Laura Burlando

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NÖRLUND MEANS OF THE SEQUENCE OF THE ITERATES OF A BOUNDED LINEAR OPERATOR, AND SPECTRAL PROPERTIES

Laura Burlando

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Abstract. We are concerned here with relating the spectral properties of a bounded linear operator T on a Banach space to the behaviour of the means $(1/s(n))\sum\limits_{k=0}^{n}(\Delta s)(n-k)T^k$, where s is a nondecreasing sequence of positive real numbers, and Δ denotes the inverse of the automorphism on the vector space of scalar sequences which maps each sequence into the sequence of its partial sums. In a previous paper, we obtained a uniform ergodic theorem for the means above, under the hypotheses $\lim\limits_{n\to\infty} s(n)=\infty$, $\lim\limits_{n\to\infty} s(n+1)/s(n)=1$, and $\Delta^q s\in l_1$ for a positive integer q: indeed, we proved that if $T^n/s(n)$ converges to zero in the uniform operator topology for such a sequence s, then the averages above converge in the same topology if and only if 1 is either in the resolvent set of T, or a simple pole of the resolvent function of T. In this paper, we prove that if $\liminf\limits_{n\to\infty} s(n+1)/s(n)=1$, and the averages above converge in the uniform operator topology, then 1 is either in the resolvent set of T, or a simple pole of the resolvent function of T. The converse is not true, even if the sequence s satisfies all the hypotheses of the theorem recalled above, except membership of $\Delta^q s$ in l_1 for a positive integer q. We also prove that if $\lim\limits_{n\to\infty}\sqrt[n]{s(n)}=1$, and the function $h_s(z)=\sum\limits_{n=0}^{\infty}s(n)z^n$ has no zeros in the open unit disk, then operator norm boundedness of the averages of the sequence T^n induced by s implies that the spectral radius of T is less than or equal to 1. This result fails if the assumption about h_s is dropped. Indeed, it may happen that the averages converge in the uniform operator topology for a sequence s satisfying $\lim\limits_{n\to\infty} s(n)=\infty$, $\lim\limits_{n\to\infty} s(n+1)/s(n)=1$, and $\Delta^q s\in l_1$ for a positive integer q, and nevertheless the spectral radius of T is strictly larger than 1.

Keywords: bounded linear operator; uniform ergodic theorem; Nörlund means of operator iterates; spectrum; pole of the resolvent

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1. Introduction

Throughout this paper, we will write $\mathbb{N}$ and $\mathbb{Z}_+$ for the sets of nonnegative integers and of strictly positive integers, respectively. Also, for each $\nu \in \mathbb{N}$ we will write $\mathbb{N}_\nu$ for the set of all nonnegative integers n satisfying $n \geqslant \nu$. For each $x \in \mathbb{R}$, $[x]$ will stand for the integer part of x. For each complex vector space V, let 0_V and I_V denote respectively the zero element of V and the identity operator on V. If V and W are complex vector spaces and $\Lambda\colon V \to W$ is a linear map, let $\mathcal{N}(\Lambda)$ and $\mathcal{R}(\Lambda)$ stand respectively for the kernel and the range of Λ.

For each complex normed space X, we will write $\|\cdot\|_X$ for the norm of X, and $L(X)$ for the complex normed algebra of all bounded linear operators on X. Henceforth, by *convergence in* $L(X)$ of a sequence of bounded linear operators on X, we will mean convergence with respect to the topology induced by $\|\cdot\|_{L(X)}$, that is, the uniform operator topology.

If X is a complex nonzero Banach space, then $L(X)$ is a complex Banach algebra—with identity I_X. For each $T \in L(X)$, let $r(T)$ and $\sigma(T)$ stand respectively for the spectral radius and for the spectrum of T. Also, let $\varrho(T)$ and $\mathfrak{R}_T$ stand respectively for the resolvent set and for the resolvent function of T. Namely, $\varrho(T) = \mathbb{C} \setminus \sigma(T)$ and $\mathfrak{R}_T\colon \varrho(T) \ni \lambda \mapsto (\lambda I_X - T)^{-1} \in L(X)$. It is well known that $\mathfrak{R}_T$ is analytic on the open set $\varrho(T)$.

The classical uniform ergodic theorem, obtained by Dunford in [4] as a special case of a result—recorded here as Theorem 2.2—about convergence of the sequence $f_n(T)$ in $L(X)$ (where $T \in L(X)$ for a complex Banach space X, and each f_n is a complex-valued function, holomorphic in an open neighborhood of $\sigma(T)$), establishes equivalence between convergence of the sequence $(1/n)\sum_{k=0}^{n-1} T^k$ in $L(X)$ and 1 being either in $\varrho(T)$ or a simple pole of $\mathfrak{R}_T$, under the hypothesis $\lim_{n\to\infty}(1/n)\|T^n\|_{L(X)} = 0$ (see 3.16 of [4], see also comments following Theorem 8 in [5]). Notice that if the sequence $(1/n)\sum_{k=0}^{n-1} T^k$ converges in $L(X)$, then $(1/n)\|T^n\|_{L(X)}$ necessarily converges to zero, as $(1/n)T^n = ((n+1)/n)\Big((1/(n+1))\sum_{k=0}^{n} T^k\Big) - (1/n)\sum_{k=0}^{n-1} T^k$ for each $n \in \mathbb{Z}_+$. Further improvements of the uniform ergodic theorem, still dealing with the arithmetic means of the sequence T^n, have been subsequently obtained in [11], [13], [10]. Examples of non power-bounded operators to which the uniform ergodic theorem applies can be found in [12], in which the relationship between convergence in $L(X)$ of the sequence of the arithmetic means above and the asymptotic behaviour of T^n is considered.

A partial extension of the uniform ergodic theorem to more general means of the sequence of the iterates of the bounded linear operator T than the arithmetical ones was obtained by Hille in [8], in which the (C,α) means $(1/A_\alpha(n))\sum_{k=0}^{n} A_{\alpha-1}(n-k)T^k$, $n\in\mathbb{N}$, are considered (where $\alpha\in(0,\infty)$, and A_α and $A_{\alpha-1}$ denote, respectively, the sequences of Cesàro numbers—whose definition is recalled here in Section 2—of order α and $\alpha-1$; notice that for $\alpha=1$ we have $(1/A_\alpha(n))\sum_{k=0}^{n} A_{\alpha-1}(n-k)T^k = (1/(n+1))\sum_{k=0}^{n} T^k$ for each $n\in\mathbb{N}$). Indeed, in Theorem 6 of [8] it is shown that if the sequence $(1/A_\alpha(n))\sum_{k=0}^{n} A_{\alpha-1}(n-k)T^k$ converges to some $E\in L(X)$ in $L(X)$, then $\|T^n\|_{L(X)}/n^\alpha\to 0$ as $n\to\infty$ and $\lim_{\lambda\to 1^+}\|(\lambda-1)\mathfrak{R}_T(\lambda)-E\|_{L(X)}=0$. Notice that the former of these two conditions yields $r(T)\leqslant 1$, and then the latter can be replaced by 1 being either in $\varrho(T)$, or a simple pole of $\mathfrak{R}_T$, and moreover E being the residue of $\mathfrak{R}_T$ at 1 (see 1.3 of [6], or 18.8.1 of [9]). Theorem 6 of [8] also provides a partial converse of this, establishing that if T is power-bounded and $\lim_{\lambda\to 1^+}\|(\lambda-1)\mathfrak{R}_T(\lambda)-E\|_{L(X)}=0$, then

$$\lim_{n\to\infty}\left\|\frac{1}{A_\alpha(n)}\sum_{k=0}^{n} A_{\alpha-1}(n-k)T^k - E\right\|_{L(X)} = 0$$

for each $\alpha\in(0,\infty)$. More recently, an improvement of this partial converse was obtained by Yoshimoto, who in Theorem 1 of [16] replaced power-boundedness of T by $\lim_{n\to\infty}\|T^n\|_{L(X)}/n^\omega=0$ (where $\omega=\min\{1,\alpha\}$).

Finally, in [6], Ed-dari was able to complete the (C,α) uniform ergodic theorem, by proving that the sequence $(1/A_\alpha(n))\sum_{k=0}^{n} A_{\alpha-1}(n-k)T^k$ converges to E in $L(X)$ if and only if $\|T^n\|_{L(X)}/n^\alpha\to 0$ as $n\to\infty$ and $\lim_{\lambda\to 1^+}\|(\lambda-1)\mathfrak{R}_T(\lambda)-E\|_{L(X)}=0$. Ed-dari's result is recorded here as Theorem 2.3.

In a previous paper (see [2]), we obtained a uniform ergodic theorem for the *Nörlund means* of the sequence T^n, that is, for the means $(1/s(n))\sum_{k=0}^{n}(\Delta s)(n-k)T^k$, $n\in\mathbb{N}$, where s is a nondecreasing sequence of strictly positive real numbers (and Δ is as in the abstract; the definition of Δ is also recalled in Section 2 here). We point out that for $s=A_\alpha$, $\alpha\in(0,\infty)$, one obtains the (C,α) means. Indeed, in 6.7 of [2], we proved that if $\lim_{n\to\infty}s(n)=\infty$, $\lim_{n\to\infty}s(n+1)/s(n)=1$, $\Delta^q s\in l_1$ for some $q\in\mathbb{Z}_+$, and $\|T^n\|_{L(X)}/s(n)\to 0$ as $n\to\infty$, then the sequence $(1/s(n))\sum_{k=0}^{n}(\Delta s)(n-k)T^k$ converges in $L(X)$ if and only if 1 is either in $\varrho(T)$ or a simple pole of $\mathfrak{R}_T$, in which case the sequence of the Nörlund means of the iterates of T converges in $L(X)$ to the residue of $\mathfrak{R}_T$ at 1 (this result is recorded here as Theorem 2.6; see also Theorem 2.1 here). Contrary to the special case of the (C,α) means, convergence of the sequence $(1/s(n))\sum_{k=0}^{n}(\Delta s)(n-k)T^k$ in $L(X)$ does not imply $\lim_{n\to\infty}\|T^n\|_{L(X)}/s(n)=0$, as a convenient example—provided in 6.10 of [2]—shows. In this paper we continue to investigate the relationships between the behaviour of the Nörlund means of the sequence T^n and the spectral properties of T. Section 2 presents some preliminaries for the purpose of making this paper as self-contained as possible. Sections 3 and 4 contain the results.

In Section 3 we derive a consequence of the above-mentioned Dunford's result about convergence of the sequence $f_n(T)$, and use this consequence to prove that if for a bounded linear operator T on a complex Banach space X and a nondecreasing sequence s of strictly positive real numbers, satisfying $\liminf_{n\to\infty} s(n+1)/s(n) = 1$, the sequence $(1/s(n)) \sum_{k=0}^{n} (\Delta s)(n-k)T^k$ converges in $L(X)$, then 1 is either in $\varrho(T)$, or a simple pole of $\mathfrak{R}_T$ (Theorem 3.2). Notice that any nondecreasing sequence s of strictly positive real numbers necessarily satisfies $\liminf_{n\to\infty} s(n+1)/s(n) \geqslant 1$. By means of a convenient example, we will show that if the following sequence $(1/s(n)) \sum_{k=0}^{n} (\Delta s)(n-k)T^k$ converges in $L(X)$ for a nondecreasing sequence s of strictly positive real numbers, satisfying $\liminf_{n\to\infty} s(n+1)/s(n) > 1$, then 1 may neither be in $\varrho(T)$, nor be a pole of $\mathfrak{R}_T$ (Example 3.5). A further example is provided in order to show that the converse of Theorem 3.2 does not hold even if the sequence s is assumed to satisfy all the hypotheses of 6.7 in [2] except membership of $\Delta^q s$ in l_1 for a positive integer q (Example 3.6). More precisely, Example 3.6 shows that in 6.7 of [2], membership of $\Delta^q s$ in l_1 for some $q \in \mathbb{Z}_+$ cannot be replaced by membership of $\Delta^r s$ in l_∞ for some $r \in \mathbb{Z}_+$ (see also Remark 4.14).

In Section 4, starting from the fact recalled above that convergence of the sequence $(1/s(n)) \sum_{k=0}^{n} (\Delta s)(n-k)T^k$ in $L(X)$ does not imply $\lim_{n\to\infty} \|T^n\|_{L(X)}/s(n) = 0$ even if s satisfies all the remaining hypotheses of 6.7 in [2], we search for conditions (on the sequence s and on the Nörlund means of the sequence T^n induced by s) which imply a weaker property than $\lim_{n\to\infty} \|T^n\|_{L(X)}/s(n) = 0$, that is, $r(T) \leqslant 1$. In Theorem 4.8 we prove that if for a bounded linear operator T on a complex Banach space X and a nondecreasing sequence s of strictly positive real numbers satisfying $\lim_{n\to\infty} \sqrt[n]{s(n)} = 1$ and such that the function $h_s(z) = \sum_{n=0}^{\infty} s(n)z^n$ has no zeros in the open unit disk, the sequence $(1/s(n)) \sum_{k=0}^{n} (\Delta s)(n-k)T^k$ is bounded in $L(X)$, then $r(T) \leqslant 1$. Also, in Theorem 4.12 we prove that if s is a nondecreasing sequence of strictly positive real numbers such that $\lim_{n\to\infty} s(n+1)/s(n) = 1$, $\lim_{n\to\infty} s(n) = \infty$, $\Delta^r s$ is bounded for a positive integer r, and the function h_s has a zero z_0 in the open unit disk, then the Nörlund means—induced by s—of the complex sequence $1/z_0^n$ converge to zero (notice that $s(0) > 0$ yields $z_0 \neq 0$). As a consequence of this, in Corollary 4.13 we derive that if s is as in Theorem 4.12, for each complex nonzero Banach space X there exists $T \in L(X)$ such that the sequence $(1/s(n)) \sum_{k=0}^{n} (\Delta s)(n-k)T^k$ converges in $L(X)$, and nevertheless $r(T) = 1/|z_0| > 1$. We conclude the paper with an example of a nondecreasing sequence s of strictly positive real numbers such that $\lim_{n\to\infty} s(n+1)/s(n) = 1$, $\lim_{n\to\infty} s(n) = \infty$, $\Delta^2 s \in l_1$, and $h_s(-\frac{1}{2}) = 0$ (Example 4.15). This, by virtue of Corollary 4.13, shows that if the sequence $(1/s(n)) \sum_{k=0}^{n} (\Delta s)(n-k)T^k$ converges in $L(X)$ for a bounded linear operator T on a complex Banach space X and a nondecreasing sequence s of strictly positive real numbers, satisfying all the hypotheses of 6.7 in [2] except $\lim_{n\to\infty} \|T^n\|_{L(X)}/s(n) = 0$, it may even happen that $r(T) > 1$.

2. Preliminaries

If X is a Banach space and Y, Z are closed subspaces of X satisfying $X = Y \oplus Z$, by the *projection of X onto Y along Z* we mean the bounded linear map $P\colon X \to X$ such that $Px \in Y$ and $x - Px \in Z$ for every $x \in X$. Notice that $I_X - P$ is the projection of X onto Z along Y, and that $P^2 = P$. On the other hand, if $E \in L(X)$ satisfies $E^2 = E$, it is easily seen that $\mathcal{R}(E)$ is closed in X, $X = \mathcal{R}(E) \oplus \mathcal{N}(E)$, and E is the projection of X onto $\mathcal{R}(E)$ along $\mathcal{N}(E)$.

The following is a classical characterization of simple poles of $\mathfrak{R}_T$.

Theorem 2.1 (V, 10.1, 10.2, 6.2–6.4, and IV, 5.10 in [15])**.** *Let X be a complex nonzero Banach space, $T \in L(X)$ and $\lambda_0 \in \mathbb{C}$. If λ_0 is a simple pole of $\mathfrak{R}_T$, then λ_0 is an eigenvalue of T, $\mathcal{N}((\lambda_0 I_X - T)^n) = \mathcal{N}(\lambda_0 I_X - T)$ and $\mathcal{R}((\lambda_0 I_X - T)^n) = \mathcal{R}(\lambda_0 I_X - T)$ for every $n \in \mathbb{Z}_+$, $\mathcal{R}(\lambda_0 I_X - T)$ is closed in X, $X = \mathcal{N}(\lambda_0 I_X - T) \oplus \mathcal{R}(\lambda_0 I_X - T)$, and the projection of X onto $\mathcal{N}(\lambda_0 I_X - T)$ along $\mathcal{R}(\lambda_0 I_X - T)$ coincides with the residue of $\mathfrak{R}_T$ at λ_0. Conversely, if $X = \mathcal{N}(\lambda_0 I_X - T) \oplus \mathcal{R}(\lambda_0 I_X - T)$, then λ_0 is either in $\varrho(T)$, or else a simple pole of $\mathfrak{R}_T$.*

If X is a complex nonzero Banach space and $T \in L(X)$, following Definition on page 310 in [15], we denote by $\mathfrak{A}(T)$ the set of all complex-valued holomorphic functions f whose domain $\operatorname{Dom}(f)$ is an open neighbourhood of $\sigma(T)$. For each $f \in \mathfrak{A}(T)$, the operator $f(T) \in L(X)$ is defined as follows:

$$f(T) = \frac{1}{2\pi \mathrm{i}} \int_{+\partial D} f(\lambda) \mathfrak{R}_T(\lambda) \,\mathrm{d}\lambda,$$

where $+\partial D$ denotes the positively oriented boundary of D, and D is any open bounded subset of $\mathbb{C}$ such that $D \supseteq \sigma(T)$, $\overline{D} \subseteq \operatorname{Dom}(f)$, D has a finite number of components with pairwise disjoint closures, and ∂D consists of a finite number of simple closed rectifiable curves, no two of which intersect; the integral above does not depend on the particular choice of D (see [15], comment 2 on pages 310–311; see also 2.2, 2.3 and 2.6 in [4]). We recall that for each polynomial $\mathfrak{p}\colon \mathbb{C} \ni \lambda \mapsto \sum_{k=0}^{n} a_k \lambda^k \in \mathbb{C}$ (where $n \in \mathbb{N}$, and $a_0, \ldots, a_n \in \mathbb{C}$), we have $\mathfrak{p}(T) = \sum_{k=0}^{n} a_k T^k$ (see [15], V, 8.1).

In the sequel, we will also need the following convergence result for the elements of $\mathfrak{A}(T)$ (due to Dunford), a special case of which is the classical uniform ergodic theorem.

Theorem 2.2 ([4], 3.16; see also comments following Theorem 8 in [5], and [2], 2.3)**.** *Let X be a complex nonzero Banach space, $T \in L(X)$, and $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{A}(T)$ such that $1 \in \operatorname{Dom}(f_n)$ for each $n \in \mathbb{N}$, $\lim_{n\to\infty} f_n(1) = 1$ and $(I_X - T)f_n(T) \to 0_{L(X)}$ in $L(X)$ as $n \to \infty$. Then the following conditions are equivalent:*

(i) *the sequence $(f_n(T))_{n\in\mathbb{N}}$ converges in $L(X)$;*
(ii) *1 is either in $\varrho(T)$, or a simple pole of $\mathfrak{R}_T$;*
(iii) *$\mathcal{R}(I_X - T)$ is closed, $X = \mathcal{N}(I_X - T) \oplus \mathcal{R}(I_X - T)$, and the sequence $(f_n(T))_{n\in\mathbb{N}}$ converges in $L(X)$ to the projection of X onto $\mathcal{N}(I_X - T)$ along $\mathcal{R}(I_X - T)$.*

For each $\alpha \in \mathbb{R}$ let $A_\alpha\colon \mathbb{N} \to \mathbb{R}$ denote the sequence of the Cesàro numbers of order α. That is,

$$A_\alpha(n) = \binom{n+\alpha}{n} = \begin{cases} 1 & \text{if } n = 0, \\ \dfrac{\prod_{j=1}^{n}(\alpha + j)}{n!} & \text{if } n \in \mathbb{Z}_+. \end{cases}$$

Notice that $A_0(n) = 1$ for all $n \in \mathbb{N}$. Also, if $\alpha > -1$, then $A_\alpha(n) > 0$ for each $n \in \mathbb{N}$. Furthermore, we point out that for each $p \in \mathbb{Z}_+$ we have $A_{-p}(n) = 0$ for every $n \in \mathbb{N}_p$. We recall that

$$\sum_{k=0}^{n} A_\alpha(k) = A_{\alpha+1}(n) \quad \text{for each } n \in \mathbb{N} \text{ and each } \alpha \in \mathbb{R} \tag{2.1}$$

and

$$\lim_{n\to\infty} \frac{A_\alpha(n)}{n^\alpha} = \frac{1}{\Gamma(\alpha+1)} \quad \text{for each } \alpha \in \mathbb{R} \setminus \{-k\colon\ k \in \mathbb{Z}_+\}, \tag{2.2}$$

where Γ denotes Euler's gamma function (see for instance [17], III, (1–11) and (1–15)).

The improvement of Hille's uniform ergodic theorem for the (C,α) means obtained by Ed-dari in [6] can be rephrased as follows, by taking also Theorem 2.1 and Theorem 2.2 (as well the latter's consequence 18.8.1 in [9]—recorded in [6] as Lemma 1.3) into account.

Theorem 2.3 (see [6], Theorem 1)**.** *Let X be a complex nonzero Banach space, $T \in L(X)$, and $\alpha \in (0,\infty)$. Then the following conditions are equivalent:*

(i) *the sequence $\Big(\sum\limits_{k=0}^{n} A_{\alpha-1}(n-k)T^k/A_\alpha(n)\Big)_{n\in\mathbb{N}}$ converges in $L(X)$;*

(ii) $\lim\limits_{n\to\infty} \|T^n\|_{L(X)}/n^\alpha = 0$ *and 1 is either in $\varrho(T)$, or a simple pole of $\mathfrak{R}_T$;*

(iii) $\lim\limits_{n\to\infty} \|T^n\|_{L(X)}/n^\alpha = 0$ *and $X = \mathcal{N}(I_X - T) \oplus \mathcal{R}(I_X - T)$;*

(iv) $\lim\limits_{n\to\infty} \|T^n\|_{L(X)}/n^\alpha = 0$*, $\mathcal{R}(I_X - T)$ is closed, and $X = \mathcal{N}(I_X - T) \oplus \mathcal{R}(I_X - T)$.*

Furthermore, if the equivalent conditions (i)–(iv) *are satisfied, and $P \in L(X)$ is such that $\sum\limits_{k=0}^{n} A_{\alpha-1}(n-k)T^k/A_\alpha(n) \to P$ in $L(X)$ as $n \to \infty$, then P is the projection of X onto $\mathcal{N}(I_X - T)$ along $\mathcal{R}(I_X - T)$.*

R e m a r k 2.4. As remarked in 2.8 of [2], it is easily seen that if a bounded linear operator T on a complex nonzero Banach space X is such that $\lim\limits_{n\to\infty} \|T^n\|_{L(X)}/n^\alpha = 0$ for some $\alpha \in (0,\infty)$, then $r(T) \leqslant 1$. The converse is not true: if $r(T) = 1$, there may exist no $\alpha \in (0,\infty)$ for which $\lim\limits_{n\to\infty} \|T^n\|_{L(X)}/n^\alpha = 0$ (if $r(T) < 1$, then clearly $\lim\limits_{n\to\infty} \|T^n\|_{L(X)}/n^\alpha = 0$ for every $\alpha \in (0,\infty)$, being $\lim\limits_{n\to\infty} \|T^n\|_{L(X)} = 0$). See for instance 6.3 of [2] for an example in which $r(T) = 1$, and nevertheless $\lim\limits_{n\to\infty} \|T^n\|_{L(X)}/n^\alpha = \infty$ for every $\alpha \in (0,\infty)$. We recall that a necessary and sufficient condition in order that $r(T) \leqslant 1$ has been provided by Allan and Ransford in [1]: $r(T) \leqslant 1$ if and only if there exists a sequence μ of strictly positive real numbers such that $\lim\limits_{n\to\infty} \mu(n+1)/\mu(n) = 1$ and $\|T^n\|_{L(X)} \leqslant \mu(n)$ for every $n \in \mathbb{N}$ (see 2.1 in [1]). We point out that the sequence μ can in fact be chosen so that it is also nondecreasing: indeed, in the power-bounded case the desired inequality is satisfied by a suitable constant—and thus nondecreasing—sequence μ (see [1], proof of 2.1). In the non power-bounded case, by applying 3.9 in [2], we conclude that the least concave majorant $(\sigma_n)_{n\in\mathbb{N}}$ of the sequence $(\varrho_n)_{n\in\mathbb{N}}$ in the proof of 2.1 in [1], is strictly increasing, being $\lim\limits_{n\to\infty} \varrho_n/n = 0$, and ϱ_n positive and unbounded. Then so is μ, as $\mu(n) = \mathrm{e}^{\sigma_n}$ for every $n \in \mathbb{N}$.

Henceforth, we will denote by $\mathbb{C}^\mathbb{N}$ the complex vector space of all sequences of complex numbers. Also, let Σ, $\Delta\colon \mathbb{C}^\mathbb{N} \to \mathbb{C}^\mathbb{N}$ denote the linear operators defined by

$$(\Sigma a)(n) = \sum_{k=0}^{n} a(k) \quad \text{and} \quad (\Delta a)(n) = \begin{cases} a(0) & \text{if } n = 0, \\ a(n) - a(n-1) & \text{if } n \in \mathbb{Z}_+ \end{cases}$$

for every $n \in \mathbb{N}$ and $a \in \mathbb{C}^{\mathbb{N}}$. We remark that both Σ and Δ are bijective. Moreover, they are mutually inverse, that is, $\Delta\Sigma = \Sigma\Delta = I_{\mathbb{C}^{\mathbb{N}}}$. We also remark that $\Delta(l_1) \subseteq l_1$. Finally, notice that if a real sequence s satisfies $\lim_{n\to\infty} s(n) = \infty$, and $q \in \mathbb{N}$ is such that $\Delta^q s \in l_1$, then we must have $q \geqslant 2$.

Following Definition 5.1 of [2], for each real sequence $a\colon \mathbb{N} \to \mathbb{R}$ we set

$$\mathcal{H}(a) = \inf\{m \in \mathbb{N}\colon \text{ the sequence } (a(n)/n^m)_{n\in\mathbb{Z}_+} \text{ is bounded from above}\}.$$

Notice that $\mathcal{H}(a) \in \mathbb{N} \cup \{\infty\}$, and the infimum above is attained if and only if $\mathcal{H}(a) < \infty$. Clearly, $\mathcal{H}(a) < \infty$ if and only if the sequence $(a(n)/n^{\beta})_{n\in\mathbb{Z}_+}$ is bounded from above for some $\beta \in [0, \infty)$. Also, a is bounded from above if and only if $\mathcal{H}(a) = 0$. We remark that $\mathcal{H}(A_\alpha) < \infty$ for every $\alpha \in \mathbb{R}$. Indeed, for each $\alpha \in (-\infty, 0]$ we have $\mathcal{H}(A_\alpha) = 0$ (because A_α is eventually constant if α is a negative integer; otherwise, by virtue of (2.2)). For each $\alpha \in (0, \infty)$, from (2.2) we derive that

$$\mathcal{H}(A_\alpha) = \begin{cases} \alpha & \text{if } \alpha \in \mathbb{Z}_+, \\ [\alpha] + 1 & \text{if } \alpha \notin \mathbb{Z}_+. \end{cases}$$

The following result—proved in [2]—relates finiteness of $\mathcal{H}(a)$ to being $\Delta^q a \in l_1$ for suitable q.

Proposition 2.5 ([2], 5.4)**.** *Let $a\colon \mathbb{N} \to \mathbb{R}$ be a real sequence, and $q \in \mathbb{Z}_+$ be such that $\Delta^q a \in l_1$. Then $\mathcal{H}(a) \leqslant q - 1$.*

The following uniform ergodic theorem for Nörlund means has been proved in [2]: indeed, by taking Proposition 2.5 into account, it is an immediate consequence of the main result of [2], that is, of [2], 6.7.

Theorem 2.6 ([2], 6.7; see also Proposition 2.5)**.** *Let X be a complex nonzero Banach space, $T \in L(X)$, and $s\colon \mathbb{N} \to \mathbb{R}$ be a nondecreasing sequence of strictly positive real numbers such that $\lim_{n\to\infty} s(n) = \infty$, $\lim_{n\to\infty} s(n+1)/s(n) = 1$, $\Delta^q s \in l_1$ for some $q \in \mathbb{N}_2$, and $\lim_{n\to\infty} \|T^n\|_{L(X)}/s(n) = 0$. Then $r(T) \leqslant 1$, and the following conditions are equivalent:*

(i) *the sequence* $\Big(\sum_{k=0}^{n} (\Delta s)(n-k)T^k/s(n)\Big)_{n\in\mathbb{N}}$ *converges in* $L(X)$*;*

(ii) *1 is either in* $\varrho(T)$ *or a simple pole of* $\mathfrak{R}_T$*;*

(iii) $X = \mathcal{N}(I_X - T) \oplus \mathcal{R}(I_X - T)$*;*

(iv) $\mathcal{R}(I_X - T)$ *is closed in* X *and* $X = \mathcal{N}(I_X - T) \oplus \mathcal{R}(I_X - T)$*.*

Finally, if the equivalent conditions (i)–(iv) are satisfied and $P \in L(X)$ is such that $\sum_{k=0}^{n} (\Delta s)(n-k)T^k/s(n) \to P$ in $L(X)$ as $n \to \infty$, then P is the projection of X onto $\mathcal{N}(I_X - T)$ along $\mathcal{R}(I_X - T)$.

R e m a r k 2.7. Fix $\alpha \in (0,\infty)$. Then A_α is a strictly increasing sequence of strictly positive real numbers (see for instance [17], III, (1–17)). Moreover, from (2.2) it follows that $\lim\limits_{n\to\infty} A_\alpha(n) = \infty$ and $\lim\limits_{n\to\infty} A_\alpha(n+1)/A_\alpha(n) = 1$. Finally, if we set

$$q = \begin{cases} \alpha + 1 & \text{if } \alpha \in \mathbb{Z}_+, \\ [\alpha] + 2 & \text{if } \alpha \notin \mathbb{Z}_+, \end{cases}$$

we have $q \in \mathbb{N}_2$, and from (2.1) we derive that $\Delta^q A_\alpha = A_{\alpha-q}$. Then $\Delta^q A_\alpha \in l_1$: if $\alpha \in \mathbb{Z}_+$, this follows being $A_{\alpha-q}(n) = A_{-1}(n) = 0$ for every $n \in \mathbb{Z}_+$; if $\alpha \notin \mathbb{Z}_+$, it follows from (2.2), being $\alpha - q < -1$.

However, as remarked in [2], Theorem 2.8 does not completely extend Theorem 2.3 from the class of all sequences of Cesàro numbers of strictly positive order to the larger one of all divergent nondecreasing sequences s of strictly positive real numbers for which $\lim\limits_{n\to\infty} s(n+1)/s(n) = 1$ and $\Delta^q s \in l_1$ for some $q \in \mathbb{N}_2$, as the condition $\lim\limits_{n\to\infty} \|T^n\|_{L(X)}/s(n) = 0$ is assumed in the hypotheses of Theorem 2.6, whereas the condition $\lim\limits_{n\to\infty} \|T^n\|_{L(X)}/n^\alpha = 0$ is not in the hypotheses of Theorem 2.3. Indeed, if X is a complex nonzero Banach space, $T \in L(X)$, and s is a nondecreasing sequence of strictly positive real numbers such that $\lim\limits_{n\to\infty} s(n) = \infty$, $\lim\limits_{n\to\infty} s(n+1)/s(n) = 1$, $\Delta^q s \in l_1$ for some $q \in \mathbb{N}_2$, and the sequence $\Big((1/s(n)) \sum\limits_{k=0}^{n} (\Delta s)(n-k)T^k\Big)_{n\in\mathbb{N}}$ converges in $L(X)$, then $\|T^n\|_{L(X)}/s(n)$ need not converge to zero as $n \to \infty$: an example, in which the sequence s is even strictly increasing, is provided in [2], 6.10.

Finally, the following classical result about real sequences will be useful to us.

Theorem 2.8 ([14], 3.37). *Let $a\colon \mathbb{N} \to \mathbb{R}$ be a sequence of strictly positive real numbers. Then*

$$\liminf_{n\to\infty} \frac{a(n+1)}{a(n)} \leqslant \liminf_{n\to\infty} \sqrt[n]{a(n)} \quad \textit{and} \quad \limsup_{n\to\infty} \sqrt[n]{a(n)} \leqslant \limsup_{n\to\infty} \frac{a(n+1)}{a(n)}.$$

In particular, if $\lim\limits_{n\to\infty} a(n+1)/a(n) = l$ for some $l \in [0,\infty]$, then also

$$\lim_{n\to\infty} \sqrt[n]{a(n)} = l.$$

3. Spectral consequences of convergence of Nörlund means in $L(X)$

We begin by deriving a consequence of Theorem 2.2.

Theorem 3.1. *Let X be a complex nonzero Banach space, $T \in L(X)$, and $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{A}(T)$ such that $1 \in \operatorname{Dom}(f_n)$ for each $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} f_n(1) = 1$. Suppose that there exist two subsequences $(f_{n_k})_{k\in\mathbb{N}}$ and $(f_{m_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$, and a sequence $(T_k)_{k\in\mathbb{N}}$ in $L(X)$ such that $T_k \to T$ in $L(X)$ and $f_{m_k}(T) - T_k f_{n_k}(T) \to 0_{L(X)}$ in $L(X)$ as $k \to \infty$. If the sequence $(f_n(T))_{n\in\mathbb{N}}$ converges in $L(X)$, then 1 is either in $\varrho(T)$, or a simple pole of $\mathfrak{R}_T$ (and consequently $\mathcal{R}(I_X - T)$ is closed, and $X = \mathcal{N}(I_X - T) \oplus \mathcal{R}(I_X - T)$). Furthermore, $(f_n(T))_{n\in\mathbb{N}}$ converges in $L(X)$ to the projection of X onto $\mathcal{N}(I_X - T)$ along $\mathcal{R}(I_X - T)$.*

Proof. Let $E \in L(X)$ be such that $f_n(T) \to E$ in $L(X)$ as $n \to \infty$. Then $f_{n_k}(T) \to E$ and $f_{m_k}(T) \to E$ in $L(X)$ as $k \to \infty$, from which we derive that

$$(I_X - T)f_{n_k}(T) = (f_{n_k}(T) - f_{m_k}(T)) + (f_{m_k}(T) - T_k f_{n_k}(T)) + (T_k - T)f_{n_k}(T) \underset{k\to\infty}{\longrightarrow} 0_{L(X)} \quad \text{in } L(X).$$

By applying Theorem 2.2 to the sequence $(f_{n_k}(T))_{k\in\mathbb{N}}$, we conclude that 1 is ether in $\varrho(T)$, or a simple pole of $\mathfrak{R}_T$ (which yields $\mathcal{R}(I_X - T)$ closed, and $X = \mathcal{N}(I_X - T) \oplus \mathcal{R}(I_X - T)$), and E is the projection of X onto $\mathcal{N}(I_X - T)$ along $\mathcal{R}(I_X - T)$. This finishes the proof. □

Now we are going to apply Theorem 3.1 to the Nörlund means of the sequence of the iterates of a bounded linear operator.

Theorem 3.2. *Let X be a complex nonzero Banach space, $T \in L(X)$, and $s\colon \mathbb{N} \to \mathbb{R}$ be a nondecreasing sequence of strictly positive real numbers such that $\liminf\limits_{n\to\infty} s(n+1)/s(n) = 1$. If the sequence $\Big(\sum\limits_{k=0}^{n} (\Delta s)(n-k)T^k/s(n)\Big)_{n\in\mathbb{N}}$ converges in $L(X)$, then 1 is either in $\varrho(T)$, or a simple pole of $\mathfrak{R}_T$ (and consequently $\mathcal{R}(I_X - T)$ is closed, and $X = \mathcal{N}(I_X - T) \oplus \mathcal{R}(I_X - T)$). Furthermore, the sequence $\Big(\sum\limits_{k=0}^{n} (\Delta s)(n-k)T^k/s(n)\Big)_{n\in\mathbb{N}}$ converges in $L(X)$ to the projection of X onto $\mathcal{N}(I_X - T)$ along $\mathcal{R}(I_X - T)$.*

Proof. Let $P \in L(X)$ be such that $\sum\limits_{k=0}^{n} (\Delta s)(n-k)T^k/s(n) \to P$ in $L(X)$ as $n \to \infty$. For each $n \in \mathbb{N}$ let $f_n\colon \mathbb{C} \to \mathbb{C}$ be defined by

$$f_n(z) = \frac{\sum_{k=0}^{n} (\Delta s)(n-k)z^k}{s(n)} \quad \text{for every } z \in \mathbb{C}.$$

Clearly, $f_n \in \mathfrak{A}(T)$. Also,

$$f_n(1) = \frac{\sum_{k=0}^{n} (\Delta s)(n-k)}{s(n)} = \frac{\sum_{j=0}^{n} (\Delta s)(j)}{s(n)} = \frac{(\Sigma \Delta s)(n)}{s(n)} = \frac{s(n)}{s(n)} = 1$$

and

$$f_n(T) = \frac{\sum_{k=0}^{n} (\Delta s)(n-k) T^k}{s(n)}.$$

Hence $\lim_{n\to\infty} f_n(1) = 1$, and $f_n(T) \to P$ in $L(X)$ as $n \to \infty$.

Now let $(n_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of nonnegative integers such that

$$\lim_{k\to\infty} \frac{s(n_k+1)}{s(n_k)} = 1. \tag{3.1}$$

We prove that $f_{n_k+1}(T) - (s(n_k)/s(n_k+1)) T f_{n_k}(T) \to 0_{L(X)}$ in $L(X)$ as $k \to \infty$. Indeed, it suffices to observe that for each $k \in \mathbb{N}$ we have

$$\begin{aligned} f_{n_k+1}(T) &- \frac{s(n_k)}{s(n_k+1)} T f_{n_k}(T) \\ &= \frac{\sum_{j=0}^{n_k+1} (\Delta s)(n_k+1-j) T^j}{s(n_k+1)} - \frac{s(n_k)}{s(n_k+1)} \frac{\sum_{j=0}^{n_k} (\Delta s)(n_k-j) T^{j+1}}{s(n_k)} \\ &= \frac{\sum_{j=0}^{n_k+1} (\Delta s)(n_k+1-j) T^j - \sum_{j=1}^{n_k+1} (\Delta s)(n_k+1-j) T^j}{s(n_k+1)} \\ &= \frac{(\Delta s)(n_k+1) I_X}{s(n_k+1)} = \left(1 - \frac{s(n_k)}{s(n_k+1)}\right) I_X, \end{aligned}$$

and now (3.1) yields the desired result. Then we are enabled to apply Theorem 3.1 with $m_k = n_k + 1$ and $T_k = (s(n_k)/s(n_k+1))T$ for every $k \in \mathbb{N}$, which finishes the proof. □

R e m a r k 3.3. Let $s\colon \mathbb{N} \to \mathbb{R}$ be a nondecreasing sequence of strictly positive real numbers. Then $s(n+1)/s(n) \geqslant 1$ for every $n \in \mathbb{N}$, and consequently, $\liminf_{n\to\infty} s(n+1)/s(n) \geqslant 1$. From Theorem 2.8 we also derive that $\liminf_{n\to\infty} \sqrt[n]{s(n)} \geqslant 1$.

Corollary 3.4. *Let X be a complex nonzero Banach space, $T \in L(X)$, and $s\colon \mathbb{N} \to \mathbb{R}$ be a nondecreasing sequence of strictly positive real numbers such that $\liminf_{n\to\infty} \sqrt[n]{s(n)} = 1$. If the sequence $\left(\sum_{k=0}^{n} (\Delta s)(n-k) T^k / s(n)\right)_{n\in\mathbb{N}}$ converges in $L(X)$, then 1 is either in $\varrho(T)$, or a simple pole of $\mathfrak{R}_T$ (and consequently $\mathcal{R}(I_X - T)$ is closed, and $X = \mathcal{N}(I_X - T) \oplus \mathcal{R}(I_X - T)$). Furthermore, the sequence $\left(\sum_{k=0}^{n} (\Delta s)(n-k) T^k / s(n)\right)_{n\in\mathbb{N}}$ converges in $L(X)$ to the projection of X onto $\mathcal{N}(I_X - T)$ along $\mathcal{R}(I_X - T)$.*

P r o o f. From Theorem 2.8 and Remark 3.3 we conclude that

$$\liminf_{n\to\infty} \frac{s(n+1)}{s(n)} = 1.$$

Now the desired result follows from Theorem 3.2. □

The next example shows that if T is a bounded linear operator on a complex nonzero Banach space X such that the sequence $\Big(\sum_{k=0}^{n} (\Delta s)(n-k)T^k/s(n)\Big)_{n\in\mathbb{N}}$ converges in $L(X)$ for a nondecreasing sequence s of strictly positive real numbers satisfying $\liminf_{n\to\infty} s(n+1)/s(n) > 1$, then 1 may neither be in $\varrho(T)$, nor be a pole of $\mathfrak{R}_T$.

Henceforth, we will denote the open unit disk in $\mathbb{C}$ by D. Also, for each $r \in [0,\infty]$ we set

$$D_r = \{z \in \mathbb{C}\colon\ |z| < r\}.$$

Notice that $D_\infty = \mathbb{C}$, $D_0 = \emptyset$, and $D_1 = D$.

E x a m p l e 3.5. First of all, fix two real numbers α and β satisfying $1 \leqslant \beta < \alpha$. Now let $a_\alpha\colon\ \mathbb{N} \to \mathbb{R}$ be the real sequence defined by

$$a_\alpha(n) = \alpha^n \quad \text{for every } n \in \mathbb{N}$$

and set $s_\alpha = \Sigma a_\alpha$. Then s_α is strictly increasing, being $a_\alpha(n) > 0$ for every $n \in \mathbb{N}$. Furthermore,

$$(\Delta s_\alpha)(n) = a_\alpha(n) = \alpha^n \quad \text{and} \quad s_\alpha(n) = \sum_{k=0}^{n} \alpha^k = \frac{\alpha^{n+1}-1}{\alpha-1} \quad \text{for every } n \in \mathbb{N},$$

and consequently,

$$\lim_{n\to\infty} \frac{s_\alpha(n+1)}{s_\alpha(n)} = \lim_{n\to\infty} \frac{\alpha^{n+2}-1}{\alpha^{n+1}-1} = \alpha > 1.$$

Finally, let $T_\beta \in L(l_2)$ be defined by $T_\beta = \beta S$, where S denotes the unilateral shift operator on l_2. Namely, $S\colon\ l_2 \to l_2$ is the bounded linear operator defined by

$$Sx = \sum_{n=0}^{\infty} x(n)e_{n+1} \quad \text{for every } x \in l_2$$

(where $\{e_n\colon\ n \in \mathbb{N}\}$ denotes the canonical orthonormal basis of l_2).

We recall that $\sigma(S) = \overline{D}$ (see for instance [7], Solution 67). Hence, $\sigma(T_\beta) = \overline{D_\beta}$ and consequently, $r(T_\beta) = \beta$. Since $\beta \geqslant 1$, it follows that 1 is neither in $\varrho(T_\beta)$, nor is a pole of $\mathfrak{R}_{T_\beta}$. Nevertheless, we prove that the sequence

$$\Big(\frac{\sum_{k=0}^{n}(\Delta s_\alpha)(n-k)T_\beta^k}{s_\alpha(n)}\Big)_{n\in\mathbb{N}}$$

converges in $L(l_2)$.

We begin by remarking that for each $n \in \mathbb{N}$ we have

$$\begin{aligned}\text{(3.2)}\quad \frac{\sum_{k=0}^{n}(\Delta s_\alpha)(n-k)T_\beta^k}{s_\alpha(n)} &= \Big(\frac{\alpha-1}{\alpha^{n+1}-1}\Big)\sum_{k=0}^{n} a_\alpha(n-k)T_\beta^k \\ &= \Big(\frac{\alpha-1}{\alpha^{n+1}-1}\Big)\sum_{k=0}^{n}\alpha^{n-k}T_\beta^k = \frac{(\alpha-1)\alpha^{n+1}}{\alpha^{n+1}-1}\sum_{k=0}^{n}\frac{T_\beta^k}{\alpha^{k+1}}.\end{aligned}$$

Since $\alpha > \beta = r(T_\beta)$, it follows that $\alpha \in \varrho(T_\beta)$ and the sequence $\Big(\sum\limits_{k=0}^{n} T_\beta^k/\alpha^{k+1}\Big)_{n\in\mathbb{N}}$ converges to $(\alpha I_{l_2} - T_\beta)^{-1}$ in $L(l_2)$ (see for instance [15], V, 3.1). Being

$$\lim_{n\to\infty}\frac{(\alpha-1)\alpha^{n+1}}{\alpha^{n+1}-1} = \alpha - 1,$$

from (3.2) we conclude that

$$\frac{\sum_{k=0}^{n}(\Delta s_\alpha)(n-k)T_\beta^k}{s_\alpha(n)} \underset{n\to\infty}{\longrightarrow} (\alpha-1)(\alpha I_{l_2} - T_\beta)^{-1} \quad \text{in } L(l_2),$$

which gives the desired result.

The following example shows that if $s\colon \mathbb{N} \to \mathbb{R}$ is a nondecreasing sequence of strictly positive real numbers satisfying $\liminf\limits_{n\to\infty} s(n+1)/s(n) = 1$, and 1 is either in $\varrho(T)$ or a simple pole of $\mathfrak{R}_T$ for a bounded linear operator T on a complex nonzero Banach space X, then the sequence $\Big(\sum\limits_{k=0}^{n}(\Delta s)(n-k)T^k/s(n)\Big)_{n\in\mathbb{N}}$ need not converge in $L(X)$ (not even if in addition s is strictly increasing, $\lim\limits_{n\to\infty} s(n) = \infty$, $\lim\limits_{n\to\infty} s(n+1)/s(n) = 1$, $\lim\limits_{n\to\infty}\|T^n\|_{L(X)}/s(n) = 0$, and $\mathcal{H}(s) < \infty$). Hence, the converse of Theorem 3.2 does not hold unless some additional conditions on the sequence s are assumed. Indeed, Theorem 2.6 provides a converse of Theorem 3.2, under the additional hypotheses $\lim\limits_{n\to\infty} s(n) = \infty$, $\lim\limits_{n\to\infty} s(n+1)/s(n) = 1$, $\lim\limits_{n\to\infty}\|T^n\|_{L(X)}/s(n) = 0$, and $\Delta^q s \in l_1$ for some $q \in \mathbb{N}_2$ (which implies $\mathcal{H}(s) < \infty$; indeed, $\mathcal{H}(s) \leqslant q-1$ by virtue of Proposition 2.5). In particular, the converse of Theorem 3.2 holds for $s = A_\alpha$, where $\alpha \in (0,\infty)$ is such that $\lim\limits_{n\to\infty}\|T^n\|_{L(X)}/n^\alpha = 0$: this can be derived either from Theorem 2.3, or from Theorem 2.6 together with Remark 2.7 and (2.2).

E x a m p l e 3.6. Let X be a complex nonzero Banach space. We set $T = -I_X$. Then $\sigma(T) = \{-1\}$, and consequently $1 \in \varrho(T)$. Now let $\tau\colon \mathbb{N} \to \mathbb{R}$ be the sequence of strictly positive real numbers defined by

$$\tau(2k) = 1 \quad \text{and} \quad \tau(2k+1) = \frac{1}{2^k} \quad \text{for every } k \in \mathbb{N}.$$

We set $s = \Sigma\tau$. Then $\Delta s = \tau$. Since $\tau(n) > 0$ for every $n \in \mathbb{N}$, it follows that s is a strictly increasing sequence of strictly positive real numbers. Furthermore, we have $s(0) = \tau(0) = 1$, and for each $k \in \mathbb{Z}_+$,

$$\begin{aligned} s(2k) &= \sum_{h=0}^{2k} \tau(h) = \sum_{j=0}^{k} \tau(2j) + \sum_{j=0}^{k-1} \tau(2j+1) = k + 1 + \sum_{j=0}^{k-1} \frac{1}{2^j} \\ &= k + 1 + 2\Big(1 - \frac{1}{2^k}\Big) = k + 3 - \frac{1}{2^{k-1}}. \end{aligned}$$

Then for each $k \in \mathbb{N}$ we have

$$s(2k) = k + 3 - \frac{1}{2^{k-1}}, \tag{3.3}$$

which in turn gives

$$s(2k+1) = s(2k) + \tau(2k+1) = k + 3 - \frac{1}{2^{k-1}} + \frac{1}{2^k} = k + 3 - \frac{1}{2^k}. \tag{3.4}$$

From (3.3) and (3.4) we conclude that for each $n \in \mathbb{N}$ we have

$$s(n) = \Big[\frac{n}{2}\Big] + 3 - \frac{1}{2^{[(n-1)/2]}} \geqslant \Big[\frac{n}{2}\Big] + 3 - \frac{1}{2^{-1}} = \Big[\frac{n}{2}\Big] + 1 > \frac{n}{2}.$$

Notice also that

$$s(n) < \Big[\frac{n}{2}\Big] + 3 \leqslant \frac{n}{2} + 3 \quad \text{for every } n \in \mathbb{N}.$$

Hence

$$\lim_{n\to\infty} s(n) = \infty \quad \text{and} \quad \mathcal{H}(s) = 1 < \infty. \tag{3.5}$$

Since

$$s(n) < s(n+1) = s(n) + \tau(n+1) \leqslant s(n) + 1 \quad \text{for every } n \in \mathbb{N},$$

from (3.5) we conclude that $\lim\limits_{n\to\infty} s(n+1)/s(n) = 1$. Finally, (3.5) also gives

$$\lim_{n\to\infty} \frac{\|T^n\|_{L(X)}}{s(n)} = \lim_{n\to\infty} \frac{\|(-1)^n I_X\|_{L(X)}}{s(n)} = \lim_{n\to\infty} \frac{1}{s(n)} = 0.$$

We prove that the sequence $\left(\sum_{k=0}^{n}(\Delta s)(n-k)T^k/s(n)\right)_{n\in\mathbb{N}}$ does not converge in $L(X)$. Since $\Delta s=\tau$, by virtue of (3.3) and (3.4) for each $k\in\mathbb{Z}_+$ we have

$$\tag{3.6} \begin{aligned} &\frac{\sum_{j=0}^{2k}(\Delta s)(2k-j)T^j}{s(2k)} \\ &= \frac{\sum_{j=0}^{2k}\tau(2k-j)(-1)^j I_X}{s(2k)} = \left(\frac{\sum_{j=0}^{2k}(-1)^j\tau(2k-j)}{s(2k)}\right)I_X \\ &= \left(\frac{\sum_{h=0}^{k}\tau(2k-2h)-\sum_{h=0}^{k-1}\tau(2k-(2h+1))}{s(2k)}\right)I_X \\ &= \left(\frac{k+1-\sum_{h=0}^{k-1}\tau(2(k-h-1)+1)}{s(2k)}\right)I_X \\ &= \left(\frac{k+1-\sum_{j=0}^{k-1}\tau(2j+1)}{s(2k)}\right)I_X = \left(\frac{k+1-\sum_{j=0}^{k-1}2^{-j}}{k+3-2^{1-k}}\right)I_X \\ &= \left(\frac{k+1-2(1-2^{-k})}{k+3-2^{1-k}}\right)I_X = \left(\frac{k-1+2^{1-k}}{k+3-2^{1-k}}\right)I_X \end{aligned}$$

and

$$\tag{3.7} \begin{aligned} &\frac{\sum_{j=0}^{2k+1}(\Delta s)(2k+1-j)T^j}{s(2k+1)} \\ &= \frac{\sum_{j=0}^{2k+1}\tau(2k+1-j)(-1)^j I_X}{s(2k+1)} = \left(\frac{\sum_{j=0}^{2k+1}(-1)^j\tau(2k+1-j)}{s(2k+1)}\right)I_X \\ &= \left(\frac{\sum_{h=0}^{k}\tau(2k+1-2h)-\sum_{h=0}^{k}\tau(2k+1-(2h+1))}{s(2k+1)}\right)I_X \\ &= -\left(\frac{\sum_{h=0}^{k}\tau(2k-2h)-\sum_{h=0}^{k}\tau(2(k-h)+1)}{s(2k+1)}\right)I_X \\ &= -\left(\frac{k+1-\sum_{j=0}^{k}\tau(2j+1)}{s(2k+1)}\right)I_X = -\left(\frac{k+1-\sum_{j=0}^{k}2^{-j}}{s(2k+1)}\right)I_X \\ &= -\left(\frac{k+1-2(1-2^{-k-1})}{s(2k+1)}\right)I_X = -\left(\frac{k-1+2^{-k}}{k+3-2^{-k}}\right)I_X. \end{aligned}$$

Now from (3.6) and (3.7) we conclude that

$$\frac{\sum_{j=0}^{2k}(\Delta s)(2k-j)T^j}{s(2k)} \underset{k\to\infty}{\longrightarrow} I_X \quad \text{and} \quad \frac{\sum_{j=0}^{2k+1}(\Delta s)(2k+1-j)T^j}{s(2k+1)} \underset{k\to\infty}{\longrightarrow} -I_X$$

in $L(X)$. Hence, the sequence $\left(\sum_{k=0}^{n}(\Delta s)(n-k)T^k/s(n)\right)_{n\in\mathbb{N}}$ does not converge in $L(X)$. Notice also that no subsequence of $\left(\sum_{k=0}^{n}(\Delta s)(n-k)T^k/s(n)\right)_{n\in\mathbb{N}}$ converges in $L(X)$ to $0_{L(X)}$, that is (being $1 \in \varrho(T)$), to the projection of X onto $\mathcal{N}(I_X - T)$ along $\mathcal{R}(I_X - T)$.

From Example 3.6 we conclude that the assumption that $\Delta^q s \in l_1$ for some $q \in \mathbb{N}_2$ cannot be dropped from Theorem 2.6, and cannot even be replaced by the weaker (see Proposition 2.5) condition $\mathcal{H}(s) < \infty$.

Remark 3.7. From Theorem 2.6 it follows that the sequence s of Example 3.6 satisfies $\Delta^q s \in l_1$ for no $q \in \mathbb{N}$. Then Example 3.6 also shows that if t is a divergent nondecreasing—or even strictly increasing—sequence of strictly positive real numbers satisfying $\lim\limits_{n\to\infty} t(n+1)/t(n) = 1$ and $\mathcal{H}(t) < \infty$, there may exist no $q \in \mathbb{N}$ for which $\Delta^q t \in l_1$. Hence, the converse of Proposition 2.5 does not hold.

4. Spectral consequences of boundedness of Nörlund means in $L(X)$

As recalled in Remark 2.7, convergence of $\Big(\sum\limits_{k=0}^{n} (\Delta s)(n-k)T^k/s(n)\Big)_{n\in\mathbb{N}}$ in $L(X)$ (where X is a complex nonzero Banach space, $T \in L(X)$, and s is a nondecreasing sequence of strictly positive real numbers such that $\lim\limits_{n\to\infty} s(n) = \infty$, $\lim\limits_{n\to\infty} s(n+1)/s(n) = 1$, and $\Delta^q s \in l_1$ for some $q \in \mathbb{N}_2$) does not imply $\lim\limits_{n\to\infty} \|T^n\|_{L(X)}/s(n) = 0$. Since if s is as above, then condition $\lim\limits_{n\to\infty} \|T^n\|_{L(X)}/s(n) = 0$ in turn implies $r(T) \leqslant 1$ (see 6.1 in [2], or 2.1 in [1], or else check it directly by using Proposition 2.5 and Remark 2.4), one could wonder whether convergence in $L(X)$ of the aforementioned sequence of Nörlund means implies the weaker (than convergence to zero of $\|T^n\|_{L(X)}/s(n)$) condition $r(T) \leqslant 1$. This question is one of our concerns in this section. Also, for more general nondecreasing sequences s of strictly positive real numbers than those satisfying the conditions above, we are going to seek for sufficient conditions in order that if for a bounded linear operator T on a complex nonzero Banach space X the sequence $\Big(\sum\limits_{k=0}^{n} (\Delta s)(n-k)T^k/s(n)\Big)_{n\in\mathbb{N}}$ is bounded in $L(X)$, then $r(T) \leqslant 1$. We begin by deriving from 2.1 in [1], and from Remark 2.4 a result that somehow suggests us the class of nondecreasing sequences s of strictly positive real numbers for which to consider the latter problem.

Proposition 4.1. *Let X be a complex nonzero Banach space, and $T \in L(X)$. Then the following conditions are equivalent:*

(i) $r(T) \leqslant 1$;

(ii) *there exists a strictly increasing sequence s of strictly positive real numbers such that $\lim\limits_{n\to\infty} s(n) = \infty$, $\lim\limits_{n\to\infty} s(n+1)/s(n) = 1$, and $\lim\limits_{n\to\infty} \|T^n\|_{L(X)}/s(n) = 0$;*

(iii) *there exists a nondecreasing sequence* s *of strictly positive real numbers such that* $\lim_{n\to\infty} \sqrt[n]{s(n)} = 1$ *and* $\lim_{n\to\infty} \|T^n\|_{L(X)}/s(n) = 0$*;*

(iv) *there exists a sequence* s *of strictly positive real numbers such that*

$$\lim_{n\to\infty} \sqrt[n]{s(n)} = 1$$

and the real sequence $(\|T^n\|_{L(X)}/s(n))_{n\in\mathbb{N}}$ *is bounded.*

Proof. If $r(T) \leqslant 1$, from 2.1 in [1] and from Remark 2.4 it follows that there exists a nondecreasing sequence s of strictly positive real numbers such that $\lim_{n\to\infty} s(n+1)/s(n) = 1$ and $\|T^n\|_{L(X)} \leqslant s(n)$ for every $n \in \mathbb{N}$. By going to the sequence $((n+1)s(n))_{n\in\mathbb{N}}$ if necessary, it is not restrictive to assume that in addition s is strictly increasing, $\lim_{n\to\infty} s(n) = \infty$, and $\lim_{n\to\infty} \|T^n\|_{L(X)}/s(n) = 0$. Hence (i) implies (ii). Furthermore, it follows from Theorem 2.8 that (ii) implies (iii). Also, it is clear that (iii) implies (iv). Finally, suppose that condition (iv) holds. Then we can proceed as in Remark 2.8 of [2], and similarly to the beginning of the proof of 2.1 in [1]: if we fix $M \in (0,\infty)$ such that $\|T^n\|_{L(X)}/s(n) \leqslant M$ for every $n \in \mathbb{N}$, we have

$$r(T) = \lim_{n\to\infty} \|T^n\|_{L(X)}^{1/n} = \lim_{n\to\infty} (\|T^n\|_{L(X)}/s(n))^{1/n} \leqslant \lim_{n\to\infty} M^{1/n} = 1$$

and therefore condition (i) is satisfied. The proof is now complete. □

In Theorem 4.8 we will give a sufficient condition on a nondecreasing sequence s of strictly positive real numbers, satisfying $\lim_{n\to\infty} \sqrt[n]{s(n)} = 1$, in order that any bounded linear operator T on a complex nonzero Banach space X for which the sequence of the Nörlund means of the powers of T induced by s is bounded in $L(X)$ must satisfy $r(T) \leqslant 1$ (which, by virtue of Proposition 4.1, is a weaker condition than convergence to zero, and even than boundedness, of $\|T^n\|_{L(X)}/s(n)$).

Definition 4.2. For each $a \in \mathbb{C}^{\mathbb{N}}$, let $\kappa(a) \in [0,\infty]$ be the radius of convergence of the power series $\sum_{n=0}^{\infty} a(n)z^n$.

We recall that

$$\kappa(a) = \begin{cases} 0 & \text{if } \limsup_{n\to\infty} \sqrt[n]{|a(n)|} = \infty, \\ \dfrac{1}{\limsup_{n\to\infty} \sqrt[n]{|a(n)|}} & \text{if } \limsup_{n\to\infty} \sqrt[n]{|a(n)|} \in (0,\infty), \\ \infty & \text{if } \lim_{n\to\infty} \sqrt[n]{|a(n)|} = 0. \end{cases}$$

Following the notation we have introduced before Example 3.5, the disk of convergence of the power series $\sum\limits_{n=0}^{\infty} a(n)z^n$ coincides with $D_{\kappa(a)}$. In particular, if $\limsup\limits_{n\to\infty} \sqrt[n]{|a(n)|} = 1$, we have $D_{\kappa(a)} = D_1 = D$.

Definition 4.3. For each $a \in \mathbb{C}^{\mathbb{N}}$, let $h_a\colon D_{\kappa(a)} \to \mathbb{C}$ be the holomorphic function defined by

$$h_a(z) = \sum_{n=0}^{\infty} a(n)z^n \quad \text{for every } z \in D_{\kappa(a)}.$$

Notice that if $\kappa(a) = 0$, then the domain of h_a is the empty set.

Lemma 4.4. *Let $a \in \mathbb{C}^{\mathbb{N}}$. Then $\kappa(\Delta a) \geqslant \kappa(a)$ (which gives $D_{\kappa(a)} \subseteq D_{\kappa(\Delta a)}$), and if $\kappa(a) \neq 1$, we have $\kappa(\Delta a) = \kappa(a)$. Furthermore,*

$$h_{\Delta a}(z) = (1-z)h_a(z) \quad \text{for every } z \in D_{\kappa(a)}, \tag{4.1}$$

and consequently, h_a and $h_{\Delta a}$ have exactly the same zeros in $D_{\kappa(a)}\setminus\{1\}$; in particular, they have exactly the same zeros in $D_{1\wedge\kappa(a)}$.

P r o o f. We begin by observing that for each $z \in \mathbb{C}$ for which the series $\sum\limits_{n=0}^{\infty} a(n)z^n$ converges, the series $\sum\limits_{n=0}^{\infty} (\Delta a)(n)z^n$ also converges. Hence, $\kappa(\Delta a) \geqslant \kappa(a)$, which in turn gives $D_{\kappa(a)} \subseteq D_{\kappa(\Delta a)}$. Also, for each $z \in D_{\kappa(a)}$ we have

$$\begin{aligned} h_{\Delta a}(z) &= \sum_{n=0}^{\infty}(\Delta a)(n)z^n = \sum_{n=0}^{\infty} a(n)z^n - \sum_{n=1}^{\infty} a(n-1)z^n \\ &= \sum_{n=0}^{\infty} a(n)z^n - z\sum_{n=0}^{\infty} a(n)z^n = (1-z)\sum_{n=0}^{\infty} a(n)z^n = (1-z)h_a(z). \end{aligned}$$

We have thus proved (4.1), of which the final claim about zeros is an immediate consequence. Now we prove that $\kappa(a) \neq 1$ implies $\kappa(\Delta a) = \kappa(a)$.

Since being $a = \Sigma\Delta a$, the power series $\sum\limits_{n=0}^{\infty} a(n)z^n$ is the Cauchy product of the two power series $\sum\limits_{n=0}^{\infty} z^n$ (whose radius of convergence is 1) and $\sum\limits_{n=0}^{\infty} (\Delta a)(n)z^n$, from [3], III, 1.6, we conclude that $\kappa(a) \geqslant 1 \wedge \kappa(\Delta a)$. Then if $\kappa(a) < 1$, it follows that $1 \wedge \kappa(\Delta a) = \kappa(\Delta a)$, and consequently, $\kappa(a) \geqslant \kappa(\Delta a)$, which in turn gives $\kappa(a) = \kappa(\Delta a)$. Finally, we suppose $\kappa(a) > 1$. Then $1 \in D_{\kappa(a)} \subseteq D_{\kappa(\Delta a)}$. Now we define the holomorphic function

$$\varphi\colon D_{\kappa(\Delta a)} \setminus \{1\} \ni z \mapsto \frac{h_{\Delta a}(z)}{1-z} \in \mathbb{C}.$$

From (4.1) it follows that $\varphi(z) = h_a(z)$ for every $z \in D_{\kappa(a)} \setminus \{1\}$. Since h_a is holomorphic in $D_{\kappa(a)}$, we conclude that φ has a removable singularity at 1. Since in turn $h_a(z) = \sum_{n=0}^{\infty} a(n)z^n$ for every $z \in D_{\kappa(a)}$, we derive that the series $\sum_{n=0}^{\infty} a(n)z^n$ converges for every $z \in D_{\kappa(\Delta a)}$, which gives $\kappa(a) \geqslant \kappa(\Delta a)$. Hence $\kappa(\Delta a) = \kappa(a)$, which finishes the proof. □

Corollary 4.5. *Let $a \in \mathbb{C}^{\mathbb{N}}$. Then $\kappa(\Sigma a) \leqslant \kappa(a)$ (which gives $D_{\kappa(\Sigma a)} \subseteq D_{\kappa(a)}$). Furthermore, we have:*
(i) *if $\kappa(a) \leqslant 1$, then $\kappa(\Sigma a) = \kappa(a)$;*
(ii) *if $\kappa(a) > 1$, then $\kappa(\Sigma a)$ coincides either with 1 or with $\kappa(a)$.*

Hence $\kappa(\Sigma a) \geqslant 1 \wedge \kappa(a)$ (which gives $D_{\kappa(\Sigma a)} \supseteq D_{1\wedge\kappa(a)}$). Finally,

$$h_{\Sigma a}(z) = \frac{h_a(z)}{1-z} \quad \text{for every } z \in D_{\kappa(\Sigma a)} \setminus \{1\}, \tag{4.2}$$

and consequently, h_a and $h_{\Sigma a}$ have exactly the same zeros in $D_{\kappa(\Sigma a)} \setminus \{1\}$; in particular, they have exacly the same zeros in $D_{1\wedge\kappa(a)}$.

Proof. Since $a = \Delta\Sigma a$, the inequality $\kappa(\Sigma a) \leqslant \kappa(a)$ (which yields $D_{\kappa(\Sigma a)} \subseteq D_{\kappa(a)}$) follows from Lemma 4.4. From Lemma 4.4 we also derive that $\kappa(a) > \kappa(\Sigma a)$ implies $\kappa(\Sigma a) = 1$. This in turn gives (i) and (ii), from which we derive that $\kappa(\Sigma a) \geqslant 1\wedge\kappa(a)$ (and consequently $D_{\kappa(\Sigma a)} \supseteq D_{1\wedge\kappa(a)}$). Finally, (4.2) is a consequence of (4.1). □

Remark 4.6. Let $a \in \mathbb{C}^{\mathbb{N}}$. Proceeding by induction, from Lemma 4.4 we derive that for each $p \in \mathbb{N}$ we have $\kappa(\Delta^p a) \geqslant \kappa(a)$, and consequently, $D_{\kappa(a)} \subseteq D_{\kappa(\Delta^p a)}$. Furthermore, if $\kappa(a) \neq 1$, it follows that $\kappa(\Delta^p a) = \kappa(a)$ for every $p \in \mathbb{N}$. If $\kappa(a) = 1$, we either have $\kappa(\Delta^p a) = 1$ for every $p \in \mathbb{N}$, or there exists $p_0 \in \mathbb{Z}_+$ such that $\kappa(\Delta^{p_0} a) > 1$ and

$$\kappa(\Delta^p a) = \begin{cases} 1 & \text{for } p = 0, \dots, p_0 - 1, \\ \kappa(\Delta^{p_0} a) & \text{for } p \in \mathbb{N}_{p_0}. \end{cases}$$

Also, from Corollary 4.5 we derive that for each $p \in \mathbb{N}$ we have $\kappa(\Sigma^p a) \leqslant \kappa(a)$, and consequently, $D_{\kappa(\Sigma^p a)} \subseteq D_{\kappa(a)}$. Furthermore, if $\kappa(a) \leqslant 1$, then $\kappa(\Sigma^p a) = \kappa(a)$ for every $p \in \mathbb{N}$. If, instead, $\kappa(a) > 1$, then either $\kappa(\Sigma^p a) = \kappa(a)$ for every $p \in \mathbb{N}$, or there exists $p_0 \in \mathbb{Z}_+$ such that

$$\kappa(\Sigma^p a) = \begin{cases} \kappa(a) & \text{for } p = 0, \dots, p_0 - 1, \\ 1 & \text{for } p \in \mathbb{N}_{p_0}. \end{cases}$$

Hence, in any case (namely, whatever is the value of $\kappa(a)$), we have $\kappa(\Sigma^p a) \geqslant 1 \wedge \kappa(a)$ for every $p \in \mathbb{N}$.

Finally, by virtue of Lemma 4.4 and Corollary 4.5, for each $p \in \mathbb{N}$ we have

$$h_{\Delta^p a}(z) = (1-z)^p h_a(z) \quad \text{for every } z \in D_{\kappa(a)}$$

and

$$h_{\Sigma^p a}(z) = \frac{h_a(z)}{(1-z)^p} \quad \text{for every } z \in D_{\kappa(\Sigma^p a)} \setminus \{1\} \supseteq D_{1\wedge\kappa(a)}.$$

Hence, h_a, $h_{\Delta^p a}$ and $h_{\Sigma^p a}$ have exactly the same zeros in $D_{1\wedge\kappa(a)}$.

Remark 4.7. If $s\colon \mathbb{N} \to \mathbb{R}$ is a nondecreasing sequence of strictly positive real numbers, since $\liminf\limits_{n\to\infty} \sqrt[n]{s(n)} \geqslant 1$ (see Remark 3.3), it follows that $\kappa(s) \leqslant 1$. Furthermore, we have

$$\kappa(s) = 1 \Leftrightarrow \limsup_{n\to\infty} \sqrt[n]{s(n)} = 1 \Leftrightarrow \lim_{n\to\infty} \sqrt[n]{s(n)} = 1,$$

in which case the domain $D_{\kappa(s)}$ of h_s is the open unit disk D.

Theorem 4.8. *Let X be a complex nonzero Banach space, $T \in L(X)$, and $s\colon \mathbb{N} \to \mathbb{R}$ be a nondecreasing sequence of strictly positive real numbers such that $\lim\limits_{n\to\infty} \sqrt[n]{s(n)} = 1$ and the holomorphic function h_s has no zeros in D. If the sequence $\Big(\sum\limits_{k=0}^{n} (\Delta s)(n-k)T^k/s(n)\Big)_{n\in\mathbb{N}}$ is bounded in $L(X)$, then $r(T) \leqslant 1$.*

Proof. Let R denote the radius of convergence of the power series $\sum\limits_{n=0}^{\infty} z^n T^n$ in $L(X)$. Since $\lim\limits_{n\to\infty} \|T^n\|_{L(X)}^{1/n} = r(T)$, it follows that

$$R = \begin{cases} 1/r(T) & \text{if } r(T) > 0, \\ \infty & \text{if } r(T) = 0. \end{cases}$$

Hence, $R \in (0,\infty]$, which gives $D_R \neq \emptyset$. Let $\Phi\colon D_R \to L(X)$ be the analytic function defined by

$$\Phi(z) = \sum_{n=0}^{\infty} z^n T^n \quad \text{for every } z \in D_R.$$

Notice that $\Phi(z) = (1/z)\mathfrak{R}_T(1/z)$ for every $z \in D_R \setminus \{0\}$. Now let $M \in (0,\infty)$ be such that

$$\left\| \frac{\sum_{k=0}^{n} (\Delta s)(n-k)T^k}{s(n)} \right\|_{L(X)} \leqslant M \quad \text{for every } n \in \mathbb{N}.$$

Then

$$\limsup_{n\to\infty}\left\|\sum_{k=0}^{n}(\Delta s)(n-k)T^k\right\|_{L(X)}^{1/n} \leqslant \lim_{n\to\infty} M^{1/n}\sqrt[n]{s(n)} = 1.$$

Hence, the power series $\sum_{n=0}^{\infty} z^n\Big(\sum_{k=0}^{n}(\Delta s)(n-k)T^k\Big)$ in $L(X)$ has radius of convergence greater than or equal to 1, and consequently converges in $L(X)$ for every $z \in D$. Let $\Psi\colon D \to L(X)$ be the analytic function defined by

$$\Psi(z) = \sum_{n=0}^{\infty} z^n\left(\sum_{k=0}^{n}(\Delta s)(n-k)T^k\right) \quad \text{for every } z \in D.$$

Since $\lim_{n\to\infty}\sqrt[n]{s(n)} = 1$, from Lemma 4.4 it follows that $\kappa(\Delta s) \geqslant \kappa(s) = 1$, and consequently, $D_{\kappa(\Delta s)} \supseteq D$. We remark that

$$h_{\Delta s}(z)\Phi(z) = \sum_{n=0}^{\infty} z^n\left(\sum_{k=0}^{n}(\Delta s)(n-k)T^k\right) = \Psi(z) \quad \text{for every } z \in D \cap D_R.$$

Since $h_{\Delta s}$ has no zeros in D by Lemma 4.4, it follows that

$$\sum_{n=0}^{\infty} z^n T^n = \Phi(z) = \frac{1}{h_{\Delta s}(z)}\Psi(z) \quad \text{for every } z \in D \cap D_R. \tag{4.3}$$

Since the function from D into $L(X)$ which maps each $z \in D$ into $(1/h_{\Delta s}(z))\Psi(z)$ is analytic in D, being so both $h_{\Delta s}$ and Ψ, from (4.3) we conclude that the power series $\sum_{n=0}^{\infty} z^n T^n$ converges in $L(X)$ for every $z \in D$. Hence $R \geqslant 1$, which in turn gives $r(T) \leqslant 1$. The proof is now complete. □

We are now going to address the first problem introduced at the beginning of this section. That is, if for a bounded linear operator T on a complex nonzero Banach space X, and a nondecreasing sequence s of strictly positive real numbers satisfying all the hypotheses stated in Theorem 2.6 except $\lim_{n\to\infty}\|T^n\|_{L(X)}/s(n) = 0$, the sequence of the Nörlund means of the powers of T induced by s converges in $L(X)$, can we conclude that $r(T) \leqslant 1$? As we shall see, the answer to this question is in the negative.

Definition 4.9. For each $n \in \mathbb{N}$ let $\tau_n\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ be the linear operator defined by

$$(\tau_n a)(k) = \begin{cases} 0 & \text{for } k = 0,\dots,n \\ a(k) & \text{for } k \in \mathbb{N}_{n+1} \end{cases} \quad \text{for every } a \in \mathbb{C}^{\mathbb{N}}.$$

Notice that $\kappa(\tau_n a) = \kappa(a)$ for every $n \in \mathbb{N}$ and every $a \in \mathbb{C}^{\mathbb{N}}$.

In the next result we give a formula—which will be useful to us in the sequel—for $\Delta^m \tau_n a$, where $m \in \mathbb{Z}_+$, $n \in \mathbb{N}$.

Lemma 4.10. *Let $a \in \mathbb{C}^{\mathbb{N}}$. Then for each $n \in \mathbb{N}$ and each $m \in \mathbb{Z}_+$ we have*

$$(\Delta^m \tau_n a)(k) = \begin{cases} 0 & \text{for } k = 0, \dots, n, \\ \displaystyle\sum_{j=1}^{k-n} (-1)^{k-n-j} \binom{m-j}{k-n-j} (\Delta^{j-1} a)(n+j) & \\ & \text{for } k = n+1, \dots, n+m, \\ (\Delta^m a)(k) & \text{for } k \in \mathbb{N}_{n+m+1}. \end{cases}$$

Proof. We proceed by induction on m. Fix $n \in \mathbb{N}$, and let S_n denote the set of all positive integers m for which the desired formula holds. We begin by remarking that

$$(\Delta \tau_n a)(k) = \begin{cases} 0 & \text{for } k = 0, \dots, n, \\ a(n+1) & \text{for } k = n+1, \\ (\Delta a)(k) & \text{for } k \in \mathbb{N}_{n+2}. \end{cases} \tag{4.4}$$

Since

$$\begin{aligned} &\sum_{j=1}^{(n+1)-n} (-1)^{(n+1)-n-j} \binom{1-j}{(n+1)-n-j} (\Delta^{j-1} a)(n+j) \\ &\quad = \sum_{j=1}^{1} (-1)^{1-j} \binom{1-j}{1-j} (\Delta^{j-1} a)(n+j) = (\Delta^0 a)(n+1) = a(n+1), \end{aligned}$$

from (4.4) we conclude that $1 \in S_n$.

Now let $m \in S_n$. We prove that $m+1 \in S_n$. From the inductive hypothesis it follows that

$$(\Delta^{m+1} \tau_n a)(k) = (\Delta(\Delta^m \tau_n a))(k) = 0 \quad \text{for } k = 0, \dots, n. \tag{4.5}$$

Furthermore,

$$\begin{aligned} &(\Delta^{m+1} \tau_n a)(n+1) \\ &\quad = (\Delta^m \tau_n a)(n+1) = \sum_{j=1}^{1} (-1)^{1-j} \binom{m-j}{1-j} (\Delta^{j-1} a)(n+j) \\ &\quad = a(n+1) = \sum_{j=1}^{1} (-1)^{1-j} \binom{m+1-j}{1-j} (\Delta^{j-1} a)(n+j) \\ &\quad = \sum_{j=1}^{(n+1)-n} (-1)^{(n+1)-n-j} \binom{m+1-j}{(n+1)-n-j} (\Delta^{j-1} a)(n+j). \end{aligned} \tag{4.6}$$

We also remark that for each $k \in \mathbb{N}$ satisfying $n+2 \leqslant k \leqslant n+m$, we have

$$
\begin{aligned}
(4.7)\ (\Delta^{m+1}\tau_n a)(k)& \\
&= (\Delta^m \tau_n a)(k) - (\Delta^m \tau_n a)(k-1) \\
&= \sum_{j=1}^{k-n} (-1)^{k-n-j} \binom{m-j}{k-n-j} (\Delta^{j-1} a)(n+j) \\
&\quad - \sum_{j=1}^{k-n-1} (-1)^{k-n-j-1} \binom{m-j}{k-n-j-1} (\Delta^{j-1} a)(n+j) \\
&= \sum_{j=1}^{k-n-1} (-1)^{k-n-j} \left(\binom{m-j}{k-n-j} + \binom{m-j}{k-n-j-1} \right) (\Delta^{j-1} a)(n+j) \\
&\quad + (\Delta^{k-n-1} a)(k) \\
&= \sum_{j=1}^{k-n-1} (-1)^{k-n-j} \binom{m+1-j}{k-n-j} (\Delta^{j-1} a)(n+j) + (\Delta^{k-n-1} a)(k) \\
&= \sum_{j=1}^{k-n} (-1)^{k-n-j} \binom{m+1-j}{k-n-j} (\Delta^{j-1} a)(n+j).
\end{aligned}
$$

In addition,

$$
\begin{aligned}
(4.8)\qquad (\Delta^{m+1}\tau_n a)&(n+m+1) \\
&= (\Delta^m \tau_n a)(n+m+1) - (\Delta^m \tau_n a)(n+m) \\
&= (\Delta^m a)(n+m+1) - \sum_{j=1}^{m} (-1)^{m-j} \binom{m-j}{m-j} (\Delta^{j-1} a)(n+j) \\
&= (\Delta^m a)(n+m+1) + \sum_{j=1}^{m} (-1)^{m+1-j} (\Delta^{j-1} a)(n+j) \\
&= \sum_{j=1}^{m+1} (-1)^{m+1-j} (\Delta^{j-1} a)(n+j) \\
&= \sum_{j=1}^{(n+m+1)-n} (-1)^{(n+m+1)-n-j} \binom{m+1-j}{(n+m+1)-n-j} (\Delta^{j-1} a)(n+j).
\end{aligned}
$$

Now from (4.6)–(4.8) we conclude that

$$
(4.9)\qquad (\Delta^{m+1}\tau_n a)(k) = \sum_{j=1}^{k-n} (-1)^{k-n-j} \binom{m+1-j}{k-n-j} (\Delta^{j-1} a)(n+j)
$$

for $k = n+1, \ldots, n+m+1$. Finally, from the inductive hypothesis we derive that

$$(4.10) \qquad \begin{aligned}(\Delta^{m+1}\tau_n a)(k) &= (\Delta^m \tau_n a)(k) - (\Delta^m \tau_n a)(k-1)\\ &= (\Delta^m a)(k) - (\Delta^m a)(k-1)\\ &= (\Delta^{m+1} a)(k) \quad \text{for every } k \in \mathbb{N}_{n+m+2}.\end{aligned}$$

Now it suffices to observe that (4.5), (4.9) and (4.10) yield $m+1 \in S_n$, which completes the proof. □

Remark 4.11. Let $s\colon \mathbb{N} \to \mathbb{R}$ be a sequence of strictly positive real numbers such that $\kappa(s) \in (0, \infty]$. Then $h_s(0) = s(0) > 0$. Indeed, we have $h_s(t) = \sum\limits_{n=0}^{\infty} s(n)t^n > 0$ for every $t \in [0, \kappa(s))$. Hence, any zero z_0 of h_s must belong to $D_{\kappa(s)} \setminus [0, \kappa(s))$.

Theorem 4.12. *Let $s\colon \mathbb{N} \to \mathbb{R}$ be a nondecreasing sequence of strictly positive real numbers such that $\lim\limits_{n\to\infty} s(n+1)/s(n) = 1$, $\lim\limits_{n\to\infty} s(n) = \infty$ and $\Delta^r s$ is bounded for some $r \in \mathbb{Z}_+$, and let $z_0 \in D$ be such that $h_s(z_0) = 0$. Then $z_0 \neq 0$ and $\lim\limits_{n\to\infty} \Big(\sum\limits_{k=0}^{n} (\Delta s)(n-k) \cdot z_0^{-k}\Big)/s(n) = 0$.*

Proof. First of all, we observe that $\kappa(s) = 1$, and consequently the domain of h_s is D (see Theorem 2.8 and Remark 4.7). From Remark 4.11 it follows that $z_0 \neq 0$ (indeed, $z_0 \in D \setminus [0,1)$). Also, from Lemma 4.4 it follows that $\kappa(\Delta s) \geqslant 1$ (and consequently $D_{\kappa(\Delta s)} \supseteq D \ni z_0$), and $h_{\Delta s}(z_0) = 0$. We prove that $\lim\limits_{n\to\infty} \Big(\sum\limits_{k=0}^{n} (\Delta s)(n-k) \cdot z_0^{-k}\Big)/s(n) = 0$.

Let $M \in [0, \infty)$ be such that $|(\Delta^r s)(n)| \leqslant M$ for every $n \in \mathbb{N}$. We begin by remarking that for each $n \in \mathbb{N}$ we have $\kappa(\tau_n \Delta s) = \kappa(\Delta s) \geqslant 1$ (from which we derive that the domain of $h_{\tau_n \Delta s}$ contains D), and besides,

$$(4.11) \qquad \begin{aligned}\sum_{k=0}^{n} (\Delta s)(n-k) \cdot \frac{1}{z_0^k} &= \frac{\sum_{k=0}^{n} (\Delta s)(n-k) z_0^{n-k}}{z_0^n} = \frac{\sum_{k=0}^{n} (\Delta s)(k) z_0^k}{z_0^n}\\ &= \frac{\sum_{k=0}^{n} (\Delta s)(k) z_0^k - h_{\Delta s}(z_0)}{z_0^n} = -\frac{\sum_{k=n+1}^{\infty} (\Delta s)(k) z_0^k}{z_0^n}\\ &= -\frac{h_{\tau_n \Delta s}(z_0)}{z_0^n}.\end{aligned}$$

If $r = 1$, we have $|\Delta s(n)| \leqslant M$ for every $n \in \mathbb{N}$. Then from (4.11) we conclude that

$$\left|\frac{\sum_{k=0}^{n} (\Delta s)(n-k) \cdot z_0^{-k}}{s(n)}\right| = \frac{\left|\sum_{k=0}^{n} (\Delta s)(n-k) \cdot z_0^{-k}\right|}{s(n)} = \frac{\left|\sum_{k=n+1}^{\infty} (\Delta s)(k) z_0^k\right|}{|z_0|^n s(n)}$$

$$\leqslant \frac{\sum_{k=n+1}^{\infty} |(\Delta s)(k)||z_0|^k}{|z_0|^n s(n)} \leqslant M\Big(\frac{\sum_{k=n+1}^{\infty} |z_0|^k}{|z_0|^n s(n)}\Big)$$
$$= \frac{M|z_0|^{n+1}}{|z_0|^n(1-|z_0|)s(n)} = \frac{M|z_0|}{1-|z_0|}\frac{1}{s(n)} \underset{n\to\infty}{\longrightarrow} 0$$

(as $\lim\limits_{n\to\infty} s(n) = \infty$), which gives the desired result.

Now suppose $r \in \mathbb{N}_2$. For each $n \in \mathbb{N}$, by virtue of Remark 4.6 we have $\kappa(\Delta^{r-1}\tau_n\Delta s) \geqslant \kappa(\tau_n\Delta s) \geqslant 1$ (which places z_0 in the domain of $h_{\Delta^{r-1}\tau_n\Delta s}$, being $z_0 \in D$); besides, from (4.11), Remark 4.6 and Lemma 4.10 we derive that

$$\begin{aligned}
&\sum_{k=0}^{n}(\Delta s)(n-k)\cdot\frac{1}{z_0^k}\\
&= -\frac{h_{\tau_n\Delta s}(z_0)}{z_0^n} = -\frac{(1-z_0)^{r-1}h_{\tau_n\Delta s}(z_0)}{z_0^n(1-z_0)^{r-1}}\\
&= -\frac{h_{\Delta^{r-1}\tau_n\Delta s}(z_0)}{z_0^n(1-z_0)^{r-1}} = -\frac{\sum_{k=n+1}^{\infty}(\Delta^{r-1}\tau_n\Delta s)(k)z_0^k}{z_0^n(1-z_0)^{r-1}}\\
&= -\frac{\sum_{k=n+1}^{n+r-1}\big(\sum_{j=1}^{k-n}(-1)^{k-n-j}\binom{r-1-j}{k-n-j}(\Delta^j s)(n+j)\big)z_0^k + \sum_{k=n+r}^{\infty}(\Delta^r s)(k)z_0^k}{z_0^n(1-z_0)^{r-1}},
\end{aligned}$$

and consequently,

$$\begin{aligned}
(4.12)\quad &\left|\frac{\sum_{k=0}^{n}(\Delta s)(n-k)\cdot z_0^{-k}}{s(n)}\right|\\
&= \frac{\big|\sum_{k=n+1}^{n+r-1}\big(\sum_{j=1}^{k-n}(-1)^{k-n-j}\binom{r-1-j}{k-n-j}(\Delta^j s)(n+j)\big)z_0^k + \sum_{k=n+r}^{\infty}(\Delta^r s)(k)z_0^k\big|}{|z_0|^n|1-z_0|^{r-1}s(n)}\\
&\leqslant \frac{\sum_{k=n+r}^{\infty}|(\Delta^r s)(k)||z_0|^k}{|z_0|^n|1-z_0|^{r-1}s(n)} + \frac{\sum_{k=n+1}^{n+r-1}\big(\sum_{j=1}^{k-n}\binom{r-1-j}{k-n-j}|(\Delta^j s)(n+j)|\big)|z_0|^k}{|z_0|^n|1-z_0|^{r-1}s(n)}\\
&\leqslant \frac{M\sum_{k=n+r}^{\infty}|z_0|^k}{|z_0|^n|1-z_0|^{r-1}s(n)} + \frac{|z_0|\sum_{k=n+1}^{n+r-1}\big(\sum_{j=1}^{k-n}\binom{r-1-j}{k-n-j}|(\Delta^j s)(n+j)|\big)|z_0|^{k-n-1}}{|1-z_0|^{r-1}s(n)}\\
&= \frac{M|z_0|^r}{(1-|z_0|)|1-z_0|^{r-1}s(n)} + \frac{|z_0|\sum_{m=1}^{r-1}\big(\sum_{j=1}^{m}\binom{r-1-j}{m-j}|(\Delta^j s)(n+j)|\big)|z_0|^{m-1}}{|1-z_0|^{r-1}s(n)}\\
&= \frac{M|z_0|^r}{(1-|z_0|)|1-z_0|^{r-1}s(n)} + \frac{|z_0|\sum_{j=1}^{r-1}|(\Delta^j s)(n+j)|\sum_{m=j}^{r-1}\binom{r-1-j}{m-j}|z_0|^{m-1}}{|1-z_0|^{r-1}s(n)}\\
&= \frac{M|z_0|^r}{(1-|z_0|)|1-z_0|^{r-1}s(n)} + \frac{|z_0|\sum_{j=1}^{r-1}|(\Delta^j s)(n+j)|\sum_{k=0}^{r-1-j}\binom{r-1-j}{k}|z_0|^{k+j-1}}{|1-z_0|^{r-1}s(n)}
\end{aligned}$$

$$= \frac{M|z_0|^r}{(1-|z_0|)|1-z_0|^{r-1}s(n)} + \frac{|z_0|\sum_{j=1}^{r-1}|(\Delta^j s)(n+j)||z_0|^{j-1}\sum_{k=0}^{r-1-j}\binom{r-1-j}{k}|z_0|^k}{|1-z_0|^{r-1}s(n)}$$
$$= \frac{M|z_0|^r}{(1-|z_0|)|1-z_0|^{r-1}}\frac{1}{s(n)} + \frac{|z_0|}{|1-z_0|^{r-1}}\sum_{j=1}^{r-1}|z_0|^{j-1}(1+|z_0|)^{r-1-j}\frac{|(\Delta^j s)(n+j)|}{s(n)}.$$

Since $\lim_{n\to\infty} s(n) = \infty$, we conclude that

$$\lim_{n\to\infty} \frac{M|z_0|^r}{(1-|z_0|)|1-z_0|^{r-1}}\frac{1}{s(n)} = 0. \tag{4.13}$$

Furthermore, since $\lim_{n\to\infty} s(n+1)/s(n) = 1$, it follows that

$$\lim_{n\to\infty}\frac{(\Delta^k s)(n)}{s(n)} = 0 \quad \text{and} \quad \lim_{n\to\infty}\frac{s(n+k)}{s(n)} = 1 \quad \text{for every } k \in \mathbb{Z}_+$$

(see [2], 6.5). Hence

$$\frac{(\Delta^j s)(n+j)}{s(n)} = \frac{(\Delta^j s)(n+j)}{s(n+j)}\cdot\frac{s(n+j)}{s(n)} \underset{n\to\infty}{\longrightarrow} 0 \quad \text{for } j = 1,\dots,r-1. \tag{4.14}$$

Now (4.13)–(4.14) together with (4.12) yield $\lim_{n\to\infty}\Big(\sum_{k=0}^{n}(\Delta s)(n-k)\cdot z_0^{-k}\Big)/s(n) = 0$, which completes the proof. □

Corollary 4.13. *Let $s\colon \mathbb{N}\to\mathbb{R}$ be a nondecreasing sequence of strictly positive real numbers such that $\lim_{n\to\infty} s(n+1)/s(n) = 1$, $\lim_{n\to\infty} s(n) = \infty$ and $\Delta^r s$ is bounded for some $r \in \mathbb{Z}_+$, and let $z_0 \in D$ be such that $h_s(z_0) = 0$. Then $0 < |z_0| < 1$. Furthermore, for each complex nonzero Banach space X,*

$$\frac{\sum_{k=0}^{n}(\Delta s)(n-k)(z_0^{-1}I_X)^k}{s(n)} \underset{n\to\infty}{\longrightarrow} 0_{L(X)} \quad \text{in } L(X),$$

and consequently there exists $T \in L(X)$ such that the sequence

$$\left(\frac{\sum_{k=0}^{n}(\Delta s)(n-k)T^k}{s(n)}\right)_{n\in\mathbb{N}}$$

converges in $L(X)$, and $r(T) = 1/|z_0| > 1$.

Proof. First of all, we recall that $0 < |z_0| < 1$ (see Theorem 4.12, or Remark 4.11). Now let X be a complex nonzero Banach space. Since for each $n \in \mathbb{N}$ we have

$$\frac{\sum_{k=0}^{n}(\Delta s)(n-k)(z_0^{-1}I_X)^k}{s(n)} = \frac{\sum_{k=0}^{n}(\Delta s)(n-k)\cdot z_0^{-k}}{s(n)} I_X,$$

from Theorem 4.12 it follows that $\Big(\sum\limits_{k=0}^{n}(\Delta s)(n-k)(z_0^{-1}I_X)^k\Big)/s(n) \underset{n\to\infty}{\longrightarrow} 0_{L(X)}$ in $L(X)$. Now, in order to obtain the final claim of the corollary, it suffices to set $T = z_0^{-1}I_X$. □

Notice that if the sequence $\Big(\Big(\sum\limits_{k=0}^{n}(\Delta s)(n-k)(\lambda I_X)^k\Big)/s(n)\Big)_{n\in\mathbb{N}}$ (where X is a complex nonzero Banach space, $\lambda \in \mathbb{C}\setminus\{1\}$, and s is a nondecreasing sequence of strictly positive real numbers satisfying $\liminf\limits_{n\to\infty} s(n+1)/s(n) = 1$) converges in $L(X)$, then it must converge to $0_{L(X)}$, by virtue of Theorem 3.2. Indeed, for $T = \lambda I_X$, being $\lambda \neq 1$ we have $1 \in \varrho(T)$, and consequently, the projection of X onto $\mathcal{N}(I_X - T)$ along $\mathcal{R}(I_X - T)$ coincides with $0_{L(X)}$.

Remark 4.14. Let $a \in \mathbb{C}^{\mathbb{N}}$. It is easily seen that if $\Delta^r a$ is bounded for some $r \in \mathbb{N}$, then $\Delta^k a$ is bounded for every $k \in \mathbb{N}_r$. Also, proceeding by induction on r (and using (2.1)), it is not difficult to verify that if $|(\Delta^r a)(n)| \leqslant M$ for every $n \in \mathbb{N}$ and for some $r \in \mathbb{N}$, $M \in [0,\infty)$, then $|a(n)| \leqslant M\binom{n+r}{n}$ for every $n \in \mathbb{N}$, and consequently, $\mathcal{H}(a) \leqslant r < \infty$.

A sufficient condition in order that $\Delta^r a$ be bounded for some $r \in \mathbb{N}$ is that $\Delta^q a \in l_1$ for some $q \in \mathbb{N}$: indeed, $\Delta^q a \in l_1$ clearly yields $\Delta^q a$ bounded; actually, if $q \in \mathbb{Z}_+$, then $\Delta^{q-1}a = \Sigma\Delta^q a$, being convergent, is also bounded, and consequently, $\Delta^k a$ is bounded for every $k \in \mathbb{N}_{q-1}$. We remark that the converse is not true. Indeed, if $\Delta^r a$ is bounded for some $r \in \mathbb{N}$, there may exist no $q \in \mathbb{N}$ for which $\Delta^q a \in l_1$: an example—with $r = 1$—is the divergent strictly increasing sequence s of strictly positive real numbers of Example 3.6 (see also Remark 3.7). Hence, Example 3.6 does not only show how in Theorem 2.6 the assumption that $\Delta^q s \in l_1$ for some $q \in \mathbb{N}_2$ cannot be replaced by $\mathcal{H}(s) < \infty$: actually, it cannot even be replaced by the assumption that $\Delta^r s$ be bounded for some $r \in \mathbb{Z}_+$.

By virtue of Corollary 4.13 and Remark 4.14, in order to conclude that the answer to the question we have posed at the beginning of this section is in the negative, it suffices to show that there exists a nondecreasing sequence s of strictly positive real numbers such that $\lim\limits_{n\to\infty} s(n) = \infty$, $\lim\limits_{n\to\infty} s(n+1)/s(n) = 1$, $\Delta^q s \in l_1$ for some $q \in \mathbb{N}_2$, and $h_s(z_0) = 0$ for some $z_0 \in D$. This is what we are going to do in the following example.

E x a m p l e 4.15. Let $s\colon \mathbb{N} \to \mathbb{R}$ be the sequence of strictly positive real numbers defined by

$$s(n) = \frac{3n+1}{2} \quad \text{for every } n \in \mathbb{N}.$$

We remark that s is strictly increasing. Furthermore, we have

$$(\Delta s)(n) = \begin{cases} \frac{1}{2} & \text{for } n = 0, \\ \frac{3}{2} & \text{for } n \in \mathbb{Z}_+ \end{cases} \quad \text{and} \quad (\Delta^2 s)(n) = \begin{cases} \frac{1}{2} & \text{for } n = 0, \\ 1 & \text{for } n = 1, \\ 0 & \text{for } n \in \mathbb{N}_2. \end{cases} \tag{4.15}$$

Hence $\Delta^2 s \in l_1$ (and Δs is bounded). We also observe that $\lim\limits_{n\to\infty} s(n) = \infty$ and $\lim\limits_{n\to\infty} s(n+1)/s(n) = 1$, which gives $\kappa(s) = 1$ (and consequently $D_{\kappa(s)} = D$) by Theorem 2.8. Finally, we prove that $h_s(z_0) = 0$ for some $z_0 \in D$.

From (4.15) it follows that $\kappa(\Delta^2 s) = \infty$. Furthermore, for each $z \in \mathbb{C}$ we have $h_{\Delta^2 s}(z) = \frac{1}{2} + z$. Hence $h_{\Delta^2 s}(-\frac{1}{2}) = 0$, which gives $h_s(-\frac{1}{2}) = 0$ by Lemma 4.4.

Example 4.15 together with Corollary 4.13 shows that convergence in $L(X)$ of the sequence $\Big(\Big(\sum\limits_{k=0}^{n} (\Delta s)(n-k)T^k\Big)/s(n)\Big)_{n\in\mathbb{N}}$, where X is a complex nonzero Banach space, $T \in L(X)$, and s is a nondecreasing sequence of strictly positive real numbers, satisfying all the hypotheses of Theorem 2.6 except $\lim\limits_{n\to\infty} \|T^n\|_{L(X)}/s(n) = 0$, does not imply $r(T) \leqslant 1$. Also, Example 4.15 and Corollary 4.13 show that the hypothesis about h_s having no zeros in D cannot be removed from Theorem 4.8.

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Author's address: *Laura Burlando*, Dipartimento di Matematica dell'Università di Genova, Via Dodecaneso 35, 16146 Genova, Italy, e-mail: `laura.burlando@unige.it`.