\bigcup_{i=1}^m D_i\right)\sum f(\xi)[g(\xi)-g(u)]\right| \\ &\leqslant \sum_{i=1}^m \frac{\varepsilon}{2^{i+2}} + \frac{\varepsilon}{2} = \frac{\varepsilon}{4}\sum_{i=1}^m \frac{1}{2^i} + \frac{\varepsilon}{2} \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

thereby completing the proof. □

**Theorem 3.9** (Sequential definition). *The function $f$ is* KHANS *integrable on $[a,b]$ with respect to $g$ if and only if there exists a sequence of positive functions $\{\delta_n\}$ and a sequence of positive numbers $\{\eta_n\}$ such that*

$$A = \lim_{n\to\infty} S(f,g,D_n,\delta_n,\eta_n), \quad \text{where } S(f,g,D_n,\delta_n,\eta_n) = (D_n)\sum f(\xi)[g(\xi)-g(u)].$$

P r o o f. Suppose $f$ is KHANS integrable on $[a,b]$ with respect to $g$, let the sequences of positive function $\delta_n$ and positive numbers $\eta_n$ be such that for any $\mathcal{PA}$ $(\delta_n,\eta_n)$-fine division $D_n$ we have

$$\left|(D_n)\sum f(\xi)[g(\xi)-g(u)]-A\right|<\frac{1}{n}$$

for all $n=1,2,3,\ldots$ Given $\varepsilon>0$, choose an integer $k$ such that $1/k<\varepsilon$, so that

$$\left|(D_k)\sum f(\xi)[g(\xi)-g(u)]-A\right|<\frac{1}{k}<\varepsilon.$$

There we have $A=\lim\limits_{n\to\infty}S(f,g,D_n,\delta_n,\eta_n)$.

Conversely, suppose there exists a sequence of positive functions $\{\delta_n\}$ and a sequence of positive numbers $\{\eta_n\}$ such that

$$A=\lim_{n\to\infty}S(f,g,D_n,\delta_n,\eta_n).$$

Suppose $f$ is not KHANS integrable to $A$. Then there exists an $\varepsilon>0$ such that for every positive function $\delta$ and every positive number $\eta$ there exists a $\mathcal{PA}$ $(\delta,\eta)$-fine division $D$, where

$$\left|(D)\sum f(\xi)[g(\xi)-g(u)]-A\right|\geqslant\varepsilon.$$

For the above $\varepsilon$, take $\delta_k\in\{\delta_n\}$ and $\eta_k\in\{\eta_n\}$ such that $D_k$ is a $\mathcal{PA}$ $(\delta_n,\eta_n)$-fine division, where

$$\left|(D_k)\sum f(\xi)[g(\xi)-g(u)]-A\right|=|A-S(f,g,D_k,\delta_k,\eta_k)|\geqslant\varepsilon.$$

This contradicts the statement that $A=\lim\limits_{n\to\infty}S(f,g,D_n,\delta_n,\eta_n)$, showing that $f$ is KHANS integrable to $A$. □

We next prove the main result of integration by substitution. In our setting, we weaken the condition of the differentiability of the integrator to left-differentiable. Compared to [7], we further weaken the condition of the boundedness of the integrator.

**Theorem 3.10** (Integration by substitution). *Let the function $f$ be* KHANS *integrable on $[a,b]$ with respect to $g$, which is left-differentiable on $[a,b]$. Then*

$$\int_a^b f(x)\,\mathrm{d}g(x)=\int_a^b f(x)g'_-(x)\,\mathrm{d}x,$$

*where $g'_-(x)$ denotes the left-sided derivative of $g$ at $x$.*

P r o o f. Given $\varepsilon > 0$, choose a positive function $\delta$ and a positive number $\eta$ such that for any $\mathcal{PA}$ $(\delta,\eta)$-fine division $D$,

$$\left|(D)\sum f(\xi)[g(\xi)-g(u)] - \int_a^b f\,\mathrm{d}g\right| < \frac{\varepsilon}{2}.$$

Let $D_k = \{\{[u,\xi],\xi\}\in D\colon\ k-1 < f(\xi) \leqslant k\}$ for integer $k$. Let the number of intervals in each $D_k$ be $n_k$. Since $g$ is left-differentiable at $\xi$ in $D_k$, where $k\neq 0$, choose $\delta(\xi) < 1/(|k|2^{|k|+3}n_k)$ such that whenever $0<\xi-u<\delta(\xi)$,

$$\left|\frac{g(\xi)-g(u)}{\xi-u} - g'_-(\xi)\right| < \varepsilon$$

or equivalently,

$$(D_k)\sum|g(\xi)-g(v)-g'_-(\xi)(\xi-u)| < (D_k)\sum\frac{\varepsilon}{|k|2^{|k|+3}n_k} = \frac{\varepsilon}{|k|2^{|k|+3}}.$$

Consider

$$\begin{aligned}\Big|(D)\sum f(\xi)g'_-(\xi)(\xi-u) &- (D)\sum f(\xi)[g(\xi)-g(u)]\Big|\\ &\leqslant (D)\sum|f(\xi)||(g'_-(\xi)(\xi-u)-g(\xi)+g(u))|\\ &\leqslant \sum_{-\infty}^{\infty}|k|(D_k)\sum|(g'_-(\xi)(\xi-u)-g(\xi)+g(u))| \leqslant \sum_{\substack{-\infty\\ k\neq 0}}^{\infty}\frac{|k|\varepsilon}{|k|2^{|k|+3}} = \frac{4\varepsilon}{8} = \frac{\varepsilon}{2}.\end{aligned}$$

Consequently,

$$\begin{aligned}\left|(D)\sum f(\xi)g'_-(\xi)(\xi-u) - \int_a^b f\,\mathrm{d}g\right|\\ \leqslant \left|(D)\sum f(\xi)g'_-(\xi)(\xi-u) - (D)\sum f(\xi)(\xi-u)\right|\\ + \left|(D)\sum f(\xi)(\xi-u) - \int_a^b f\,\mathrm{d}g\right|\\ \leqslant \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

showing that $\int_a^b f(x)\,\mathrm{d}g(x) = \int_a^b f(x)g'_-(x)\,\mathrm{d}x$. □

## 4. Kurzweil-Henstock variational anticipative integral

If the KHANS integral is likened to the definite integral in the classical calculus, the Kurzweil-Henstock variational anticipative integral (or KHVA) introduced in this section is the counterpart of the indefinite integral.

**Definition 4.1** (Bounded variation). A real-valued function $f$ defined on $[a, b]$ is said to be of bounded variation on $[a, b]$ if for any real number $M$,

$$\sup \sum |f(v) - f(u)| \leqslant M,$$

where the supremum is taken over all the possible partitions of $[a, b]$. If the set

$$S = \left\{ \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \colon \{a = x_0, x_1, \dots, x_n = b\} \text{ is a partition of } [a, b] \right\}$$

is bounded, then the total variation of $f$ on $[a, b]$ is defined to be $V(f, [a, b]) = \sup S$. In other words, a function $f$ is said to be of bounded variation on $[a, b]$ if $V(f, [a, b])$ is finite.

Let $\mathcal{I}[a, b]$ denote the collection of all closed subintervals $[u, v] \subset [a, b]$. Then Definition 4.1 can be defined on real-valued function $F$ defined on $\mathcal{I}[a, b]$ as: A real-valued function $F$ defined on $\mathcal{I}[a, b]$ is said to be of bounded variation on $[a, b]$ if for any real number $M$,

$$\sup \sum |F[u, v]| \leqslant M,$$

where the supremum is taken over all the possible partitions of $[a, b]$.

We shall use these two interchangeably in our discussion below whenever necessary.

**Definition 4.2** (Zero variation). A real-valued function $F$ defined on $\mathcal{I}[a, b]$ is said to be of zero variation if for any $\varepsilon > 0$ there exists a positive function $\delta > 0$ such that for any partition of $\delta$-fine partial division $D = \{([u, v]\}$ of $[a, b]$ with $(D) \sum |v - u| < \delta$, we have

$$\left| (D) \sum F[u, v] \right| < \varepsilon.$$

It is instructional to check that $F[u, v] \equiv K$ for all $[u, v] \in \mathcal{I}[a, b]$ has zero variation for any constant $K$. The function $G[u, v] = |v - u|^2$ for all $[u, v] \in \mathcal{I}[a, b]$ also has zero variation on $[a, b]$.

**Definition 4.3** (KHVA integral). A real-valued function $f$ defined on $[a, b]$ is said to be Kurzweil-Henstock variational anticipative KHVA integrable with respect to $g$ to a function $F \colon \mathcal{I} \to \mathbb{R}$ if for every $\varepsilon > 0$ there exists a positive function $\delta > 0$ such that for any partial anticipative $\delta$-fine division $D = \{([u, \xi], \xi)\}$ on $[a, b]$, we have

$$\left| (D) \sum \{ f(\xi)[g(\xi) - g(u)] - F[u, \xi] \} \right| < \varepsilon.$$

We next state without proof the basic properties of the integrals. The proofs of the following theorems are similar to that for KHANS integral, hence omitted.

**Theorem 4.1** (Uniqueness of KHVA integral). *If $f$ is* KHVA *integrable on $[a, b]$, then its primitive on $\mathcal{I}[a, b]$ is said to be unique up to zero variation. That is, if both $F$ and $G$ are the primitives of $f$ with respect to $g$ on $[a, b]$, then $F - G$ is of zero variation.*

We denote the KHVA integral by $F$, where $F[u, v] = (\mathrm{KHVA}) \int_u^v f \, \mathrm{d}g$ for all $[u, v] \subset \mathcal{I}[a, b]$. If no ambiguity exists, we shall use $\int_u^v f \, \mathrm{d}g$ to denote $F[u, v]$. Note that the KHANS integral is a real-valued function on $[a, b]$ while the KHVA integral is a real-valued function on $\mathcal{I}[a, b]$. Also readers are reminded that the equality is only up to zero variation.

**Theorem 4.2** (Integrability of sum and difference). *Suppose $f$ and $h$ are* KHVA *integrable functions on $[a, b]$. Then $f \pm h$ is* KHVA *integrable on $[a, b]$. Furthermore,*

$$\int_a^b (f(t) \pm h(t)) \, \mathrm{d}g(t) = \int_a^b f(t) \, \mathrm{d}g(t) \pm \int_a^b h(t) \, \mathrm{d}g(t).$$

**Theorem 4.3** (Integrability of scalar multiple). *Suppose $f$ is* KHVA *integrable on $[a, b]$ and $k \in \mathbb{R}$. Then $kf$ is* KHVA *integrable on $[a, b]$. Furthermore,*

$$\int_a^b k f(t) \, \mathrm{d}g(t) = k \int_a^b f(t) \, \mathrm{d}g(t).$$

**Theorem 4.4** (Integration by substitution). *Let $f$ be* KHVA *integrable on $[a, b]$ and $g$ be left-differentiable on $[a, b]$. Then*

$$\int_a^b f(x) \, \mathrm{d}g(x) = \int_a^b f(x) g'_-(x) \, \mathrm{d}x.$$

**Theorem 4.5** (Integration-by-parts). *Let the function $f$ be left-continuous on $[a, b]$ and the function $g$ be of bounded variation on $[a, b]$. If* (KHVA) $\int_a^b f \, \mathrm{d}g$ *exists, then so does* (KHVA) $\int_a^b g \, \mathrm{d}f$, *and*

$$(\mathrm{KHVA}) \int_a^b g(t) \, \mathrm{d}f(t) = f(b)g(b) - f(a)g(a) - (\mathrm{KHVA}) \int_a^b f(t) \, \mathrm{d}g(t).$$

Proof. Given $\varepsilon > 0$, choose a positive function $\delta_1$ and such that for any $\mathcal{PA}$ $\delta_1$-fine division $D_1$ of $[a, b]$ we have

$$\left| (D_1) \sum f(\xi)[g(\xi) - g(u)] - F[u, \xi] \right| < \frac{\varepsilon}{2}.$$

Let $M$ be the total variation of $g$. Choose $\delta_2 > 0$ such that whenever $0 < \xi - u < \delta_2$, we have $|f(\xi) - f(u)| < \varepsilon / 2M$. Take $\delta = \min(\delta_1, \delta_2)$ and consider any $\mathcal{PA}$ $\delta$-fine division $D_2$ on $[a,b]$.

$$
\begin{aligned}
&\left|(D_2)\sum g(\xi)[f(\xi)-f(u)]+(D_2)\sum F[u,\xi]-[f(b)g(b)-f(a)g(a)]\right|\\
&\quad=\left|(D_2)\sum f(u)[g(u)-g(\xi)]+(D_2)\sum F[\xi,u]\right|\\
&\quad=\left|(D_2)\sum[f(u)-f(\xi)][g(u)-g(\xi)]\right.\\
&\qquad\left.+(D_2)\sum[f(\xi)g(u)-g(\xi)]+(D_2)\sum F[\xi,u]\right|\\
&\quad\leqslant(D_2)\sum\frac{\varepsilon}{2M}|g(u)-g(\xi)|+\frac{\varepsilon}{2}\leqslant\frac{\varepsilon}{2M}(M)+\frac{\varepsilon}{2}=\varepsilon,
\end{aligned}
$$

thereby completing the proof. □

Note that the proof of the above theorem is still true if the roles of $f$ and $g$ are swapped, that is, $f$ is of bounded variation on $[a,b]$ and $g$ is left-continuous on $[a,b]$. We next establish the relation between KHANS integral and KHVA integral.

**Theorem 4.6.** *If $f$ is* KHANS *integrable on $[a,b]$, then it is also* KHVA *integrable there.*

P r o o f. This follows directly from Henstock's Lemma. □

**Theorem 4.7.** *If $f$ is* KHVA *integrable on $[a,b]$ and the primitive $F$ is absolutely continuous on $[a,b]$, then $f$ is* KHANS *integrable there.*

P r o o f. Given $\varepsilon>0$, choose a positive function $\delta>0$ and such that for any $\mathcal{PA}$ $\delta$-fine division $D=\{([u,\xi],\xi\}$ of $[a,b]$ we have

$$
\left|(D)\sum\{f(\xi)[g(\xi)-g(u)]-F[u,\xi]\}\right|<\frac{\varepsilon}{2}. \tag{4.1}
$$

Choose $\mu>0$ such that whenever $\{[\alpha,\beta]\}$ is a collection of subintervals of $[a,b]$ with $\sum|\beta-\alpha|<\mu$, we have

$$
\sum|F[\alpha,\beta]|<\frac{\varepsilon}{2}. \tag{4.2}
$$

Let $\overline{[a,b]\setminus\bigcup_D[u,\xi]}$ be denoted by $\bigcup_{i=1}^{m}[\alpha_i,\beta_i]$. We can let $\eta=\mu>0$ such that the $\delta$-fine partial division $D$ given above is a $\eta$-fine partial division on $[a,b]$. Note that in this case, both (4.1) and (4.2) hold. Hence,

$$\begin{aligned}
&\left|(D)\sum f(\xi)[g(\xi)-g(u)]-F[a,b]\right| \\
&\quad=\left|(D)\sum f(\xi)[g(\xi)-g(u)]-\left\{(D)\sum F[u,\xi]+\sum_{i=1}^{m}F[\alpha_i,\beta_i]\right\}\right| \\
&\quad\leqslant\left|(D)\sum\{f(\xi)[g(\xi)-g(u)]-F[u,\xi]\}\right|+\left|\sum_{i=1}^{m}F[\alpha_i,\beta_i]\right| \\
&\quad\leqslant\frac{\varepsilon}{2}+\sum_{i=1}^{m}|F[\alpha_i,\beta_i]|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}<\varepsilon,
\end{aligned}$$

therefore completing the proof. □

## 5. Convergence theorems

In this section, we shall study the integrability of the limit of a sequence of integrable functions. Roughly speaking, a convergence theorem states that integrability is preserved under taking limits. In the second part, we prove the equivalence of KHAN integral and Lebesgue Integral. This was largely motivated by the study of convergence theorems for the stochastic integrals (see [9]–[15]).

**Definition 5.1** (Pointwise convergence)**.** Let $f$ and $f^{(n)}$, $n=1,2,3,\ldots$, be real-valued functions on $[a,b]$. Then $\{f^{(n)}\}$ converges pointwise to $f$ if for any $\varepsilon>0$ there exists $N(t)>0$ such that we have $|f^{(n)}(t)-f(t)|<\varepsilon$ whenever $n\geqslant N(t)$.

**Definition 5.2** (Uniform convergence)**.** Let $f$ and $f^{(n)}$, $n=1,2,3,\ldots$, be real-valued functions on $[a,b]$. Then $\{f^{(n)}\}$ converges uniformly to $f$ if for any $\varepsilon>0$ there exists $N>0$ such that we have $|f^{(n)}(t)-f(t)|<\varepsilon$ whenever $n\geqslant N$.

Note that the difference between uniform convergence and pointwise convergence is that in the latter, $N$ does not depend on $t$ but in pointwise convergence, $N(t)$ is dependent on $t$.

**Definition 5.3** (Variational convergence)**.** Let $F$ and $F^{(n)}$, $n=1,2,3,\ldots$, be real-valued functions on $\mathcal{I}[a,b]$. Then $\{F^{(n)}\}$ is said to converge variationally to $F$ if for any $\varepsilon>0$ there exists a positive function $\delta>0$ and a positive number $N>0$ such that for any $\delta$-fine partial division $D=\{(u,\xi),\xi)\}$ we have

$$\left|(D)\sum\{F^{(n)}[u,\xi]-F[u,\xi]\}\right|<\varepsilon$$

if $n\geqslant N$.

**Theorem 5.1** (Variational convergence theorem). *Let $f$ and $f^{(n)}$, $n = 1, 2, 3, \ldots$, be real-valued functions on $[a, b]$ such that $\{f^{(n)}\}$ converge pointwise to $f$. Suppose that each $f^{(n)}$ is* KHVA *integrable to $F^{(n)}$ on $[a, b]$ and $\{F^{(n)}\}$ converges variationally to $F$. Further, let $g$ be of bounded variation on $[a, b]$. Then $f$ is* KHVA *integrable to $F$ on $[a, b]$.*

Proof. Let $M$ be the total variation of $g$ and $\varepsilon > 0$ be given. For each integer $n > 0$, choose a positive function $\delta^{(n)}$ on $[a, b]$ such that for any $\mathcal{PA}$ $\delta^n$-fine division $D_n = \{[u, \xi], \xi\}$ we have

$$\left|(D)\sum\{f^{(n)}(\xi)[g(\xi) - g(u)] - |F^{(n)}(\xi) - F^{(n)}(u)|\}\right| < \varepsilon.$$

Without loss of generality, assume that for $k = 1, 2, 3, \ldots$, $f^{(k)}$ is HVA integrable to $F^{(k)}$, with the associated $\delta^{(k)} > 0$ such that for all $\delta^{(k)}$-fine partial division $D_k$ we have

$$\left|(D_k)\sum\{f^{(k)}(\xi)[g(\xi) - g(u)] - |F^{(k)}(\xi) - F^{(k)}(u)|\}\right| < \frac{\varepsilon}{2^{k+2}}. \tag{5.1}$$

Let $\delta > 0$ be a positive function on $[a, b]$ and $N > 0$ be a positive number such that for any $\mathcal{PA}$ $\delta$-fine division $D_n = \{[u, \xi], \xi\}$ we have

$$\left|(D)\sum\{[F^{(n)}(\xi) - F^{(n)}(u)] - [F(\xi) - F(u)]\}\right| < \varepsilon$$

for all $n \geqslant N$. There exists a subsequence $n_k$ of $n$ such that

$$\left|(D)\sum\{[F^{(n_k)}(\xi) - F^{(n_k)}(u)] - [F(\xi) - F(u)]\}\right| < \frac{\varepsilon}{2^{k+2}}.$$

We re-index the sequence $\{n_k\}$ by $\{k\}$, so that there exists a positive function $\delta^k > 0$ and a positive integer $k > 0$ such that for any $\delta^k$-fine partial division $D_k = \{[u, \xi], \xi\}$ we have

$$\left|(D)\sum\{[F^{(k)}(\xi) - F^{(k)}(u)] - [F(\xi) - F(u)]\}\right| < \frac{\varepsilon}{2^{k+2}}. \tag{5.2}$$

Choose $n(\xi) > 0$ such that $f^{n(\xi)}(\xi) - f(\xi) < \varepsilon/2M$. Thus, we have

$$\begin{aligned}&\left|(D)\sum\{f(\xi)[g(\xi) - g(u)] - [F(\xi) - F(u)]\}\right| \\ &\qquad \leqslant \left|(D)\sum\{f(\xi)[g(\xi) - g(u)] - f^{n(\xi)}(\xi)[g(\xi) - g(u)]\}\right|\end{aligned}$$

$$
\begin{aligned}
&\quad + \left|(D)\sum\{f^{n(\xi)}(\xi)[g(\xi)-g(u)] - [F^{n(\xi)}(\xi) - F^{n(\xi)}(\xi)(u)]\}\right| \\
&\quad + \left|(D)\sum\{[F^{n(\xi)}(\xi) - F^{n(\xi)}(\xi)(u)] - [F(\xi)-F(u)]\}\right| \\
&= I_1 + I_2 + I_3, \\
I_1 &= \left|(D)\sum\{f(\xi)[g(\xi)-g(u)] - f^{n(\xi)}(\xi)[g(\xi)-g(u)]\}\right| \\
&< (D)\sum\left|\left\{\left(\frac{\varepsilon}{2M}\right)[g(\xi)-g(u)]\right\}\right| \\
&< \left(\frac{\varepsilon}{2M}\right)\cdot(D)\sum|\{[g(\xi)-g(u)]\}| \leqslant \left(\frac{\varepsilon}{2M}\right)\cdot M = \frac{\varepsilon}{2}, \\
I_2 &\leqslant \sum_{k=1}^{\infty}\left|(D_k)\sum\{f^k(\xi)[g(\xi)-g(u)] - [F^k(\xi) - F^k(\xi)(u)]\}\right| \\
&\leqslant \sum_{k=1}^{\infty}\frac{\varepsilon}{2^{k+2}} = \frac{\varepsilon}{4} \quad \text{from (5.1)}, \\
I_3 &\leqslant \sum_{k=1}^{\infty}\left|(D_k)\sum\{[F^{n(\xi)}(\xi) - F^{n(\xi)}(\xi)(u)] - [F(\xi)-F(u)]\}\right| \\
&\leqslant \sum_{k=1}^{\infty}\frac{\varepsilon}{2^{k+2}} = \frac{\varepsilon}{4} \quad \text{from (5.2)}.
\end{aligned}
$$

Thus, $I_1 + I_2 + I_3 \leqslant \varepsilon/2 + \varepsilon/4 + \varepsilon/4 = \varepsilon$, showing that $f$ is KHVA integrable to $F$ on $[a,b]$. □

**Theorem 5.2** (Monotone convergence theorem)**.** *If the following conditions are satisfied, then $f$ is* KHAN *integrable to $A$ on $[a,b]$:*

(i) *$\{f_n(x)\}$ converges pointwise to $f(x)$ almost everywhere in $[a,b]$ as $n\to\infty$, where each $f_n$ is* KHAN *integrable on $[a,b]$ to $F_n$;*

(ii) *$f_1(x)\leqslant f_2(x)\leqslant\ldots$ for almost all $x\in[a,b]$;*

(iii) *$F_n$ converges to $A$ on $[a,b]$ as $n\to\infty$.*

P r o o f. Without loss of generality, assume $f_n(x)$ converges pointwise to $f(x)$ everywhere in $[a,b]$. Given $\varepsilon>0$, choose positive integer $N(\xi)>0$ such that whenever $n\geqslant N(\xi)$, we have $|f_n(\xi)-f(\xi)|<\varepsilon/(b-a)$.

Given each $f_n$ is KHAN integrable on $[a,b]$ to $F_n$ by Theorem 3.7, choose $\delta_n>0$ and $\eta_n>0$ such that for any $\mathcal{PA}$ $(\delta_n,\eta_n)$-fine division $D_n=\{([u,\xi],\xi)\}$ on $[a,b]$, we have

$$(D_n)\sum|f_n(\xi)[\xi-u]-F_n(u,\xi)|<\frac{\varepsilon}{2^n}.$$

Now take $\delta(\xi) = \delta_{N(\xi)}(\xi)$ and $\eta = \eta_{N(\xi)}$ and consider any $\mathcal{PA}$ $(\delta, \eta)$-fine division $D = \{([u, \xi], \xi)\}$ on $[a, b]$. Then

$$\begin{aligned}\left|(D)\sum f(\xi)[\xi-u]-A\right| &\leqslant \left|(D)\sum f(\xi)(\xi-u)-f_{N(\xi)}(\xi)(\xi-u)\right| \\ &\quad+\left|(D)\sum f_{N(\xi)}(\xi)(\xi-u)-F_{N(\xi)}(u,\xi)\right| \\ &\quad+\left|(D)\sum F_{N(\xi)}(u,\xi)-A\right| \\ &= I_1+I_2+I_3.\end{aligned}$$

Then

$$\begin{aligned}I_1 &\leqslant (D)\sum |f(\xi)-f_{N(\xi)}(\xi)||\xi-u| \\ &\leqslant (D)\sum \frac{\varepsilon}{(b-a)}|\xi-u| \leqslant \frac{\varepsilon}{(b-a)}(b-a)=\varepsilon, \\ I_2 &\leqslant (D)\sum |f_{N(\xi)}(\xi)(\xi-u)-F_{N(\xi)}(u,\xi)| \\ &\leqslant \sum_{n=1}^{\infty}(D_n)\sum |f_n(\xi)(\xi-u)-F_n(u,\xi)| \leqslant \sum_{n=1}^{\infty}\frac{\varepsilon}{2^n}=\varepsilon.\end{aligned}$$

Since the sequence $\{f_n(x)\}$ is monotone increasing, the sequence $\{F_n(a, b)\}$ is also monotone increasing. Further, the number of the associated points $\xi$ in $D$ is finite, and so is the number of those different $N(\xi)$ in the above sum over $D$. Let $k_0$ denote the maximum of those $N(\xi)$. Since $F_{k_0}(a, b) \leqslant A$, let $k_1$ be such that $|F_k(a, b)-A| < \varepsilon$ whenever $k \geqslant k_1$. Take $k_2 = \max\{k_0, k_1\}$, and we have

$$I_3 \leqslant \left|(D)\sum F_{N(\xi)}(u,\xi)-A\right| \leqslant \left|(D)\sum F_{k_2}(u,\xi)-A\right| \leqslant |F_{k_2}(a,b)-A| < \varepsilon,$$

completing the proof of the Monotone Convergence Theorem. □

**Lemma 5.1** (Max-Min Lemma)**.** *If $f_1$ and $f_2$ are* KHAN *integrable on $[a, b]$ and if $g(x) \leqslant f_i(x) \leqslant h(x)$ almost everywhere for $i = 1, 2$, where $g$ and $h$ are also* KHAN *integrable on $[a, b]$, then* $\max(f_1, f_2)$ *and* $\min(f_1, f_2)$ *are both* KHAN *integrable on $[a, b]$.*

The proof of Lemma 5.1 is modelled after the Henstock case (see [6]), hence omitted here.

**Lemma 5.2.** *If $f_1$ and $f_2$ are* KHAN *integrable on $[a, b]$ and if their primitives $F_1$ and $F_2$ are both of bounded variation on $[a, b]$, then* $\max(f_1, f_2)$ *and* $\min(f_1, f_2)$ *are both* KHAN *integrable on $[a, b]$.*

P r o o f. Let $M_i$ be the total variation of $f_i$ on $[a,b]$ for $i=1,2$. Define $F^*(u,v) = \max\{F_1(u,v), F_2(u,v)\}$ for any $(u,v)\in\mathcal{I}$. For any $\mathcal{PA}$ $(\delta,\eta)$-fine division $D = \{[u,\xi],\xi\}$ of $[a,b]$,

$$\begin{aligned}(D)\sum F^*(u,\xi) &\leqslant (D)\sum F_1(u,\xi) + (D)\sum F_2(u,\xi)\\ &= (D)\sum(F_1(\xi)-F_1(u)) + (D)\sum(F_2(\xi)-F_2(u))\\ &\leqslant (D)\sum|F_1(\xi)-F_1(u)| + (D)\sum|F_2(\xi)-F_2(u)| \leqslant M_1+M_2.\end{aligned}$$

Hence, $(D)\sum F^*(u,\xi)$ is bounded. The rest of the proof is similar to that of Lemma 5.1, hence omitted. □

**Theorem 5.3** (Dominated convergence theorem)**.** *If the following conditions are satisfied, then $f$ is* KHAN *integrable on $[a,b]$ and $\int_a^b f_n\,\mathrm{d}x$ converges to $\int_a^b f\,\mathrm{d}x$ as $n\to\infty$:*

(i) *$f_n(x)$ converges pointwise to $f(x)$ almost everywhere in $[a,b]$ as $n\to\infty$ where each $f_n$ is* KHAN *integrable on $[a,b]$;*

(ii) *$g(x)\leqslant f_n(x)\leqslant h(x)$ for almost all $x\in[a,b]$ and all $n$, where $g$ and $h$ are also* KHAN *integrable on $[a,b]$.*

P r o o f. Let ${f_j}^* = \min\{f_n\colon\ i\leqslant n\leqslant j\}$ for $j=i,\, i+1,\, i+2,\dots$ By Lemma 5.1, each ${f_j}^*$ is KHAN integrable on $[a,b]$. Consider the sequence $\{\int_a^b -{f_j}^*\,\mathrm{d}x\}$. So $\{-{f_j}^*\}$, and hence $\{\int_a^b -{f_j}^*\,\mathrm{d}x\}$ are monotone increasing. Since each $f_n\geqslant g$, each $\int_a^b -{f_j}^*\,\mathrm{d}x$ is bounded above by $\int_a^b g\,\mathrm{d}x$, so $\{\int_a^b -{f_j}^*\,\mathrm{d}x\}$ is convergent. By Theorem 5.2, $\lim\limits_{j\to\infty} {f_j}^*$ is KHAN integrable on $[a,b]$. Hence

$$\inf\{f_n\colon\ n\geqslant i\} = \lim_{j\to\infty}\min\{f_n\colon\ i\leqslant n\leqslant j\} = \lim_{j\to\infty}{f_j}^*$$

is KHAN integrable on $[a,b]$. Similarly, $\sup\{f_n\colon\ n\geqslant i\}$ is also KHAN integrable on $[a,b]$. Hence

$$\int_a^b\Bigl(\inf_{n\geqslant i} f_n\Bigr) \leqslant \inf_{n\geqslant i}\int_a^b f_n \leqslant \sup_{n\geqslant i}\int_a^b f_n \leqslant \int_a^b\Bigl(\sup_{n\geqslant i} f_n\Bigr).$$

Note that $f_n(x)$ converges pointwise to $f(x)$ as $n\to\infty$ if and only if

$$\lim_{i\to\infty}\Bigl\{\inf_{n\geqslant i} f_n(x)\Bigr\} = f(x) = \lim_{i\to\infty}\Bigl\{\sup_{n\geqslant i} f_n(x)\Bigr\},$$

and denote respectively the upper and lower limits above by

$$\liminf_{n\to\infty} f_n(x) = \lim_{i\to\infty}\Bigl\{\inf_{n\geqslant i} f_n(x)\Bigr\} \text{ and } \limsup_{n\to\infty} f_n(x) = \lim_{i\to\infty}\Bigl\{\sup_{n\geqslant i} f_n(x)\Bigr\}.$$

Since the sequence $\{\inf\{f_n\colon n \geqslant i\}\}_{i=1}^{\infty}$ is monotone increasing, the corresponding integral sequence $\{\int_a^b \inf\{f_n\colon n \geqslant i\}\,\mathrm{d}x\}_{i=1}^{\infty}$ is also monotone increasing. Note that each $\int_a^b \inf\{f_n\colon n \geqslant i\}\,\mathrm{d}x$ is bounded above by $\int_a^b h\,\mathrm{d}x$. Hence, the sequence $\{\int_a^b \inf\{f_n\colon n \geqslant i\}\,\mathrm{d}x\}_{i=1}^{\infty}$ is convergent. By Theorem 5.2, $\{\inf\{f_n\colon n \geqslant i\}\}_{i=1}^{\infty}$, $f$ is KHAN integrable on $[a,b]$. Hence

$$\int_a^b f\,\mathrm{d}x \leqslant \liminf_{n\to\infty} \int_a^b f_n\,\mathrm{d}x \leqslant \limsup_{n\to\infty} \int_a^b f_n\,\mathrm{d}x \leqslant \int_a^b f\,\mathrm{d}x.$$

By Squeeze Theorem,

$$\liminf_{n\to\infty} \int_a^b f_n\,\mathrm{d}x = \limsup_{n\to\infty} \int_a^b f_n\,\mathrm{d}x \to \lim_{n\to\infty} \int_a^b f_n\,\mathrm{d}x = \int_a^b f\,\mathrm{d}x,$$

thereby completing the proof. □

We are now ready to prove that the KHAN integral and the classical Lebesgue integral on a closed and bounded interval $[a,b]$ agree. This approach was inspired by that used to prove that the Henstock's stochastic integral and the classical stochastic integrals agree by using convergence theorems [3], [9]–[15].

**Theorem 5.4** (KHAN integral of simple measurable functions). *Let $f\colon [a,b]\to\mathbb{R}$ be a simple measurable function such that $f$ is Lebesgue integrable. Then $f$ is also* KHAN *integrable and*

$$(L)\int f\,\mathrm{d}\mu = (\text{KHAN})\int_a^b f\,\mathrm{d}x.$$

Proof. It is sufficient to prove for the case of a characteristic function of a set $E$ that it is measurable on $[a,b]$, denoted by $\chi_E$. By definition, the Lebesgue integral $(L)\int_a^b \chi_E\,\mathrm{d}x = \mu(E)$, where $\mu$ is the Lebesgue measure. We just need to prove that

$$(\text{KHAN})\int_a^b \chi_E\,\mathrm{d}x = \mu(E) = (L)\int_a^b \chi_E\,\mathrm{d}x.$$

Given $\varepsilon > 0$, choose a $\mathcal{PA}$ $(\delta,\eta)$-fine division $D = \{[u,\xi],\xi\}$. Since $E$ is a measurable subset of $\mathbb{R}$, let open intervals $\{I_n\} = I_1, I_2, \ldots$ be such that $E \subset \bigcup\limits_{n=1}^{\infty} I_n$ and $\sum\limits_{n=1}^{\infty} \mu(I_n \setminus E) < \varepsilon$. Choose $\delta$ such that if $\xi \in I_n$ for some $n$, then $(u,\xi) \subset (\xi-\delta, \xi+\delta) \subset I_n$. Then

$$\Big|(D)\sum \chi_E(\xi)(\xi-u) - \mu(E)\Big| \leqslant \Big|(D)\sum \mu[u,\xi] - \mu(E)\Big| \leqslant \sum_{n=1}^{\infty} \mu(I_n \setminus E) < \varepsilon.$$

Hence, $(\text{KHAN})\int_{\mathbb{R}} \chi_E\,\mathrm{d}x = \mu(E)$. □

**Theorem 5.5.** *Let $f\colon [a,b]\to\mathbb{R}$ be a measurable function such that $f$ is Lebesgue integrable. Then $f$ is also* KHAN *integrable on $[a,b]$ and*

$$(L)\int f\,\mathrm{d}\mu = (KHAN)\int_a^b f\,\mathrm{d}x.$$

P r o o f. Since $f$ is measurable, there exists a sequence $f_1, f_2, \ldots$ of real-valued functions on $[a,b]$ such that $\lim\limits_{n\to\infty} f_n = f$, where each $f_n$ is a simple measurable function. Let $g(x) = \sup\limits_{n\geqslant 1}\{\max\limits_{x\in[a,b]}\{|f_n(x)|\}\}$ such that $|f_n(x)|\leqslant g(x)$ for all $n=1,2,3,\ldots$ By Theorem 5.4, we have that each $f_n$ and $g$ are both Lebesgue and KHAN integrable. Hence, the Dominated Convergence Theorem for both the Lebesgue and KHAN integral result in

$$(L)\int f\,\mathrm{d}\mu = \lim_{n\to\infty}(L)\int f_n\,\mathrm{d}\mu \text{ and } (\text{KHAN})\int f\,\mathrm{d}x = \lim_{n\to\infty}(\text{KHAN})\int f_n\,\mathrm{d}x.$$

By Theorem 5.4, for every $n=1,2,3,\ldots$,

$$\begin{aligned}(L)\int f_n\,\mathrm{d}\mu = (\text{KHAN})\int_a^b f_n\,\mathrm{d}x \to \lim_{n\to\infty}(L)\int f_n\,\mathrm{d}\mu = \lim_{n\to\infty}(HIA)\int_a^b f_n\,\mathrm{d}x \\ \to (L)\int_a^b f\,\mathrm{d}x = (\text{KHAN})\int f\,\mathrm{d}\mu.\end{aligned}$$

□

## 6. Conclusion

In this note, we have studied non-stochastic integrals using the Kurzweil-Henstock approach used by Toh and Chew (see [3], [9]–[15] in their study of stochastic integrals. Motivated by [2], we fixed the tag to be the right-hand point of the interval, hence anticipating in the sense of stochastic integral. We prove the equivalence of the classical Lebesgue integral and the Kurzweil-Henstock anticipating integral using convergence theorems.

Another approach in using stochastic approach to study non-stochastic integral could be the Kurzweil-Henstock approach in defining the Stratonovich integral (see [16], [17]), where the tag is the mid-point of the interval. A study of non-stochastic integral using this approach will appear in a paper sometime in the future.

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*References*

[1] *R. G. Bartle, D. R. Sherbert*: Introduction to Real Analysis. John Wiley & Sons, New York, 1992. zbl MR

[2] *T.-S. Chew, Z. Huang, C. S. Wang*: The non-uniform Riemann approach to anticipating stochastic integrals. Stochastic Anal. Appl. *22* (2004), 429–442. zbl MR doi

[3] *T.-S. Chew, J.-Y. Tay, T.-L. Toh*: The non-uniform Riemann approach to Itô's integral. Real Anal. Exch. *27* (2002), 495–514. zbl MR doi

[4] *H. L. Lebesgue*: Leçons sur l'intégration et la recherche des fonctions primitives. Gauthier-Villars, Paris, 1904. (In French.) zbl

[5] *P.-Y. Lee*: Lanzhou Lectures on Henstock Integration. Series in Real Analysis 2. World Scientific, London, 1989. zbl MR

[6] *P. Y. Lee, R. Výborný*: The Integral: An Easy Approach After Kurzweil and Henstock. Australian Mathematical Society Lecture Series 14. Cambridge University Press, Cambridge, 2000. zbl MR

[7] *C. Y. Y. Lim, T. L. Toh*: A note on Henstock-Itô's non-stochastic integral. Real Anal. Exch. *47* (2022), 443–460. zbl MR doi

[8] *B. Riemann*: Collected Papers. Kendrick Press, Heber City, 2004. zbl MR

[9] *T.-L. Toh, T. S. Chew*: A variational approach to Itô's intgegral. Trends in Probability and Related Analysis. World Scientific, Singapore, 1999, pp. 291–299. zbl MR

[10] *T.-L. Toh, T.-S. Chew*: The Riemann approach to stochastic integration using non-uniform meshes. J. Math. Anal. Appl. *280* (2003), 133–147. zbl MR doi

[11] *T.-L. Toh, T.-S. Chew*: Henstock's multiple Wiener integral and Henstock's version of Hu-Meyer theorem. Math. Comput. Modelling *42* (2005), 139–149. zbl MR doi

[12] *T.-L. Toh, T.-S. Chew*: On belated differentiation and a characterisation of Henstock-Kurzweil-Itô integrable processes. Math. Bohem. *130* (2005), 63–72. zbl MR doi

[13] *T.-L. Toh, T.-S. Chew*: On Itô-Kurzweil-Henstock integral and integration-by-parts formula. Czech. Math. J. *55* (2005), 653–663. zbl MR doi

[14] *T. L. Toh, T. S. Chew*: Henstock's version of Itô's formula. Real Anal. Exch. *35* (2010), 375–389. zbl MR doi

[15] *T.-L. Toh, T.-S. Chew*: The Kurzweil-Henstock theory of stochastic integration. Czech. Math. J. *62* (2012), 829-848. zbl MR doi

[16] *H. Yang, T. L. Toh*: On Henstock method to Stratonovich integral with respect to continuous semimartingale. Int. J. Stoch. Anal. *2014* (2014), Article ID 534864, 7 pages. zbl MR doi

[17] *H. Yang, T. L. Toh*: On Henstock-Kurzweil method to Stratonovich integral. Math. Bohem. *141* (2016), 129–142. zbl MR doi

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