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Abstract. We consider a new family of recurrence sequences, the (q,k)-generalized Fibonacci numbers. These sequences naturally extend the well-known sequences of k-generalized Fibonacci numbers and generalized k-order Pell numbers. Further, we obtain the Binet formula and study the asymptotic behavior of the dominant root of the characteristic equation. The proof methods exploit pairs of characteristic polynomials which allow several auxiliary results.

Keywords: generalized Fibonacci number; generalized Pell number; recurrence sequence; Binet formula

MSC 2020: 11B37, 11B39

1. Introduction

The study of recurrence sequences has implications in many areas such as Diophantine equations, combinatorial problems and others [3], [4], [6], [7], [8], [11], [16]. For instance, the Fibonacci sequence and its generalizations have been widely studied due to interesting results obtained by the Fibonacci recursion, Binet formula, generating function and matrix methods [9], [10], [11], [14], [15], [16].

The Fibonacci numbers have been generalized in a variety of ways, some of which are reviewed below. The aim of this paper is to define and prove properties of a new family of generalized recurrence sequences. For integers $k \geqslant 2$ and $q \geqslant 3$, the (q,k)-generalized Fibonacci numbers are defined recursively by

$$F_{q,n}^{(k)} = qF_{q,n-1}^{(k)} + F_{q,n-2}^{(k)} + \ldots + F_{q,n-k}^{(k)} \quad \forall\, n \geqslant 2, \tag{1.1}$$

with initial conditions $F_{q,-(k-2)}^{(k)} = F_{q,-(k-3)}^{(k)} = \ldots = F_{q,0}^{(k)} = 0$ and $F_{q,1}^{(k)} = 1$.

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When $q = 1$ and $k = 2$ in (1.1), we have the well-known sequence of Fibonacci numbers $(F_n)_{n\geqslant 0}$ defined recursively by $F_{n+1} = F_n + F_{n-1}$ with initial conditions $F_0 = 0$ and $F_1 = 1$. The Fibonacci sequence appears in The Online Encyclopedia of Integers Sequences, OEIS, [18], A000045 and has been studied and generalized by many authors [9], [11], [12], [16], [17].

When $q = 2$ and $k = 2$ in (1.1), we obtain the Pell numbers [18], A000129, defined recursively by

$$P_n = 2P_{n-1} + P_{n-2} \quad \forall\, n \geqslant 2, \tag{1.2}$$

with initial conditions $P_0 = 0$ and $P_1 = 1$, see [9], [12].

When $q = 1$ and $k \geqslant 2$ in (1.1), we obtain the k-generalized Fibonacci sequences defined recursively by

$$F_n^{(k)} = F_{n-1}^{(k)} + \ldots + F_{n-k}^{(k)} \quad \forall\, n \geqslant 2, \tag{1.3}$$

with initial conditions $F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \ldots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$, see [10], [14]. For example, when $k = 3$, we obtain the Tribonacci numbers that appear in OEIS [18], A000073. Moreover, for $k = 4$ we get the Tetranacci numbers [18], A000078.

For integers $q \geqslant 2$ and $k = 2$ in (1.1), we get

$$F_{q,n} = qF_{q,n-1} + F_{q,n-2} \quad \forall\, n \geqslant 2, \tag{1.4}$$

with initial conditions $F_{q,0} = 0$ and $F_{q,1} = 1$, see [9], [12]. For example, when $q = 3$ and $q = 4$, this sequence appears in OEIS [18], A006190 and A001076.

Moreover, when $q = 2$ and $k \geqslant 3$ in (1.1), we obtain the sequences of order-k Pell numbers recursively defined by

$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \ldots + P_{n-k}^{(k)} \quad \forall\, n \geqslant 2, \tag{1.5}$$

with initial conditions $P_{-(k-2)}^{(k)} = P_{-(k-3)}^{(k)} = \ldots = P_0^{(k)} = 0$ and $P_1^{(k)} = 1$, see [5], [6]. For example, the case $k = 3$ appears in OEIS [18], A077939.

Reference [6] presents some combinatorial interpretations for (1.5), while [5] determines the Binet formula for these sequences. Moreover, several results are proved about the asymptotic behavior of the dominant roots of their characteristic polynomials.

The generating function for (1.1) is given by

$$f_{q,k}(x) = \sum_{n=0}^{\infty} F_{q,n}^{(k)} x^n = \frac{x}{1 - qx - x^2 - \ldots - x^k}.$$

In the case $q = 3$, the first terms of (1.1) are given by

$k \backslash n$	3	4	5	6	7	8	9	10	...
2	10	33	109	360	1 189	3 927	12 970	42 837	...
3	10	34	115	389	1 316	4 452	15 061	50 951	...
4	10	34	116	395	1 345	4 580	15 596	53 108	...
5	10	34	116	396	1 351	4 609	15 724	53 644	...

while for $q = 4$ in (1.1) we get:

$k \backslash n$	3	4	5	6	7	8	9	10	...
2	17	72	305	1 292	5 473	23 184	98 209	416 020	...
3	17	73	313	1 342	5 754	24 671	105 780	453 545	...
4	17	73	314	1 350	5 804	24 953	107 280	461 227	...
5	17	73	314	1 351	5 812	25 003	107 562	462 728	...

Importantly, some of these sequences do not appear in OEIS, for example when $q = 4$ and $k = 3$.

The main goal of this paper is to generalize the results on asymptotic behavior presented in [5] to the sequences defined by (1.1). We have the following main theorem that is proved in Section 3.

Theorem 1.1 (Main theorem). *With the notations of* (1.1) *we have*

(a)

$$F_{q,n}^{(k)} = \sum_{i=1}^{k} g_{q,k}(\gamma_i)\gamma_i^n \quad \forall n \geqslant -(k-2), \tag{1.6}$$

where $\gamma_1, \gamma_2, \ldots, \gamma_k$ *are roots of the characteristic polynomial* $\Phi_{q,k}(t)$*, given by*

$$\Phi_{q,k}(t) = t^k - qt^{k-1} - t^{k-2} - \ldots - t - 1, \tag{1.7}$$

and

$$g_{q,k}(x) := \frac{x-1}{(k+1)x^2 - (q+1)kx + (q-1)(k-1)}. \tag{1.8}$$

(b)

$$|F_{q,n}^{(k)} - g_{q,k}(\gamma)\gamma^n| \leqslant \frac{1}{q} \quad \forall n \geqslant -(k-2), \tag{1.9}$$

where $\gamma = \gamma_1$ is the dominant root of $\Phi_{q,k}(t)$. Moreover,

$$\gamma^{n-2} < \gamma^{n-1}\Big(\frac{q-1}{q}\Big) < F_{q,n}^{(k)} < \gamma^{n-1}\Big(\frac{q+2}{q}\Big) < \gamma^{n} \tag{1.10}$$

for all $n \geqslant 1$.

2. Proof method

We would like to note that the cases $q = 1$ and $q = 2$ of Theorem 1.1 were proved in [10] and [5], respectively. Thus, for the rest of the paper let

$$q \geqslant 3. \tag{2.1}$$

To study (1.1), we consider (1.7) and the auxiliary function

$$h_{q,k}(t) = (t-1)\Phi_{q,k}(t) = t^{k+1} - (q+1)t^{k} + (q-1)t^{k-1} + 1. \tag{2.2}$$

The technique of considering characteristic polynomials and auxiliary functions is similar to those found in [5], [10], [13]. Since $\Phi_{q,k}(t)$ divides $h_{q,k}(t)$, we obtain the first identity involving (1.1), given by the following theorem.

Theorem 2.1. *For all integer $k \geqslant 2$ we have*

$$F_{q,n}^{(k)} = (q+1)F_{q,n-1}^{(k)} - (q-1)F_{q,n-2}^{(k)} - F_{q,n-k-1}^{(k)} \quad \forall\, n \geqslant 3. \tag{2.3}$$

Proof. Indeed, since $(F_{q,n}^{(k)})$ is a linear recurrence of order k with the characteristic polynomial $\Phi_{q,k}(t)$ and $\Phi_{q,k}(t)$ divides the auxiliary function $h_{q,k}(t)$, we deduce that $(F_{q,n}^{(k)})$ is also a linear recurrence of order $k+1$ with the characteristic polynomial $h_{q,k}(t)$. This completes the proof. $\square$

Theorem 2.1 motivates considering the following recursive sequences, which give us an alternative way to compute $F_{q,n}^{(k)}$. Let $(U_{q,n})_{n\geqslant 1}$ be the sequence given by

$$U_{q,n} = (q+1)U_{q,n-1} - (q-1)U_{q,n-2} \quad \forall\, n \geqslant 3,$$

with $U_{q,1} = 1$ and $U_{q,2} = q$; these sequences are considered in [1]. The Binet formula for $(U_{q,n})_{n\geqslant 1}$ is given by

$$U_{q,n} = \frac{((q-3)+\sqrt{q^2-2q+5})\alpha_q^{n} + ((3-q)+\sqrt{q^2-2q+5})\beta_q^{n}}{2(q-1)\sqrt{q^2-2q+5}},$$

where α_q and β_q are the roots of

$$t^2 - (q+1)t + (q-1) = 0 \tag{2.4}$$

given by

$$\alpha_q = \frac{(q+1) + \sqrt{q^2 - 2q + 5}}{2} \quad \text{and} \quad \beta_q = \frac{(q+1) - \sqrt{q^2 - 2q + 5}}{2}. \tag{2.5}$$

For future reference, we note that

$$q < \alpha_q < q+1 \quad \text{and} \quad 0 < \beta_q < 1. \tag{2.6}$$

We similarly define the sequence $(V_{q,n})_{n\geqslant 1}$ by

$$V_{q,n} = (q+1)V_{q,n-1} - (q-1)V_{q,n-2} \quad \forall\, n \geqslant 3,$$

with $V_{q,1} = 1$ and $V_{q,2} = q+1$. There are important relationships between these sequences and (1.1). For example, we have that $F_{q,n}^{(k)} \leqslant U_{q,n}$ for all $n \geqslant 1$. More generally, we have the following theorem.

Theorem 2.2. *For all integers $k \geqslant 2$ we have*

$$F_{q,n}^{(k)} = U_{q,n} \quad \forall\, 1 \leqslant n \leqslant k+1,$$

and

$$F_{q,n}^{(k)} = U_{q,n} - \sum_{j=1}^{n-k-1} V_{q,j} F_{q,n-k-j}^{(k)} \quad \forall\, n \geqslant k+2.$$

Proof. The proof of the first identity is an immediate consequence of Theorem 2.1, and the second identity may be proven inductively using (2.3) for n. Since this proof is completely analogous to the proof of Theorem 2.2 in [5], its details are omitted. □

In the next sections, in order to prove the Main Theorem, we determine the Binet formula and study the asymptotic behavior of the dominant root of (1.1).

3. Proof of main theorem – part (a)

3.1. Binet formula. Kalman in [14] proved that if $(u_n)_{n\geqslant 0}$ is a linear recurrence sequence of order $k \geqslant 2$ satisfying the recurrence

$$u_{n+k} = c_{k-1}u_{n+k-1} + c_{k-2}u_{n+k-2} + \ldots + c_1u_{n+1} + c_0u_n \quad \forall\, n \geqslant 0,$$

with initial condition $u_0 = u_1 = \ldots = u_{k-2} = 0$ and $u_{k-1} = 1$, where $c_0, c_1, \ldots, c_{k-1}$ are constants, then

$$u_n = \sum_{i=1}^{k} \frac{\alpha_i^n}{P'(\alpha_i)},$$

where $P(t) = t^k - c_{k-1}t^{k-1} - \ldots - c_1t - c_0$ is the characteristic polynomial of $(u_n)_{n\geqslant 0}$ and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are the distinct roots of $P(t)$.

Taking the sequence $(u_n)_{n\geqslant 0} = (F^{(k)}_{q,n-(k-2)})_{n\geqslant 0}$, we get $P(t) = \Phi_{q,k}(t)$. By Brauer's criterion [2], we have that (1.7) is irreducible over $\mathbb{Z}[t]$; moreover, (1.7) is primitive over $\mathbb{Z}[t]$. Thus, by Gauss's lemma, we conclude that (1.7) is irreducible over $\mathbb{Q}[t]$; hence, it has no repeated zeros in $\mathbb{C}$. Therefore

$$F^{(k)}_{q,n} = \sum_{i=1}^{k} \frac{\gamma_i^{n+(k-2)}}{\Phi'_{q,k}(\gamma_i)} \tag{3.1}$$

with $\gamma_1, \gamma_2, \ldots, \gamma_k$ the distinct roots of (1.7). Using (2.2), we have

$$\Phi_{q,k}(t) = \frac{h_{q,k}(t)}{t-1}.$$

Differentiating this we get

$$\Phi'_{q,k}(t) = \frac{h'_{q,k}(t)(t-1) - h_{q,k}(t)}{(t-1)^2}.$$

Using (2.2) again and noting that for each i, $1 \leqslant i \leqslant k$, $h_{q,k}(\gamma_i) = 0$, we obtain for each $1 \leqslant i \leqslant k$

$$\Phi'_{q,k}(\gamma_i) = \frac{(k+1)\gamma_i^k - (q+1)k\gamma_i^{k-1} + (q-1)(k-1)\gamma_i^{k-2}}{\gamma_i - 1}. \tag{3.2}$$

By (3.1) and (3.2), we conclude that

$$F^{(k)}_{q,n} = \sum_{i=1}^{k} g_{q,k}(\gamma_i)\gamma_i^n,$$

where $g_{q,k}$ is given by (1.8). This proves item (a).

3.2. Asymptotic behavior. For integers $k \geqslant 2$ and $n \geqslant 2-k$ we define $E_{q,n}^{(k)}$ as the error of the approximation of the nth (q,k)-generalized Fibonacci number with the dominant term of (1.6), i.e.,

$$E_{q,n}^{(k)} = F_{q,n}^{(k)} - g_{q,k}(\gamma)\gamma^n \tag{3.3}$$

for $\gamma = \gamma_1$ the dominant root of $\Phi_{q,k}$.

It follows by (3.3) that $E_{q,n}^{(k)}$ satisfies (1.1) with $F_{q,n}^{(k)}$ replaced by $E_{q,n}^{(k)}$. Moreover, by (2.3),

$$E_{q,n}^{(k)} = (q+1)E_{q,n-1}^{(k)} - (q-1)E_{q,n-2}^{(k)} - E_{q,n-k-1}^{(k)}. \tag{3.4}$$

By [19], we have that for all integer numbers $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_m \geqslant 1$ with $m \geqslant 2$, the polynomial

$$f(x) = x^m - a_1x^{m-1} - a_2x^{m-2} - \ldots - a_1x - a_m$$

has exactly one positive real zero α with $a_1 < \alpha < a_1 + 1$ and the other $m-1$ zeros of $f(x)$ lie in the unit circle. Thus, (1.7) has a dominant root $q < \gamma < q+1$ and the other roots are in the unit circle.

Using the fact that $\lim_{n\to\infty} |\gamma_i|^n = 0$ for $2 \leqslant i \leqslant k$ and taking into account that

$$|E_{q,n}^{(k)}| \leqslant \sum_{j=2}^{k} |g_{q,k}(\gamma_j)||\gamma_j|^n,$$

we also deduce that

$$\lim_{n\to\infty} |E_{q,n}^{(k)}| = 0. \tag{3.5}$$

Lemma 3.1. *Let $\gamma(l)$ and $\gamma(k)$ be the dominant roots of $\Phi_{q,l}(t)$ and $\Phi_{q,k}(t)$, respectively. Then*

(i) *for $l > k$ we have that $\gamma(l) > \gamma(k)$;*

(ii)
$$\alpha_q\left(1 - \frac{1}{q^k}\right) < \gamma(k) < \alpha_q.$$

In particular,
$$\lim_{k\to\infty} \gamma(k) = \alpha_q.$$

P r o o f. For the proof of item (i), we proceed by contradiction. Let us assume that $\gamma(k) \geqslant \gamma(l)$. Thus, $\gamma(k)^{-i} \leqslant \gamma(l)^{-i}$ holds for all $i \geqslant 1$. Taking into account that $\Phi_{q,k}(\gamma(k)) = 0$, we get

$$\gamma(k)^k = q\gamma(k)^{k-1} + \gamma(k)^{k-2} + \ldots + \gamma(k) + 1,$$

and the same conclusion remains valid for $\gamma(l)$. Since $k < l$, we have that

$$\begin{aligned} 1 &= \frac{q}{\gamma(k)} + \frac{1}{\gamma(k)^2} + \ldots + \frac{1}{\gamma(k)^k} \\ &< \frac{q}{\gamma(l)} + \frac{1}{\gamma(l)^2} + \ldots + \frac{1}{\gamma(l)^k} + \frac{1}{\gamma(l)^{k+1}} + \ldots + \frac{1}{\gamma(l)^l} = 1, \end{aligned}$$

which is a contradiction. Thus, we conclude that $\gamma(l) > \gamma(k)$ and this proves item (i).

Next we prove item (ii). By (2.2), (2.4) and (2.5),

$$\Phi_{q,k}(\alpha_q) = \frac{1}{\alpha_q - 1} > 0,$$

while by (1.7) and (2.1)

$$\Phi_{q,k}(q) = -q^{k-2} - q^{k-3} - \ldots - q - 1 < 0.$$

Since $\gamma = \gamma(k)$ is the only root of (1.7) bigger than 1, we obtain $q < \gamma < \alpha_q$.

By (2.4) and (2.5)

$$\alpha_q^2 - (q+1)\alpha_q + (q-1) = 0,$$

while by (2.2)

$$\gamma^2 - (q+1)\gamma + (q-1) = \frac{-1}{\gamma^{k-1}}.$$

Taking the difference of these two equations, we obtain

$$(\alpha_q - \gamma)(\alpha_q + \gamma - (q+1)) = \frac{1}{\gamma^{k-1}}.$$

Since $\alpha_q > \gamma > q$ and $(\alpha_q + \gamma - (q+1)) > q/\alpha_q$, we obtain that $\alpha_q - \gamma < \alpha_q q^{-k}$. Hence,

$$\gamma > \alpha_q\Big(1 - \frac{1}{q^k}\Big),$$

and this concludes the proof of (ii). $\square$

In preparation for the next lemma giving properties of $g_{q,k}$, (1.8), since α_q is a root of (2.4), we get

$$g_{q,k}(\alpha_q) = \frac{\alpha_q - 1}{\alpha_q^2 - (q-1)}.$$

In particular, using (2.6),

$$\frac{1}{q+1} < g_{q,k}(\alpha_q) < \frac{1}{q}. \tag{3.6}$$

Lemma 3.2. *The rational function $g_{q,k}$ has a vertical asymptote at*

$$c_{q,k} := \frac{(q+1)k + \sqrt{k^2(q^2-2q+5)+4(q-1)}}{2(k+1)}. \tag{3.7}$$

Moreover, $g_{q,k}(x)$ is positive, continuous, and decreasing for all x in $(c_{q,k}, \infty)$.

Proof. Since $c_{q,k}$ is the largest root of the denominator of (1.8), we have that the denominator is different from 0 in $(c_{q,k}, \infty)$. As both the numerator and the denominator of (1.8) are positive and continuous, we conclude that (1.8) is positive and continuous in this interval. Further,

$$g'_{q,k}(x) = \frac{-[(k+1)(x-1)^2 + q + k - 2]}{[(k+1)x^2 - (q+1)kx + (q-1)(k-1)]^2}$$

is negative in $(c_{q,k}, \infty)$. Indeed, the denominator of $g'_{q,k}$ is positive for all $x > c_{q,k}$ and $-[(k+1)(x-1)^2 + q + k - 2] < 0$ for all $k \geqslant 2$ and x real. Hence, $g_{q,k}(x)$ is decreasing in the same interval. □

Taking advantage of this approach, we can prove the following technical lemma.

Lemma 3.3. *Let γ be the dominant root of $\Phi_{q,k}(t)$. Then*

$$\frac{1}{q+1} < g_{q,k}(\gamma) < \frac{1}{q}.$$

Proof. In order to prove this lemma, we consider three cases. First, we consider $k = 2$. In this case we have that

$$g_{q,2}(x) = \frac{x-1}{3x^2 - 2(q+1)x + (q-1)},$$

and γ is the largest root of $t^2 - qt - 1 = 0$ given by $\gamma = \bigl(q + \sqrt{q^2+4}\bigr)/2$. Therefore, we get

$$\frac{1}{q+1} < g_{q,2}(\gamma) < \frac{1}{q}.$$

Second, we consider the case $3 \leqslant k \leqslant q$. By (3.7) and (2.1),

$$c_{q,k} = \frac{(q+1) + \sqrt{q^2 - 2q + 5 + 4k^{-2}(q-1)}}{2}\Bigl(1 - \frac{1}{k+1}\Bigr).$$

By (2.5) and the inequality

$$\sqrt{q^2 - 2q + 5 + 4k^{-2}(q-1)} < \sqrt{q^2 - 2q + 5} + \frac{2\sqrt{q-1}}{k},$$

we get

$$c_{q,k} < \left(\alpha_q + \frac{\sqrt{q-1}}{k}\right)\Bigl(1 - \frac{1}{k+1}\Bigr) < \alpha_q - \frac{q - \sqrt{q-1}}{k+1} < \alpha_q - \frac{1}{q(q-1)}. \tag{3.8}$$

On the other hand, by (2.6), we have that

$$\alpha_q\left(1 - \frac{1}{q^k}\right) > \alpha_q - \frac{q+1}{q^k} \geqslant \alpha_q - \frac{q+1}{q^3} > \alpha_q - \frac{1}{q(q-1)}. \tag{3.9}$$

By (3.8), (3.9) and Lemma 3.1, we conclude

$$c_{q,k} < \alpha_q - \frac{1}{q(q-1)} < \alpha_q\left(1 - \frac{1}{q^k}\right) < \gamma < \alpha_q.$$

Therefore, using Lemma 3.2 and (3.6), we get

$$\frac{1}{q+1} < g_{q,k}(\alpha_q) < g_{q,k}(\gamma) < G_{q,k}, \tag{3.10}$$

where $G_{q,k}$ denotes $g_{q,k}(\alpha_q - 1/(q(q-1)))$. However,

$$\begin{aligned} G_{q,k} &= \frac{\alpha_q - \dfrac{1}{q(q-1)} - 1}{(k+1)\left(\alpha_q - \dfrac{1}{q(q-1)}\right)^2 - (q+1)k\left(\alpha_q - \dfrac{1}{q(q-1)}\right) + (q-1)(k-1)} \\ &= \frac{\alpha_q - \dfrac{1}{q(q-1)} - 1}{q\alpha_q + (\alpha_q - q) - q + 2\left(1 - \dfrac{\alpha_q(k+1)}{q(q-1)}\right) + \dfrac{k(q+1)}{q(q-1)} + \dfrac{k+1}{q^2(q-1)^2}}. \end{aligned}$$

By (2.5),

$$(\alpha_q - q) + 2\left(1 - \frac{\alpha_q(k+1)}{q(q-1)}\right) + \frac{k(q+1)}{q(q-1)} + \frac{k+1}{q^2(q-1)^2} > -\frac{1}{q-1}.$$

From this we derive

$$G_{q,k} < \frac{\alpha_q - \dfrac{1}{q(q-1)} - 1}{q\left(\alpha_q - \dfrac{1}{q(q-1)} - 1\right)} = \frac{1}{q}. \tag{3.11}$$

Hence, by (3.10) and (3.11), we conclude that $g_{q,k}(\gamma) < 1/q$ in this case.

Finally, we consider the case $k \geqslant q+1$. By (3.8),

$$c_{q,k} < \alpha_q - \frac{1}{k(k+1)}.$$

By (2.6), we get $k(k+1)\alpha_q < q^k$. Using this inequality and Lemma 3.1, we obtain

$$c_{q,k} < \alpha_q - \frac{1}{k(k+1)} < \alpha_q\left(1 - \frac{1}{q^k}\right) < \gamma < \alpha_q.$$

Therefore, by Lemma 3.2 and (3.6) we obtain

$$\frac{1}{q+1} < g_{q,k}(\alpha_q) < g_{q,k}(\gamma) < \overline{G}_{q,k}, \tag{3.12}$$

where $\overline{G}_{q,k}$ denotes $g_{q,k}(\alpha_q - 1/(k(k+1)))$. We claim that $\overline{G}_{q,k} < 1/q$. Indeed, we have that

$$\begin{aligned}\overline{G}_{q,k} &= \frac{\alpha_q - \dfrac{1}{k(k+1)} - 1}{(k+1)\left(\alpha_q - \dfrac{1}{k(k+1)}\right)^2 - (q+1)k\left(\alpha_q - \dfrac{1}{k(k+1)}\right) + (q-1)(k-1)} \\ &= \frac{\alpha_q - \dfrac{1}{k(k+1)} - 1}{q\alpha_q + (\alpha_q - q) - q + 2\left(1 - \dfrac{\alpha_q}{k}\right) + \dfrac{k^2(q+1)+1}{k^2(k+1)}}.\end{aligned}$$

By (2.5),

$$(\alpha_q - q) + 2\left(1 - \frac{\alpha_q}{k}\right) + \frac{k^2(q+1)+1}{k^2(k+1)} > -\frac{q}{k(k+1)}.$$

Therefore,

$$\overline{G}_{q,k} < \frac{\alpha_q - \dfrac{1}{k(k+1)} - 1}{q\left(\alpha_q - \dfrac{1}{k(k+1)} - 1\right)} = \frac{1}{q}. \tag{3.13}$$

Hence, by (3.12) and (3.13), we obtain that $g_{q,k}(\gamma) < 1/q$, concluding the proof of the lemma. □

4. Proof of main theorem – part (b)

We first prove (1.9). By (1.1) and (3.3), we have that

$$E_{q,n}^{(k)} = -g_{q,k}(\gamma)\gamma^n$$

for all $2-k \leqslant n \leqslant 0$. Suppose $n=0$. By Lemma 3.3, we get

$$|E_{q,0}^{(k)}| = g_{q,k}(\gamma) < 1/q.$$

Moreover, if $2-k \leqslant n \leqslant -1$, then $\gamma^n \leqslant \gamma^{-1} < 1$ and

$$g_{q,k}(\gamma)\gamma^n \leqslant g_{q,k}(\gamma) < 1/q$$

for all $k \geqslant 2$.

Using Lemma 3.3, we have that $\gamma/(q+1) < g_{q,k}(\gamma)\gamma < \gamma/q$. Since $q < \gamma < q+1$, we get $1 - 1/(q+1) < g_{q,k}(\gamma)\gamma < 1 + 1/q$ and

$$-\frac{1}{q} < F_{q,1}^{(k)} - g_{q,k}(\gamma)\gamma < \frac{1}{q+1},$$

where we use that $F_{q,1}^{(k)} = 1$. Hence, we obtain that $|E_{q,1}^{k}| < 1/q$.

We give a proof by contradiction. Assume to the contrary that $|E_{q,n}^{(k)}| \geqslant 1/q$ for an integer $n \geqslant 2$. Let n_0 be the smallest positive integer with this property. Since $|E_{q,n_0-1}^{(k)}| < 1/q$ and $|E_{q,n_0-k}| < 1/q$, we get

$$|(q-1)E_{q,n_0-1}^{(k)} + E_{q,n_0-k}| < 1.$$

By (3.4),

$$|E_{q,n_0+1}^{(k)}| \geqslant (q+1)|E_{q,n_0}^{(k)}| - |(q-1)E_{q,n_0-1}^{(k)} + E_{q,n_0-k}^{(k)}|.$$

Hence,

$$|E_{q,n_0+1}^{(k)}| - |E_{q,n_0}^{(k)}| \geqslant q|E_{q,n_0}^{(k)}| - |(q-1)E_{q,n_0-1}^{(k)} + E_{q,n_0-k}^{(k)}| > 0,$$

implying

$$|E_{q,n_0+1}^{(k)}| > |E_{q,n_0}^{(k)}|.$$

Since $n_0 - k + 1 < n_0$, we infer that

$$|E_{q,n_0-k+1}^{(k)}| \leqslant \frac{1}{q} < |E_{q,n_0}^{(k)}| < |E_{q,n_0+1}^{(k)}|,$$

and therefore

$$|(q-1)E^{(k)}_{q,n_0}+E^{(k)}_{q,n_0-k+1}|<q|E^{(k)}_{q,n_0+1}|.$$

By (3.4),

$$|E^{(k)}_{q,n_0+2}|\geqslant (q+1)|E^{(k)}_{q,n_0+1}|-|(q-1)E^{(k)}_{q,n_0}+E^{(k)}_{q,n_0-k+1}|,$$

and we obtain that $|E^{(k)}_{q,n_0+2}|>|E^{(k)}_{q,n_0+1}|$.

Suppose that $|E^{(k)}_{q,n_0}|<|E^{(k)}_{q,n_0+1}|<\ldots<|E^{(k)}_{n_0+i-1}|$ for an integer $i\geqslant 3$. We distinguish two cases according to whether $n_0+i-k-1<n_0$ or $n_0\leqslant n_0+i-k-1$. First, if $n_0+i-k-1<n_0$, then we get

$$|E^{(k)}_{q,n_0+i-k-1}|<\frac{1}{q}\leqslant|E^{(k)}_{q,n_0}|<|E^{(k)}_{q,n_0+1}|<\ldots<|E^{(k)}_{n_0+i-1}|.$$

If $n_0\leqslant n_0+i-k-1<n_0+i-1$, then we obtain that

$$|E^{(k)}_{q,n_0+i-k-1}|<|E^{(k)}_{q,n_0+i-1}|.$$

In either case, we conclude that $|E^{(k)}_{q,n_0+i-k-1}|<|E^{(k)}_{q,n_0+i-1}|$, implying

$$|(q-1)E^{(k)}_{q,n_0+i-2}+E^{(k)}_{q,n_0+i-k-1}|<q|E^{(k)}_{q,n_0+i-1}|.$$

Using (3.4) again, we get

$$|E^{(k)}_{q,n_0+i}|\geqslant (q+1)|E^{(k)}_{n_0+i-1}|-|(q-1)E^{(k)}_{q,n_0+i-2}+E^{(k)}_{q,n_0+i-k-1}|>|E^{(k)}_{q,n_0+i-1}|.$$

Therefore, $|E^{(k)}_{q,n_0}|<|E^{(k)}_{q,n_0+1}|<\ldots<|E^{(k)}_{n_0+i-1}|<|E^{(k)}_{n_0+i}|$ contradicting (3.5). Hence, we conclude that $|E^{(k)}_{q,n}|<1/q$ for all integers $n\geqslant 2-k$, proving (1.9).

We next prove (1.10). By (1.9),

$$g_{q,k}(\gamma)\gamma^n-\frac{1}{q}<F^{(k)}_{q,n}<g_{q,k}(\gamma)\gamma^n+\frac{1}{q}.$$

By Lemma 3.3, we obtain that

$$\frac{\gamma^n}{q+1}-\frac{1}{q}<F^{(k)}_{q,n}<\frac{\gamma^n}{q}+\frac{1}{q}.$$

Hence,

$$\gamma^{n-2}<\gamma^{n-1}\Big(\frac{q-1}{q}\Big)<F^{(k)}_{q,n}<\gamma^{n-1}\Big(\frac{q+2}{q}\Big)<\gamma^n$$

for all $n\geqslant 1$, completing the proof of (1.10) and Theorem 1.1.

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