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ON QUASIRECURRENT MANIFOLDS

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Abstract. We introduce a type of Riemannian manifolds (namely, quasirecurrent manifold) and study its several geometric properties. Among others, we prove that the scalar curvature of such a manifold is constant, and that the manifold is Einstein under certain condition. In addition, we deal with a quasirecurrent product manifold. Finally, we ensure the existence of quasirecurrent manifold by a proper example.

Keywords: quasirecurrent manifold; associated vector field; constant scalar curvature; Ricci symmetry; Einstein; cyclic Ricci symmetry; conformally flat; quasirecurrent product manifold; space of constant curvature

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1. Introduction

Cartan [3] introduced the notion of (locally) symmetric manifolds as a generalization of the notion of a space of constant curvature and he obtained a classification of such manifolds. The study on generalization of locally symmetric manifolds started in 1946 and continued to date in different directions. For instance the notions of recurrent manifold and conformally recurrent manifold were introduced by Ruse [8], [9], [10] and Walker [11]; Adati and Miyazawa [1], respectively. A Riemannian manifold (M^n, g) is called a recurrent manifold provided that its curvature tensor R satisfies the relation

$$(\nabla_X R)(Y, Z, U, V) = A(X)R(Y, Z, U, V), \tag{1.1}$$

where ∇ denotes the Levi-Civita connection and A is a nonzero 1-form.

Also a Riemannian manifold (M^n, g) is called a conformally recurrent manifold provided that its conformal curvature tensor C satisfies the relation

$$(\nabla_X C)(Y, Z, U, V) = A(X)C(Y, Z, U, V), \tag{1.2}$$

where A is a nonzero 1-form.

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The author has recently studied a type of weakly symmetric structure on a Riemannian manifold [4]. In [6], Pokhariyal defined some curvature tensors with the help of Weyl's projective curvature tensor and studied their physical and geometric properties. One of the curvature tensors introduced in [6] was the W_2-curvature tensor defined by

$$W_2(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}(g(X,Z)\mathrm{Ric}(Y,U) - g(Y,Z)\mathrm{Ric}(X,U)), \tag{1.3}$$

where Ric denotes the Ricci tensor.

In [5], [7], Pokhariyal et al. introduced the notion of W_2-recurrent manifold and studied its geometric properties. A Riemannian manifold (M^n, g) is said to be W_2-recurrent provided that its W_2-curvature tensor satisfies the relation

$$(\nabla_X W_2)(Y,Z,U,V) = A(X)W_2(Y,Z,U,V), \tag{1.4}$$

where A is a nonzero 1-form.

Motivated by the above studies, as a sequel to [4], we introduce a type of Riemannian manifold called quasirecurrent manifold and study its several geometric properties.

A Riemannian manifold (M^n, g) is called a quasirecurrent manifold provided that its curvature tensor R satisfies the relation

$$(\nabla_X R)(Y,Z,U,V) = A(X)W_2(Y,Z,U,V), \tag{1.5}$$

where A is a nonzero 1-form.

A quasirecurrent manifold with covariantly constant Ricci tensor is W_2-recurrent, while a quasirecurrent manifold with vanishing Ricci tensor reduces to a recurrent manifold. Hence, it is worthwhile to undertake the study of a quasirecurrent manifold. This paper is organized as follows. In Section 2, we give a sufficient condition for a quasirecurrent manifold to be Einstein. Section 3 is concerned with quasirecurrent product manifold. In addition, a proper example of quasirecurrent manifold is provided.

2. Some properties of quasirecurrent manifold

Let (M^n, g) be a quasirecurrent manifold. Contracting (1.3) with respect to X and U, we get

$$W_2(Y,Z) = \frac{n}{n-1}\Big(\mathrm{Ric}(Y,Z) - \frac{s}{n}g(Y,Z)\Big), \tag{2.1}$$

where s denotes the scalar curvature.

Contracting (1.5) with respect to Y and V, from (2.1) we have

$$(\nabla_X \mathrm{Ric})(Z,U) = A(X)\frac{n}{n-1}\Big(\mathrm{Ric}(Z,U) - \frac{s}{n}g(Z,U)\Big). \tag{2.2}$$

Now we have the following theorem.

Theorem 2.1. *Let (M^n, g) be a quasirecurrent manifold of dimension n. Then its scalar curvature s is constant.*

P r o o f. Contracting (2.2) with respect to Z and U, we get

$$Xs = 0,$$

showing that the scalar curvature s of a quasirecurrent manifold is constant. □

A Riemannian manifold (M^n, g) is said to be Ricci-symmetric provided that its Ricci tensor is covariantly constant. In particular, a Riemannian manifold (M^n, g) is called an Einstein manifold provided that its Ricci tensor is proportional to the metric tensor, i.e., $\mathrm{Ric} = sg/n$. In this case, its scalar curvature s is constant [2].

Theorem 2.2. *Let (M^n, g) be a quasirecurrent manifold which is Ricci-symmetric. Then the manifold is Einstein.*

P r o o f. By taking account of (2.2) and $A \neq 0$, we know that the manifold is Einstein. □

Now we verify some relationships among recurrent manifold, quasirecurrent manifold, W_2-recurrent manifold and conformally recurrent manifold.

Theorem 2.3. *Let (M^n, g) be a Riemannian manifold with covariantly constant Ricci tensor. Then condition* (1.5) *of quasirecurrent manifold is equivalent to condition* (1.4) *of W_2-recurrent manifold. In particular, if a Riemannian manifold (M^n, g) is Einstein, then condition* (1.5) *of quasirecurrent manifold is equivalent to condition* (1.2) *of conformally recurrent manifold. Furthermore, if a Riemannian manifold (M^n, g) has vanishing Ricci tensor, then condition* (1.5) *of quasirecurrent manifold is equivalent to condition* (1.1) *of recurrent manifold.*

P r o o f. By taking account of (1.3), we have from $\nabla \mathrm{Ric} = 0$ the equivalence between relations (1.4) and (1.5). In particular, if the manifold is Einstein, it is well known [2] that

$$C(X,Y,Z,U) = R(X,Y,Z,U) - \frac{s}{n(n-1)}(g(X,U)g(Y,Z) - g(X,Z)g(Y,U)),$$

which yields

$$C(X,Y,Z,U)=W_2(X,Y,Z,U) \tag{2.3}$$

and

$$(\nabla_X C)(Y,Z,U,V)=(\nabla_X R)(Y,Z,U,V) \tag{2.4}$$

because of (1.3) and $\mathrm{Ric}=sg/n$ (and hence $s=$ constant). Therefore by taking account of (2.3) and (2.4), we conclude that relation (1.5) holds if and only if relation (1.2) holds. Furthermore, considering $\mathrm{Ric}=0$ and (1.3), we can easily see that relation (1.5) holds true when relation (1.1) holds, and vice versa. □

From now on, a vector field $A^\sharp$ on a quasirecurrent manifold (M^n,g) denotes a vector field associated with the 1-form A in (1.5), i.e., $g(A^\sharp,X)=A(X)$.

Theorem 2.4. *Let (M^n,g) be a quasirecurrent manifold. Then s/n is an eigenvalue of the Ricci tensor corresponding to the vector field $A^\sharp$.*

P r o o f. By virtue of the second Bianchi identity, we have

$$(\nabla_X R)(Y,Z,U,V)+(\nabla_Z R)(X,Y,U,V)+(\nabla_Y R)(Z,X,U,V)=0,$$

which using (1.5) yields

$$A(X)W_2(Y,Z,U,V)+A(Z)W_2(X,Y,U,V)+A(Y)W_2(Z,X,U,V)=0.$$

Contracting the last relation with respect to Y and V, from (2.1) we get

$$A(X)\Big(\frac{n}{n-1}\Big)\Big[\mathrm{Ric}(Z,U)-\frac{s}{n}g(Z,U)\Big]+A(Z)\Big(\frac{n}{n-1}\Big)\Big[-\mathrm{Ric}(X,U)+\frac{s}{n}g(X,U)\Big] + W_2(Z,X,U,A^\sharp)=0.$$

Contracting the last relation with respect to Z and U, we have

$$\Big(\frac{2n}{n-1}\Big)\Big[-\mathrm{Ric}(X,A^\sharp)+\frac{s}{n}g(X,A^\sharp)\Big]=0,$$

which yields

$$\mathrm{Ric}(X,A^\sharp)=\frac{s}{n}g(X,A^\sharp). \tag{2.5}$$

□

Now we give some sufficient conditions for a quasirecurrent manifold to be Einstein.

Theorem 2.5. *Let (M^n, g) be a quasirecurrent manifold which is conformally flat. Then the manifold is Einstein.*

P r o o f. It is well known [2] that a conformally flat manifold (M^n, g) satisfies the relation

$$(\nabla_X \mathrm{Ric})(Y,Z) - (\nabla_Z \mathrm{Ric})(Y,X) = \frac{1}{2(n-1)}[g(Y,Z)\,\mathrm{d}s(X) - g(X,Y)\,\mathrm{d}s(Z)].$$

From Theorem 2.1, it follows that the last relation yields

$$(\nabla_X \mathrm{Ric})(Y,Z) = (\nabla_Z \mathrm{Ric})(Y,X).$$

By virtue of (2.2), the last relation leads to

$$A(X)\Big(\frac{n}{n-1}\Big)\Big[\mathrm{Ric}(Y,Z) - \frac{s}{n}g(Y,Z)\Big] = A(Z)\Big(\frac{n}{n-1}\Big)\Big[\mathrm{Ric}(Y,X) - \frac{s}{n}g(Y,X)\Big],$$

which using $X = A^\sharp$ and (2.5) yields

$$\|A\|^2\frac{n}{n-1}\Big[\mathrm{Ric}(Y,Z) - \frac{s}{n}g(Y,Z)\Big] = 0.$$

Hence, from the last relation and $A \neq 0$ it follows that

$$\mathrm{Ric}(Y,Z) = \frac{s}{n}g(Y,Z),$$

showing that the manifold is Einstein. □

A Riemannian manifold is called a cyclic Ricci symmetric manifold provided that its Ricci tensor Ric satisfies the relation:

$$(\nabla_X \mathrm{Ric})(Y,Z) + (\nabla_Z \mathrm{Ric})(X,Y) + (\nabla_Y \mathrm{Ric})(Z,X) = 0. \tag{2.6}$$

Now we get the following theorem.

Theorem 2.6. *Let (M^n, g) be a quasirecurrent manifold which is of cyclic Ricci symmetry. Then the manifold is Einstein.*

P r o o f. Taking account of (2.2) and (2.6), we have

$$A(X)[\mathrm{Ric}(Y,Z)-\frac{s}{n}g(Y,Z)]+A(Z)\Big[\mathrm{Ric}(X,Y)-\frac{s}{n}g(X,Y)\Big] + A(Y)\Big[\mathrm{Ric}(Z,X)-\frac{s}{n}g(Z,X)\Big]=0. \tag{2.7}$$

In Walker's lemma [11], it is said that if $a(X)$ and $b(X,Y)$ are the numbers satisfying $b(X,Y)=b(Y,X)$ and $a(X)b(Y,Z)+a(Y)b(Z,X)+a(Z)b(X,Y)=0$ for all X, Y, Z, then either all the $a(X)$ are zero or all the $b(X,Y)$ are zero. Therefore from (2.7) and Walker's lemma, we get either $A(X)=0$ or $\mathrm{Ric}(Y,Z)-(s/n)g(Y,Z)=0$. However, $A(X)=0$ is inadmissible by the defining condition of quasirecurrent manifold and hence we find

$$\mathrm{Ric}(Y,Z)-\frac{s}{n}g(Y,Z)=0,$$

showing that the manifold is Einstein. □

A vector field V on a Riemannian manifold (M^n,g) is said to be torse-forming provided that it satisfies

$$\nabla_X V = fX+\omega(X)V, \tag{2.8}$$

where f is a nonzero scalar and ω is a 1-form.

Concerning a torse-forming vector field in a quasirecurrent manifold, we get the following theorem.

Theorem 2.7. *Let (M^n,g) be a quasirecurrent manifold with nonvanishing scalar curvature s. If its associated vector field $A^\sharp$ is a unit torse-forming vector field, then the integral curve α of $A^\sharp$ in M^n is geodesic.*

P r o o f. Taking account of (2.2) and (2.5), we have

$$(\nabla_X \mathrm{Ric})(Y,A^\sharp)=A(X)\frac{n}{n-1}\Big[\mathrm{Ric}(Y,A^\sharp)-\frac{s}{n}g(Y,A^\sharp)\Big]=0. \tag{2.9}$$

On the other hand, we get

$$(\nabla_X \mathrm{Ric})(Y,A^\sharp)=\nabla_X[\mathrm{Ric}(Y,A^\sharp)]-\mathrm{Ric}(\nabla_X Y,A^\sharp)-\mathrm{Ric}(Y,\nabla_X A^\sharp). \tag{2.10}$$

From (2.5), (2.9), (2.10) and Theorem 2.1, it follows that

$$0=\frac{s}{n}g(Y,\nabla_X A^\sharp)-\mathrm{Ric}(Y,\nabla_X A^\sharp),$$

which using torse-forming vector field $A^\sharp$ yields

$$0 = \frac{s}{n}(\nabla_X A)(Y) - f\,\mathrm{Ric}(Y, X) - \omega(X)\,\mathrm{Ric}(Y, A^\sharp). \tag{2.11}$$

Putting $Y = A^\sharp$ in (2.11), from (2.5) and $g(A^\sharp, A^\sharp) = A(A^\sharp) = 1$ we get

$$0 = \frac{s}{n}(\nabla_X A)(A^\sharp) - f\frac{s}{n}A(X) - \frac{s}{n}\omega(X). \tag{2.12}$$

From (2.12), $s \neq 0$ and $(\nabla_X A)(A^\sharp) = -A(\nabla_X A^\sharp)$, it follows that

$$A(\nabla_X A^\sharp) = -f A(X) - \omega(X). \tag{2.13}$$

However, from torse-forming vector field $A^\sharp$ we know that

$$\nabla_X A^\sharp = fX + \omega(X)A^\sharp$$

and hence

$$A(\nabla_X A^\sharp) = f A(X) + \omega(X). \tag{2.14}$$

Taking account of (2.13) and (2.14), we have

$$f A(X) + \omega(X) = 0.$$

Putting $X = A^\sharp$ in the last relation, we get

$$f = -\omega(A^\sharp),$$

which using torse-forming vector field $A^\sharp$ yields

$$\nabla_X A^\sharp = -\omega(A^\sharp)X + \omega(X)A^\sharp.$$

Putting $X = A^\sharp$ in the last relation, we have

$$\nabla_{A^\sharp} A^\sharp = 0,$$

showing that the integral curve α of $A^\sharp$ in M^n is geodesic. □

3. Quasirecurrent product manifolds

Let (M^n, g) be a Riemannian product manifold $(M^p \times M^{n-p}, \widehat{g} + \widetilde{g})$. In local coordinates, we adopt the Latin indices (or the Greek indices) for tensor components which are constructed on $(M^p, \widehat{g})$ (or $(M^{n-p}, \widetilde{g})$). Therefore, the Latin indices take the values from $1, \ldots, p$, whereas the Greek indices run over the range $p+1, \ldots, n$. A Riemannian product manifold $(M^p \times M^{n-p}, \widehat{g} + \widetilde{g})$ is called a quasirecurrent product manifold provided that the product manifold is quasirecurrent. Now we have the following theorem.

Theorem 3.1. *Let $(M^p \times M^{n-p}, \widehat{g} + \widetilde{g})$ be a quasirecurrent product manifold. Then either one decomposition manifold $(M^p, \widehat{g})$ is locally symmetric or the other decomposition manifold $(M^{n-p}, \widetilde{g})$ is Einstein.*

Proof. Since any tensor components of R and its covariant derivatives with both Latin and Greek indices together should be zero, from (1.5) we have

$$0 = R_{\alpha\beta\gamma\delta;p} = A_p W_{2\alpha\beta\gamma\delta},$$

which leads to either

$$A_p = 0 \tag{3.1}$$

or

$$W_{2\alpha\beta\gamma\delta} = 0. \tag{3.2}$$

Here semicolon ";" indicates covariant differentiation.

In the case of $A_p = 0$, from (1.5) we have

$$R_{ijkl;p} = 0,$$

showing that the manifold $(M^p, \widehat{g})$ is locally symmetric.

On the other hand, if we assume that $A_p \neq 0$, then from (1.3) and (3.2) we have

$$R_{\alpha\beta\gamma\delta} + \frac{1}{n-1}(g_{\alpha\gamma} \operatorname{Ric}_{\beta\delta} - g_{\beta\gamma} \operatorname{Ric}_{\alpha\delta}) = 0. \tag{3.3}$$

Contracting (3.3) over α, δ, we obtain

$$\frac{n}{n-1}\Big(\operatorname{Ric}_{\beta\gamma} - \frac{s}{n} g_{\beta\gamma}\Big) = 0,$$

which implies

$$\operatorname{Ric}_{\beta\gamma} = \frac{s}{n} g_{\beta\gamma},$$

showing that $(M^{n-p}, \widetilde{g})$ is Einstein.

Therefore, either one decomposition manifold $(M^p, \widehat{g})$ is locally symmetric or the other decomposition manifold $(M^{n-p}, \widetilde{g})$ is Einstein. □

Now we show a proper example of quasirecurrent manifold.

Example 3.1. Let (M^n, g_c) be a space of constant curvature. It is well known [2] that a space of constant curvature is Einstein. Therefore from (1.3) and $\mathrm{Ric} = sg/n$, we have

$$W_2(X,Y,Z,U) = R(X,Y,Z,U) + \frac{s}{n(n-1)}(g(X,Z)g(Y,U) - g(Y,Z)g(X,U)),$$

which leads to

$$W_2 = 0 \tag{3.4}$$

because (M^n, g_c) is a space of constant curvature.

On the other hand, it is also well known [2] that a space of constant curvature is locally symmetric, that is,

$$\nabla R = 0. \tag{3.5}$$

Hence, both (3.4) and (3.5) imply that (1.5) holds for any nonzero 1-form A.

Therefore (M^n, g_c) is a quasirecurrent manifold.

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