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AN EXAMPLE GINSBURG SAID IN 1984
HE WAS “UNABLE TO FIND” AND A FORBIDDEN SUBPOSET
CHARACTERIZATION OF SUBSETS OF REGULAR POSETS

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Abstract. In 1984, Ginsburg wrote, “We have been unable to find an example of an ordered set P having the properties of [being complete, densely ordered, with no antichain other than $\{0\}$ and $\{1\}$ that is a cutset] and in which all antichains are countable.” In this very brief note, such an example is shown. Posets that can be embedded in regular posets are characterized as posets that do not contain $\omega \times \{0, 1\}$ or its dual as a subposet. Any such poset P can be embedded in a regular poset that can be embedded in any other regular poset containing P .

Keywords: regular poset; (minimal) cutset; (maximal) chain; (maximal) antichain; lexicographic sum; complete lattice

MSC 2020: 06A06

1. DEFINITIONS AND MOTIVATION

Our notation and terminology largely come from [1], [2], and [13]. We use subscripts to indicate which poset we are referring to—for instance, $\bigvee_R S$ is the supremum of the set S with respect to the poset R .

A subset D of a poset is *directed* if $D \neq \emptyset$ and, for all $d, d' \in D$, there exists $e \in D$ such that $d, d' \leq e$. Let $\mathcal{I}(P)$ denote the family of directed down-sets of a poset P , ordered by set-inclusion. For an element p of a poset P , $\overset{\circ}{\downarrow} p := \{q \in P : q < p\}$. Let $\partial_P : P \rightarrow P^\partial$ be the canonical anti-isomorphism from a poset P to its dual.

A poset P is *sup-pre-regular* if for every directed subset $D \subseteq P$ such that $\bigvee D$ exists and for all $p \in P$ such that $p < \bigvee D$, there exists $d \in D$ such that $p < d$.

A poset is *sup-regular* if every nonempty chain has a supremum and for every nonempty chain C and every element p such that $p < \bigvee C$, there exists $c \in C$ such that $p < c$.

Dually, we define *inf-pre-regular* and *inf-regular*. A poset is *regular* if it is sup- and inf-regular.

One finds the definitions of “sup-regular,” and “regular” posets in Grillet’s article (see [7]). Other articles discussing regular posets include [3], [8], [9], [10] and [12].

Proposition 1.1. *A poset is sup-regular if and only if it is sup-pre-regular and every directed subset has a supremum.*

Proof. For the nontrivial direction, assume P is sup-regular. By [11], Corollary 2, every directed subset has a supremum.

We now prove by transfinite induction on $|D|$ that if D is a directed set and $p \in P$ such that $p < \bigvee D$, then there exists $d \in D$ with $p < d$. This is clear if D is finite (so D has a greatest element $\bigvee D \in D$). If D is infinite, [11], Theorem 1, implies there is a sequence $(D_\alpha)_{\alpha < |D|}$ of directed subsets of D such that, for all $\alpha < |D|$, D_α is either finite or $|D_\alpha| = |\alpha|$; if $\alpha, \beta < |D|$ and $\alpha < \beta$, then $D_\alpha \subseteq D_\beta$; and $D = \bigcup_{\alpha < |D|} D_\alpha$.

Let $c_\alpha = \bigvee D_\alpha$ ($\alpha < |D|$). Then $\{c_\alpha : \alpha < |D|\}$ is a nonempty chain, and $\bigvee \{c_\alpha : \alpha < |D|\} = \bigvee D$. By sup-regularity, there exists $\alpha < |D|$ such that $p < c_\alpha$. By induction, there exists $d_\alpha \in D_\alpha \subseteq D$ such that $p < d_\alpha$. $\square$

A *cutset* of a poset is a subset that intersects every maximal chain. A poset is *CAC* if every maximal antichain is a cutset. Grillet showed that every regular poset E can be embedded in a regular CAC poset $\overline{E}$ in a “minimal” way: no subposet $F \subsetneq \overline{E}$ containing E is CAC and, if G is CAC, every order-preserving map from E to G can be extended to $\overline{E}$. He conjectured that any poset had such a “completion.”

Maltby observed that no regular poset can contain $\omega \times \mathbf{2}$ (where $\mathbf{2}$ is the chain $\{0, 1\}$) or its dual $(\omega \times \mathbf{2})^\partial \cong \omega^\partial \times \mathbf{2}$ (see Figure 1).

We repeat (the dual of) Maltby’s argument:

Observation 1.2 (Maltby [10], page 394). *Let P be a poset containing $\omega \times \mathbf{2}$. Then P cannot be embedded in a sup-regular poset.*

Proof. Assume $\omega \times \mathbf{2}$ is contained in a sup-regular poset R . Let $r_i = \bigvee_R \{(n, i) : n < \omega\}$ for $i \in \{0, 1\}$. Then $r_0 \leq r_1$.

Case 1. $r_0 < r_1$. Then, by sup-regularity, there exists $n < \omega$ such that $r_0 < (n, 1)$. But $(n + 1, 0) \leq r_0 < (n, 1)$ gives a contradiction.

Case 2. $r_0 = r_1$. Then $(42, 1) < r_0$, so, by sup-regularity, there exists $n < \omega$ such that $(42, 1) < (n, 0)$, a contradiction. $\square$

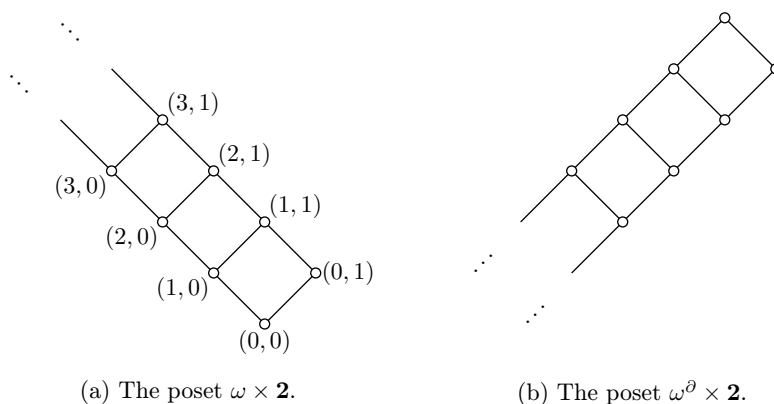


Figure 1.

Observation 1.3. *Let P be a poset. Let $a_0, a_1, a_2, \dots, b_0, b_1, b_2, \dots \in P$ be such that $a_0 < a_1 < a_2 < \dots$ and $b_0 < b_1 < b_2 < \dots$ and $a_i < b_i$ and $a_{i+1} \not\leq b_i$ ($i = 0, 1, 2, \dots$) and $b_i \not\leq a_j$ ($i, j = 0, 1, 2, \dots$). Then $\{a_0, a_1, \dots, b_0, b_1, \dots\}$ is order-isomorphic to $\omega \times 2$.*

The poset $\omega \times 2 \cup \{(\omega, 0)\}$ (Figure 7 (a)) appears in [6], Theorem 1.4 as a forbidden subposet for a certain class of posets we will not describe (the “chain-complete” posets whose space of maximal chains is compact).

We prove the converse of Maltby’s observation: *a poset can be embedded in a regular poset if it does not contain $\omega \times 2$ or its dual*. Indeed, there is a “least” such “regular completion.” We prove an analogous result for sup-regular posets and posets excluding $\omega \times 2$ (Theorem 6.8 and Corollary 6.9). Lemma 4.1 makes it all work.

While Ginsburg does not define the term, a poset P is *densely ordered* if, for all $x, y \in P$ such that $x < y$, there exists $z \in P$ such that $x < z < y$, and P is a *complete ordered set* if it is a complete lattice. He calls an antichain $\{0\}$ or $\{1\}$ of a bounded poset *trivial*.

In his 1984 article [6], Example 2.1, Ginsburg exhibits a “complete ordered set P which is densely ordered... and in which no nontrivial antichain is a cutset.” He adds, “We have been unable to find an example of an ordered set P having the properties of Example 2.1 and in which all antichains are countable.”

We provide such an example in Section 7 (Theorem 7.24), by modestly altering a poset Higgs created for another purpose. Additional terminology related to the example will be in Section 7, which with Section 1 is self-contained.

2. MORE TERMINOLOGY: FILTERED LIMITS OF POSETS

Let $\mathbf{Poset}$ be the category of posets + order-preserving maps. Let $(C_i)_{i < \omega}$ be a family of posets and

$$(f_{ij} : C_j \rightarrow C_i)_{i, j < \omega; j \leq i}$$

a family of order-preserving maps with the following properties:

- (1) $f_{ii} = \text{id}_{C_i}$ for all $i < \omega$;
- (2) $f_{ij} \circ f_{jk} = f_{ik}$ for all $i, j, k < \omega$ such that $k \leq j \leq i$.

Then

$$\mathcal{S} = ((C_i)_{i < \omega}, (f_{ij} : C_j \rightarrow C_i)_{i, j < \omega; j \leq i})$$

is a *filtered system* in $\mathbf{Poset}$.

Assume C is a poset and $(f_i : C_i \rightarrow C)_{i < \omega}$ is a family of order-preserving maps such that, for $i, j < \omega$, $j \leq i$ implies $f_i \circ f_{ij} = f_j$. Then

$$(C, (f_i : C_i \rightarrow C)_{i < \omega})$$

is *compatible* with $\mathcal{S}$. It is a *filtered limit* of $\mathcal{S}$ if we assume further that whenever $(C', (f'_i : C_i \rightarrow C')_{i < \omega})$ is compatible with $\mathcal{S}$, there is a unique order-preserving map $f : C \rightarrow C'$ such that $f \circ f_i = f'_i$ for all $i < \omega$.

A construction of a filtered limit is given in [5], Proposition 4.1. The reader can find proofs of the following in [5], Proposition 4.2, Lemma 4.4, and Lemma 4.5.

Proposition 2.1. *Let*

$$(C, (f_i : C_i \rightarrow C)_{i < \omega})$$

be the filtered limit of the filtered system

$$\mathcal{S} = ((C_i)_{i < \omega}, (f_{ij} : C_j \rightarrow C_i)_{i, j < \omega; j \leq i}).$$

Then for all $c \in C$ there exist $i < \omega$ and $c_i \in C_i$ such that $f_i(c_i) = c$. For $c, d \in C$, $c \leq d$ is equivalent to each of the following:

- (1) *if $i, j < \omega$, $c_i \in C_i$, $c_j \in C_j$, and $f_i(c_i) = c$ and $f_j(c_j) = d$, then there exists $h < \omega$ such that $i, j \leq h$ and $f_{hi}(c_i) \leq f_{hj}(c_j)$;*
- (2) *there exist $h, i, j < \omega$, $c_i \in C_i$, $c_j \in C_j$ such that $i, j \leq h$, $f_i(c_i) = c$, $f_j(c_j) = d$, and $f_{hi}(c_i) \leq f_{hj}(c_j)$.*

For (1) or (2), any $h' < \omega$ such that $h \leq h'$ also works.

Lemma 2.2. *Let*

$$(C, (f_i: C_i \rightarrow C)_{i < \omega})$$

be a filtered limit of the filtered system

$$\mathcal{S} = ((C_i)_{i < \omega}, (f_{ij}: C_j \rightarrow C_i)_{i, j < \omega; j \leq i}).$$

(1) *Let*

$$(C', (f'_i: C_i \rightarrow C')_{i < \omega})$$

be compatible with $\mathcal{S}$. Let $f: C \rightarrow C'$ be the order-preserving map such that $f \circ f_i = f'_i$ for all $i < \omega$. If f'_i is an order-embedding for all $i < \omega$, then f is an order-embedding.

(2) *If $f_{i+1,i}$ is an order-embedding for all $i < \omega$, then f_i is an order-embedding for all $i < \omega$.*

Lemma 2.3. *Let C_i be a poset for all $i < \omega$. Let $f_{i+1,i}: C_i \rightarrow C_{i+1}$ be an order-preserving map for all $i < \omega$. For all $i, j < \omega$ such that $j < i$, define $f_{ij}: C_j \rightarrow C_i$ as $f_{i,i-1} \circ f_{i-1,i-2} \circ \dots \circ f_{j+2,j+1} \circ f_{j+1,j}$ and let $f_{jj}: C_j \rightarrow C_j$ be id_{C_j} . Then*

$$\mathcal{S} = ((C_i)_{i < \omega}, (f_{ij}: C_j \rightarrow C_i)_{i, j < \omega; j \leq i})$$

is a filtered system in Poset.

Let C be a poset. Let $f_i: C_i \rightarrow C$ be an order-preserving map for all $i < \omega$. Then

$$(C, (f_i: C_i \rightarrow C)_{i < \omega})$$

is compatible with $\mathcal{S}$ if and only if $f_{i+1} \circ f_{i+1,i} = f_i$ for all $i < \omega$.

Lemma 2.4. *Let*

$$\mathcal{S} = ((C_i)_{i < \omega}, (f_{ij}: C_j \rightarrow C_i)_{i, j < \omega; j \leq i})$$

be a filtered system in Poset with filtered limit

$$\mathcal{T} = (C, (f_i: C_i \rightarrow C)_{i < \omega}).$$

Fix $k < \omega$. Then

$${}_k\mathcal{S} := ((C_i)_{k \leq i < \omega}, (f_{ij}: C_j \rightarrow C_i)_{i, j < \omega; k \leq j \leq i})$$

is a filtered system in Poset with filtered limit

$${}_k\mathcal{T} = (C, (f_i: C_i \rightarrow C)_{k \leq i < \omega}).$$

Proof. Let ${}_k\mathcal{T}' = (C', (f'_i: C_i \rightarrow C')_{k \leq i < \omega})$ be compatible with ${}_k\mathcal{S}$. For $i < k$, let $f'_i := f'_k \circ f_{ki}: C_i \rightarrow C'$. Let $\mathcal{T}' := (C', (f'_i: C_i \rightarrow C')_{i < \omega})$. We show $\mathcal{T}'$ is compatible with $\mathcal{S}$. Let $i, j < \omega$ be such that $j \leq i$. If $i < k$, then $f'_i \circ f_{ij} = f'_k \circ f_{ki} \circ f_{ij} = f'_k \circ f_{kj} = f'_j$ since $j < k$. If $j < k \leq i$, then $f'_i \circ f_{ij} = f'_i \circ f_{ik} \circ f_{kj} = f'_k \circ f_{kj} = f'_j$ since $j < k$. If $k \leq j$, then $f'_i \circ f_{ij} = f'_j$. Thus, there exists a unique order-preserving map $f: C \rightarrow C'$ such that, for all $i < \omega$, $f'_i = f \circ f_i$.

Now assume $g: C \rightarrow C'$ is an order-preserving map such that $k \leq i < \omega$ implies $f'_i = g \circ f_i$. Let $i < k$. We have $g \circ f_i = g \circ f_k \circ f_{ki} = f'_k \circ f_{ki} = f'_i$. Thus $g = f$. $\square$

3. ONE STEP OF THE PROCESS OF MAKING POSETS SUP-PRE-REGULAR

What makes a poset P not be sup-pre-regular is the existence of a directed subset D with a join $\bigvee D$ and an element $p < \bigvee D$ such that $p \not\leq d$ for all $d \in D$. We fix the problem by making the “join” of D be less than $\bigvee D$ in a new poset P^+ .

We start with results that have trivial proofs.

Definition 3.1. Let P be a poset. Let

$$\mathcal{D} := \left\{ D \in \mathcal{I}(P): \bigvee D \text{ exists and there exists } p \in P \right. \\ \left. \text{with } p < \bigvee D \text{ but for all } d \in D, p \not\leq d \right\}.$$

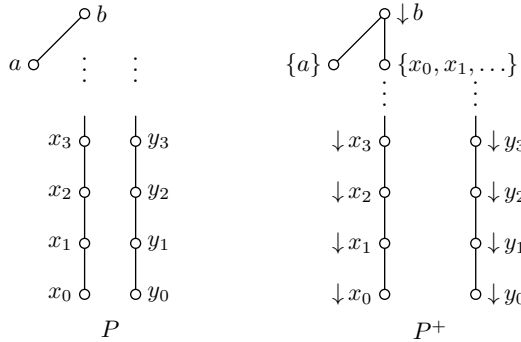
Let $P^+ := \{\downarrow p: p \in P\} \cup \mathcal{D}$ be ordered by set-inclusion.

Let $\iota_P^+: P \rightarrow P^+$ be the order-embedding $p \mapsto \downarrow p$ ($p \in P$).

Let $\eta^-: \mathcal{P}(P^+) \rightarrow \mathcal{P}(P)$ be given as follows: for all $B \subseteq P^+$,

$$\eta^-(B) = \bigcup B = \{p \in P: p \in b \text{ for some } b \in B\}.$$

(Remember every *element* of P^+ is a *subset* of P .)



(a) The poset P .

(b) The poset P^+ .

Figure 2. The poset P .

Remark 3.2.

- (1) For all $p \in P$, $\downarrow p \notin \mathcal{D}$, since no $D \in \mathcal{D}$ has a maximal element.
- (2) The map η^- is $\subseteq$ -preserving.

Lemma 3.3. *Let P be a poset. Let D be a set. For each $d \in D$, let $B_d \in \mathcal{P}(P^+)$. Then $\eta^-\left(\bigcup_{d \in D} B_d\right) = \bigcup_{d \in D} \eta^-(B_d)$.*

Proof. By Remark 3.2 (2), $\bigcup_{d \in D} \eta^-(B_d) \subseteq \eta^-\left(\bigcup_{d \in D} B_d\right)$.

Now let $p \in \eta^-\left(\bigcup_{d \in D} B_d\right)$, so there exists $b \in \bigcup_{d \in D} B_d$ such that $p \in b$. Hence, there exists $d \in D$ such that $b \in B_d$, so $p \in \eta^-(B_d)$. $\square$

Lemma 3.4. *Let P be a poset. For all $x \in P^+$, $\eta^-(\{x\}) = \eta^-(\downarrow_{P^+} x)$.*

Proof. By Remark 3.2 (2), $\eta^-(\{x\}) \subseteq \eta^-(\downarrow_{P^+} x)$.

Now let $z \in \eta^-(\downarrow_{P^+} x) = \bigcup \downarrow_{P^+} x = \bigcup \{y \in P^+ : y \leq_{P^+} x\}$. Thus, there exists $y \in P^+$ such that $y \subseteq x$ and $z \in y$, so $z \in x$. Hence $z \in \bigcup \{x\} = \eta^-(\{x\})$. $\square$

Lemma 3.5. *Let P be a poset. If $B \subseteq P^+$ is a directed subset of P^+ , then $\eta^-(B)$ is a directed subset of P .*

Proof. Since $B \neq \emptyset$, there exists $b \in B$, and every element of B is nonempty. Thus, there exists $c \in b$, and $c \in \bigcup B = \eta^-(B)$, so $\eta^-(B) \neq \emptyset$.

Now let $p, p' \in \eta^-(B) = \bigcup B$. Then there exist $b, b' \in B$ such that $p \in b$ and $p' \in b'$. Since B is directed, there exists $b'' \in B$ such that $b, b' \subseteq b''$, so $p, p' \in b''$. As every element of P^+ is a directed subset of P , there exists $p'' \in b''$ such that $p, p' \leq_P p''$. Also $p'' \in \bigcup B = \eta^-(B)$. $\square$

Lemma 3.6. *Let P be a poset. Let D be a directed set such that, for all $d \in D$, $B_d \in \mathcal{I}(P)$. Assume that for all $d, d' \in D$, if $d \leq_D d'$, then $B_d \subseteq B_{d'}$. Then $\bigcup_{d \in D} B_d \in \mathcal{I}(P)$.*

Proof. Since D is directed, $D \neq \emptyset$, so take $\tilde{d} \in D$. Since $B_{\tilde{d}}$ is directed, then $B_{\tilde{d}} \neq \emptyset$, so $\bigcup_{d \in D} B_d \neq \emptyset$.

Now let $p, p' \in \bigcup_{d \in D} B_d$. Then there exist $d, d' \in D$ such that $p \in B_d$ and $p' \in B_{d'}$. There exists $d'' \in D$ such that $d, d' \leq_D d''$, so $B_d, B_{d'} \subseteq B_{d''}$ and hence $p, p' \in B_{d''}$. Since $B_{d''}$ is directed, there exists $p'' \in B_{d''}$ such that $p, p' \leq_P p''$. Hence $\bigcup_{e \in D} B_e$ is directed. $\square$

Lemma 3.7. *Let B be a subset of P^+ . Then $\eta^-(B)$ is a down-set of P .*

Proof. Let $p \in \eta^-(B)$. Let $p' \in P$ be such that $p' \leq_P p$. As $p \in \bigcup B$, there exists $b \in B$ such that $p \in b$. Since every element of P^+ is a down-set of P , $p' \in b$, so $p' \in \bigcup B = \eta^-(B)$. $\square$

Lemma 3.8. *Let P be a poset.*

- (1) *Let $B \subseteq P^+$ be a down-set of P^+ . Let $p \in \eta^-(B)$. Then $\iota_P^+(p) \in B$.*
- (2) *Let $p \in P$, $B \subseteq P^+$. Assume there is a $b \in B$ such that $b \not\subseteq_{\downarrow_P} p$. Then there is $a \in \eta^-(B)$ such that $a \not\leq_P p$.*

Proof. (1) Since $p \in \eta^-(B) = \bigcup B$, there exists $b \in B$ such that $p \in b$. Every element of P^+ is a down-set of P , so $\downarrow_P p \subseteq b$. As $\downarrow_P p \in P^+$ and B is a down-set of P^+ and $b \in B$, then $\downarrow_P p \in B$.

(2) There is $a \in b$ such that $a \not\leq_P p$. But $a \in \bigcup B = \eta^-(B)$. $\square$

Lemma 3.9. *Let P be a poset. Let $B \subseteq P^+$. Let $c \in P$ be an upper bound of $\eta^-(B)$ in P . Then $\downarrow_P c$ is an upper bound of B in P^+ .*

Proof. For all $p \in \eta^-(B) = \bigcup B$ we have $p \leq_P c$. Thus, for all $b \in B$ and for all $p \in b$ we have $p \leq_P c$, i.e., for all $b \in B$ and for all $p \in b$ we have $p \in \downarrow_P c$. Hence, for all $b \in B$ we have $b \subseteq_{\downarrow_P} c = \iota_P^+(c)$. This means that $\iota_P^+(c)$ is an upper bound of B in P^+ . $\square$

Lemma 3.10. *Let P be a poset. Let $b \in P^+$. Then*

$$b = \bigvee_{P^+} \{\downarrow_P c : c \in b\} = \bigcup \{\downarrow_P c : c \in b\}.$$

Lemma 3.11. *Let P be a poset. Let R be a sup-regular poset. Let $f: P \rightarrow R$ be an order-embedding. Define $\bar{f}: P^+ \rightarrow R$ by $\bar{f}(D) = \bigvee_R \{f(d) : d \in D\}$ for all $D \in P^+$. The map $\bar{f}$ is an order-embedding such that $\bar{f} \circ \iota_P^+ = f$.*

Proof. Any $D \in P^+$ is directed, so $\{f(d) : d \in D\}$ is directed and hence, $\bigvee_R \{f(d) : d \in D\}$ exists. If $D = \iota_P^+(p)$ for some $p \in P$, then $\{f(d) : d \in D\}$ has a greatest element $f(p)$, which is the supremum. Thus, $\bar{f}$ is well-defined and $\bar{f} \circ \iota_P^+ = f$.

Now we show $\bar{f}$ is an order-embedding. Let $D, D' \in P^+$. If $D \subseteq D'$, then $\bar{f}(D) \leq_R \bar{f}(D')$.

Now assume $\bar{f}(D) \leq_R \bar{f}(D')$.

Case 1. $D' = \downarrow_P p'$, where $p' \in P$. Then $\bar{f}(D) \leq_R f(p')$ and hence, for all $d \in D$, $f(d) \leq_R f(p')$, so, for all $d \in D$, $d \leq_P p'$. Thus $D \subseteq_{\downarrow_P} p' = D'$.

Case 2. $D' \in \mathcal{I}(P)$ has no maximal element; $\bigvee_P D'$ exists and there exists $a \in P$ with $a <_P \bigvee_P D'$ but $a \not<_P d'$ for all $d' \in D'$. Assume for a contradiction that $D \not\subseteq D'$. Then there exists $\tilde{d} \in D \setminus D'$. Yet $f(\tilde{d}) \leq_R \bigvee_{d' \in D'} f(d')$.

Case 2(a). $f(\tilde{d}) = \bigvee_{d' \in D'} f(d')$. In this case, $f(d') \leq_R f(\tilde{d})$ for all $d' \in D'$, so, since f is an order-embedding, $d' \leq_P \tilde{d}$ for all $d' \in D'$. As D is a down-set, $D' \subseteq \downarrow \tilde{d} \subseteq D$. Hence $a <_P \bigvee_P D' \leq \tilde{d}$, so $f(a) <_R f(\tilde{d}) = \bigvee_{d' \in D'} f(d')$. By sup-regularity, there exists $\tilde{d}' \in D'$ such that $f(a) <_R f(\tilde{d}')$ and so $a <_P \tilde{d}'$, because f is an embedding. This is a contradiction.

Case 2(b). $f(\tilde{d}) <_R \bigvee_{d' \in D'} f(d')$. By sup-regularity, there exists $\tilde{d}' \in D'$ such that $f(\tilde{d}) <_R f(\tilde{d}')$. Since f is an order-embedding, $\tilde{d} <_P \tilde{d}'$, so $\tilde{d} \in D'$, since D' is a down-set, a contradiction. Hence $D \subseteq D'$. This proves $\bar{f}$ is an order-embedding. $\square$

4. CHAINS OF DIRECTED DOWN-SETS OF POSETS NOT CONTAINING $\omega \times 2$ OR $\omega^\partial \times 2$

Lemma 4.1. *Let P be a poset not containing $\omega \times 2$. Let $D, D' \in \mathcal{I}(P)$ be such that $D \subsetneq D'$. Then there exists $d' \in D'$ such that $D \subseteq \bigcirc_P d'$.*

Proof. *Case 1.* $D' = \downarrow d'$. Since $D \subsetneq D'$, then $d' \notin D$. Thus, for all $d \in D$, $d < d'$, so $D \subseteq \bigcirc_P d'$.

Case 2. D' has no maximal element. This means that for any $d' \in D'$, there exists $d'' \in D'$ such that $d' < d''$. Pick $\bar{d}' \in D' \setminus D$. Pick $d_0 \in D$. Let $d'_0 \in D'$ be such that $\bar{d}', d_0 < d'_0$.

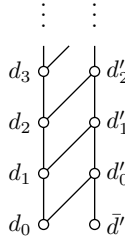


Figure 3.

Now assume there exist $n \in \mathbb{N}_0$, $d_0, d_1, \dots, d_n \in D$, $d'_0, d'_1, \dots, d'_n \in D'$ such that $d_0 < d_1 < \dots < d_n$ and $\bar{d}' < d'_0 < \dots < d'_n$ and $d_i < d'_i$ ($i = 0, 1, \dots, n$) and $d_{i+1} \not< d'_i$ ($i = 0, 1, \dots, n-1$). If $D \subseteq \bigcirc_P d'_n$, then since there exists $d'' \in D'$ such that $d'_n < d''$, we have $D \subseteq \bigcirc_P d''$ and we are done.

Otherwise, there exists $\bar{d} \in D$ such that $\bar{d} \not\leq d'_n$. Then there exists $d_{n+1} \in D$ such that $\bar{d}, d_n \leq d_{n+1}$. As $d_n \leq d'_n$ but $d_{n+1} \not\leq d'_n$, we have $d_n < d_{n+1}$. We then have $d'_{n+1} \in D'$ such that $d_{n+1}, d'_n < d'_{n+1}$. By induction, we have $d_0, d_1, d_2, \dots \in D$ and $\bar{d}', d'_0, d'_1, \dots \in D'$, where $d_0 < d_1 < d_2 < \dots$ and $\bar{d}' < d'_0 < d'_1 < \dots$, where $\bar{d}' \notin D$ (so $d'_0, d'_1, \dots \notin D$) and $d_i < d'_i$ and $d_{i+1} \not\leq d'_i$ ($i = 0, 1, 2, \dots$). For $i, j \in \{0, 1, 2, \dots\}$, $d'_i \not\leq d_j$. Use Observation 1.3. $\square$

Proposition 4.2. *Let P be a poset. Then P contains $\omega \times \mathbf{2}$ if and only if $\mathcal{I}(P)$ contains $\omega \times \mathbf{2}$.*

Proof. One direction is trivial. For the other, let $D_0, D_1, D_2, \dots, E_0, E_1, E_2, \dots \in \mathcal{I}(P)$ be such that $D_0 \subsetneq D_1 \subsetneq D_2 \subsetneq \dots$ and $E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \dots$ and $D_i \subsetneq E_i$ and $D_{i+1} \not\subseteq E_i$ ($i = 0, 1, 2, \dots$) and $E_i \not\subseteq D_j$ ($i, j = 0, 1, 2, \dots$).

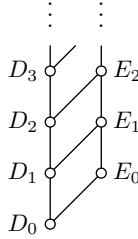


Figure 4.

Let $d_0 \in D_0$. There exists $e'_0 \in E_0$ such that $D_0 \subseteq \downarrow^\circ e'_0$ by Lemma 4.1, so $d_0 < e'_0$. As $E_0 \not\subseteq D_1$, there exists $e''_0 \in E_0 \setminus D_1$. Let $e_0 \in E_0$ be such that $e'_0, e''_0 \leq e_0$. Thus $d_0 < e_0$ and $e_0 \notin D_1$.

Now assume $n \in \mathbb{N}_0$, $d_i \in D_i$ ($i = 0, 1, \dots, n$) and $e_i \in E_i$ ($i = 0, 1, \dots, n$), where $d_0 < d_1 < d_2 < \dots < d_n$ and $e_0 < e_1 < \dots < e_n$ and $d_i < e_i$ ($i = 0, 1, \dots, n$) and $d_{i+1} \not\leq e_i$ ($i = 0, 1, \dots, n-1$). Further assume that $E_i \subseteq \downarrow^\circ e_{i+1}$ ($i = 0, 1, \dots, n-1$) and $D_i \subseteq \downarrow^\circ e_i$ ($i = 0, 1, \dots, n$). Finally, assume $e_n \notin D_{n+1}$. Since $D_{n+1} \not\subseteq E_n$, there exists $d'_{n+1} \in D_{n+1} \setminus E_n$. Thus, there exists $d_{n+1} \in D_{n+1}$ such that $d_n, d'_{n+1} \leq d_{n+1}$. Since E_n is a down-set and $d_n \in E_n$, $d_n < d_{n+1}$. Also, $d_{n+1} \not\leq e_n$.

Let $e'_{n+1} \in E_{n+1}$ be such that $D_{n+1}, E_n \subseteq \downarrow^\circ e'_{n+1}$ (using Lemma 4.1). As $E_{n+1} \not\subseteq D_{n+2}$, there exists $e''_{n+1} \in E_{n+1} \setminus D_{n+2}$. Let $e_{n+1} \in E_{n+1}$ be such that $d_{n+1}, e'_{n+1}, e''_{n+1} \leq e_{n+1}$. Since $d_{n+1} \in D_{n+2}$ but $e_{n+1} \notin D_{n+2}$, then $d_{n+1} < e_{n+1}$. Also, $e_n < e'_{n+1} \leq e_{n+1}$, so $e_n < e_{n+1}$. Thus, we have $d_i \in D_i$, $e_i \in E_i$ ($i = 0, 1, 2, \dots$) such that $d_i < d_{i+1}$ and $e_i < e_{i+1}$ and $d_i < e_i$ ($i = 0, 1, 2, \dots$). Also, $d_{i+1} \not\leq e_i$ ($i = 0, 1, 2, \dots$) and $E_i \subseteq \downarrow^\circ e_{i+1}$ and $D_i \subseteq \downarrow^\circ e_i$ and $e_i \notin D_{i+1}$ ($i = 0, 1, 2, \dots$).

Assume that $e_{i+1} \leq d_{j+1}$ for some $i, j \in \{0, 1, 2, \dots\}$. Then $E_i \subseteq D_{j+1}$, a contradiction. Thus P contains $\omega \times \mathbf{2}$. $\square$

Lemma 4.3. *Let P be a poset not containing $\omega \times \mathbf{2}$. If $\mathcal{I}(P)$ contains $\omega^\partial \times \mathbf{2}$, then P contains $\omega^\partial \times \mathbf{2}$.*

Proof. Let $D_0, D_1, D_2, \dots, D'_0, D'_1, D'_2, \dots \in \mathcal{I}(P)$ be such that $D_i \supsetneq D_{i+1}$ and $D'_i \supsetneq D'_{i+1}$ and $D'_i \supsetneq D_i$ and $D'_{i+1} \not\supseteq D_i$ ($i = 0, 1, 2, \dots$) and $D_i \not\supseteq D'_j$ ($i, j = 0, 1, 2, \dots$).

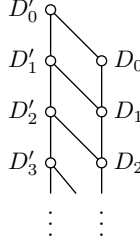


Figure 5.

For each $i \in \{0, 1, 2, \dots\}$, let $\bar{d}_i \in D_i \setminus D'_{i+1}$ and, using Lemma 4.1, let $\bar{\bar{d}}_i \in D_i$ be such that $D_{i+1} \subseteq \bar{\bar{d}}_i$; let $\bar{d}'_i \in D'_i$ be such that $D'_{i+1} \subseteq \bar{d}'_i$; let $\bar{\bar{d}}'_i \in D'_i$ be such that $D_i \subseteq \bar{\bar{d}}'_i$; let $d_i \in D_i$ be such that $\bar{d}_i, \bar{\bar{d}}_i \leq d_i$; let $d'_i \in D'_i$ be such that $\bar{d}'_i, \bar{\bar{d}}'_i \leq d'_i$.

Claim 1. For $i \in \{0, 1, 2, \dots\}$, $d_{i+1} < d_i$ and $d_i \notin D'_{i+1}$.

Proof of Claim 1. Since $d_{i+1} \in D_{i+1}$, we have $d_{i+1} < \bar{\bar{d}}_i \leq d_i$. Also, if $d_i \in D'_{i+1}$, then $\bar{d}_i \in D'_{i+1}$, a contradiction. $\square$

Claim 2. For $i \in \{0, 1, 2, \dots\}$, $d'_{i+1} < d'_i$ and $d_i \not\leq d'_{i+1}$.

Proof of Claim 2. Since $d'_{i+1} \in D'_{i+1}$ but $d_i \notin D'_{i+1}$ by Claim 1, we have $d_i \not\leq d'_{i+1}$. Since $d'_{i+1} \in D'_{i+1}$ and $\bar{\bar{d}}'_i \leq d'_i$, we have $\bar{\bar{d}}'_i \subseteq \bar{\bar{d}}'_i$, so $D'_{i+1} \subseteq \bar{\bar{d}}'_i$. Hence $d'_{i+1} < d'_i$. $\square$

Claim 3. For $i \in \{0, 1, 2, \dots\}$, $d_i < d'_i$.

Proof of Claim 3. As $d_i \in D_i \subseteq \bar{\bar{d}}'_i$, we have $d_i < \bar{\bar{d}}'_i \leq d'_i$. $\square$

Claim 4. For $i, j \in \{0, 1, 2, \dots\}$, $d'_i \not\leq d_j$.

Proof of Claim 4. Assume for a contradiction that $d'_i \leq d_j$. As $\bar{\bar{d}}'_i \leq d'_i$, we have $D'_{i+1} \subseteq \bar{\bar{d}}'_i$, so $D'_{i+1} \subseteq \bar{\bar{d}}'_i \subseteq d_j \subseteq D_j$, a contradiction. $\square$

Hence $\{d_i, d'_i : i = 0, 1, 2, \dots\} \cong \omega^\partial \times \mathbf{2}$ by Observation 1.3. $\square$

Corollary 4.4. *Let P be a poset not containing $\omega \times \mathbf{2}$ and not containing $\omega^\partial \times \mathbf{2}$. Then P^+ does not contain $\omega \times \mathbf{2}$ and does not contain $\omega^\partial \times \mathbf{2}$.*

Lemma 4.5. *Let P be a poset not containing $\omega \times 2$ such that every nonempty chain has an infimum. Let $\mathcal{C}$ be a nonempty chain in $\mathcal{I}(P)$. Then there exists $E \in \mathcal{I}(P)$ such that $E \subseteq D$ for all $D \in \mathcal{C}$, and if $F \subseteq D$ for all $D \in \mathcal{C}$, then $F \subseteq E$. Moreover, if $\mathcal{C}$ has no least element, then $E = \downarrow_P e$ for some $e \in P$.*

Now suppose P is also inf-regular. Then if $A \in \mathcal{I}(P)$ and $A \not\supseteq E$, then $A \not\supseteq D$ for some $D \in \mathcal{C}$.

Remark 4.6. The above set F need not be in $\mathcal{I}(P)$.

Proof. The chain $\mathcal{C}$ has a dually well-ordered co-initial subset, so we may assume $\mathcal{C} = \{D_\beta : \beta < \alpha\}$ for a nonzero ordinal α , where for all $\beta, \beta' < \alpha$ such that $\beta < \beta'$, we have $D_\beta \not\supseteq D_{\beta'}$. If α is a successor, we are done, so assume α is a limit. By Lemma 4.1, for all $\beta < \alpha$ there exists $d_\beta \in D_\beta$ such that

$$D_{\beta+1} \subseteq \overset{\circ}{\downarrow} d_\beta.$$

Claim. The map $\beta \mapsto d_\beta$ ($\beta < \alpha$) is strictly order-reversing.

Proof of the claim. Fix $\gamma < \alpha$. Let $\gamma < \eta < \alpha$. We show $d_\gamma > d_\eta$. If $\eta = \gamma + 1$, then since $d_\eta \in D_{\gamma+1} \subseteq \overset{\circ}{\downarrow} d_\gamma$, we have $d_\eta < d_\gamma$.

Now assume $\eta = \zeta + 1$, where $\gamma < \zeta < \alpha$ and $d_\gamma > d_\zeta$. We know $d_\zeta > d_\eta$, so $d_\gamma > d_\eta$.

Now assume η is a limit. As $d_\eta \in D_{\zeta+1}$ for all $\zeta < \eta$, we have $d_\eta \in \overset{\circ}{\downarrow} d_\zeta$ for all $\zeta < \eta$, so $d_\gamma > d_\eta$. $\square$

For every limit $\beta \leq \alpha$, let $e_\beta := \bigwedge_P \{d_\gamma : \gamma < \beta\}$. This is well-defined by the claim, and for every limit $\beta < \alpha$, $d_\beta \leq e_\beta$.

Let $E = \downarrow e_\alpha \in \mathcal{I}(P)$. For every $\beta < \alpha$, $e_\alpha \leq d_\beta \in D_\beta$, so $E \subseteq D_\beta$. If $x \in \bigcap_{\beta < \alpha} D_\beta$, then for all $\beta < \alpha$, $x \in D_{\beta+1} \subseteq \overset{\circ}{\downarrow} d_\beta$, so

$$x \leq \bigwedge_P \{d_\beta : \beta < \alpha\} = e_\alpha,$$

so $x \in E$. Hence, if $F \subseteq \bigcap_{\beta < \alpha} D_\beta$, then $F \subseteq E$.

Now assume P is also inf-regular. Assume $A \in \mathcal{I}(P)$ and $A \not\supseteq E$. Then there exists $a \in A$ such that $a > e_\alpha$. By inf-regularity, there exists $\beta < \alpha$ such that $a > d_\beta$, and hence $D_{\beta+1} \subseteq \overset{\circ}{\downarrow} a$. Thus $A \not\supseteq D_{\beta+1}$. $\square$

Lemma 4.7. *Let P be a poset not containing $\omega \times 2$. If every nonempty chain of P has an infimum, then every nonempty chain of P^+ has an infimum. If P is inf-regular, then P^+ is inf-regular.*

5. MAKING POSETS SUP-PRE-REGULAR

In this section, let P_0 be a poset. If $n < \omega$ and P_n is defined, let $P_{n+1} = P_n^+$. For $i < \omega$, let $\varphi_{i+1,i}: P_i \rightarrow P_{i+1}$ be $\iota_{P_i}^+$. As per Lemma 2.3, define a filtered system

$$((P_i)_{i < \omega}, (\varphi_{ij}: P_j \rightarrow P_i)_{i,j < \omega; j \leq i}).$$

Let

$$(P_\omega, (\varphi_i: P_i \rightarrow P_\omega)_{i < \omega})$$

be its filtered limit. By Lemma 2.2 (2), φ_i is an order-embedding for all $i < \omega$. We will use ι_n^+ , $\leq_n$, $\downarrow_n$, $\bigvee_n$, $\leq_\omega$, and $\bigvee_\omega$ for $\iota_{P_n}^+$, $\leq_{P_n}$, $\downarrow_{P_n}$, $\bigvee_{P_n}$, $\leq_{P_\omega}$, and $\bigvee_{P_\omega}$, but η_n^- and η_ω^- will not have the meanings one might think they do ($n < \omega$).

Definition 5.1. Let $n < \omega$ be nonzero. If $p_n \in P_n$, define $\eta_n^-(p_n)$ by $\eta_n^-(\eta_n^-(\dots \eta_n^-(\{p_n\}) \dots))$.
 n times

Note that the domain of η_n^- is P_n and its range is $\mathcal{P}(P_0)$, whereas the rightmost η^- above has domain $\mathcal{P}(P_n)$.

Lemma 5.2. Assume $0 < j \leq i < \omega$. Suppose $p_j \in P_j$. Then $\eta_i^-(\varphi_{ij}(p_j)) = \eta_j^-(p_j)$.

Proof. We may assume $j < i$. First assume $i = j + 1$. Then

$$\begin{aligned} \eta_i^-(\varphi_{ij}(p_j)) &= \eta_{j+1}^-(\iota_j^+(p_j)) = \eta^-(\eta^-(\dots \eta^-(\{\downarrow_j p_j\}) \dots)) \\ &\quad \text{ $j+1$ times} \\ &= \eta^-(\eta^-(\dots \eta^-(\downarrow_j p_j) \dots)) \\ &\quad \text{ j times} \\ &= \eta^-(\eta^-(\dots \eta^-(\{p_j\}) \dots)) \quad \text{by Lemma 3.4} \\ &\quad \text{ j times} \\ &= \eta_j^-(p_j). \end{aligned}$$

Now assume $\eta_i^-(\varphi_{ij}(p_j)) = \eta_j^-(p_j)$. Then

$$\begin{aligned} \eta_{i+1}^-(\varphi_{i+1,j}(p_j)) &= \eta_{i+1}^-(\varphi_{i+1,i}(\varphi_{ij}(p_j))) = \eta_{i+1}^-(\iota_i^+(\varphi_{ij}(p_j))) \\ &= \eta_i^-(\varphi_{ij}(p_j)) \quad \text{by the above} \\ &= \eta_j^-(p_j). \end{aligned}$$

□

Definition 5.3. For $p_\omega \in P_\omega$, define $\eta_\omega^-(p_\omega)$ as $\eta_j^-(p_j)$ for any $p_j \in P_j$ such that $0 < j < \omega$ and $\varphi_j(p_j) = p_\omega$.

Lemma 5.4. *The map η_ω^- is well-defined.*

Proof. Assume $0 < j \leq i < \omega$. Let $p_\omega \in P_\omega$. Let $p_i \in P_i$ and $p'_j \in P_j$ be such that $\varphi_i(p_i) = p_\omega = \varphi_j(p'_j) = \varphi_i[\varphi_{ij}(p'_j)]$. Since φ_i is one-to-one, $p_i = \varphi_{ij}(p'_j)$. From Lemma 5.2, $\eta_i^-(p_i) = \eta_j^-(p'_j)$. $\square$

Lemma 5.5. *Assume $0 < i < \omega$ and let $p_i \in P_i$. Then $\eta_i^-(p_i) \in \mathcal{I}(P_0)$. Hence, for all $p_\omega \in P_\omega$, $\eta_\omega^-(p_\omega) \in \mathcal{I}(P_0)$.*

Proof. Use Lemmas 3.4, 3.5, and 3.7. $\square$

Lemma 5.6. *Let $x_\omega, y_\omega \in P_\omega$. Assume $x_\omega \leq_\omega y_\omega$. Then $\eta_\omega^-(x_\omega) \subseteq \eta_\omega^-(y_\omega)$.*

Proof. By Lemma 5.4, we may assume there exist nonzero $i < \omega$ and $x_i, y_i \in P_i$ such that $\varphi_i(x_i) = x_\omega$, $\varphi_i(y_i) = y_\omega$, $\eta_\omega^-(x_\omega) = \eta_i^-(x_i)$ and $\eta_\omega^-(y_\omega) = \eta_i^-(y_i)$. As φ_i is an order-embedding, $x_i \leq_i y_i$. As

$$\eta_\omega^-(x_\omega) = \eta^-(\underbrace{\eta^-(\dots \eta^-(\{x_i\}) \dots)}_{i \text{ times}}) = \eta^-(\underbrace{\eta^-(\dots \eta^-(\downarrow_i x_i) \dots)}_{i \text{ times}})$$

by Lemma 3.4 and

$$\eta_\omega^-(y_\omega) = \eta^-(\underbrace{\eta^-(\dots \eta^-(\downarrow_i y_i) \dots)}_{i \text{ times}}),$$

the result follows from $\downarrow_i x_i \subseteq \downarrow_i y_i$ and Remark 3.2 (2). $\square$

Corollary 5.7. *Let D be a directed subset of P_ω . Then $\bigcup_{d_\omega \in D} \eta_\omega^-(d_\omega) \in \mathcal{I}(P_0)$.*

Proof. Use Lemma 5.5, Lemma 5.6, and Lemma 3.6. $\square$

Lemma 5.8. *Assume $p_\omega \in P_\omega$. Then p_ω is an upper bound of*

$$\{\varphi_0(x_0) : x_0 \in \eta_\omega^-(p_\omega)\}.$$

Indeed, if $0 < i < \omega$ and $p_i \in P_i$ and $\varphi_i(p_i) = p_\omega$, then $\varphi_{i0}(x_0) \leq_i p_i$ for all $x_0 \in \eta_\omega^-(p_\omega)$. If $c_\omega \in P_\omega$ is also an upper bound of

$$\{\varphi_0(x_0) : x_0 \in \eta_\omega^-(p_\omega)\}$$

and $i \leq j < \omega$ and $c_j \in P_j$ and $\varphi_j(c_j) = c_\omega$ (there is such a j), then $\varphi_{j0}(x_0) \leq_j c_j$ for all $x_0 \in \eta_\omega^-(p_\omega)$.

P r o o f. Let i be such that $0 < i < \omega$ and there is $p_i \in P_i$ with $\varphi_i(p_i) = p_\omega$ and $\eta_\omega^-(p_\omega) = \eta_i^-(p_i)$. Take any

$$x_0 \in \eta_\omega^-(p_\omega) = \eta^-(\underbrace{\eta^-(\dots \eta^-(\{p_i\}) \dots)}_{i \text{ times}})) = \eta^-(\underbrace{\eta^-(\dots \eta^-(\downarrow_i p_i) \dots)}_{i \text{ times}})) \text{ (by Lemma 3.4).}$$

Thus, $\varphi_{10}(x_0) = \downarrow_0 x_0 \in \eta^-(\underbrace{\eta^-(\dots \eta^-(\downarrow_i p_i) \dots)}_{i-1 \text{ times}}))$ (by Lemmas 3.7 and 3.8) and $\varphi_{20}(x_0) = \downarrow_1 (\downarrow_0 x_0) \in \eta^-(\underbrace{\eta^-(\dots \eta^-(\downarrow_i p_i) \dots)}_{i-2 \text{ times}}))$, etc., till we get $\varphi_{i0}(x_0) \in \downarrow_i p_i$, that is, $\varphi_{i0}(x_0) \leq_i p_i$. Applying φ_i , we get

$$\varphi_0(x_0) \leq_\omega p_\omega.$$

Hence, p_ω is an upper bound in P_ω of

$$\{\varphi_0(x_0) : x_0 \in \eta_\omega^-(p_\omega)\}.$$

Now assume $c_\omega \in P_\omega$ is an upper bound in P_ω of

$$\{\varphi_0(x_0) : x_0 \in \eta_\omega^-(p_\omega)\}.$$

By Proposition 2.1, there exist $j < \omega$ and $c_j \in P_j$ such that $\varphi_j(c_j) = c_\omega$. If $j < i$, then $\varphi_i(\varphi_{ij}(c_j)) = \varphi_j(c_j) = c_\omega$. Thus, we may assume $i \leq j$.

For $x_0 \in \eta_\omega^-(p_\omega)$, let $x_j = \varphi_{j0}(x_0)$, so $\varphi_j(x_j) = \varphi_0(x_0) \leq_\omega c_\omega = \varphi_j(c_j)$. Since φ_j is an order-embedding, $x_j \leq_j c_j$. $\square$

Let $\circledast$ be the following statement, related to a specific positive integer k :

$\circledast$ Assume P_0 does not contain $\omega \times \mathbf{2}$. Assume that whenever i and m are integers such that $0 < i < k$ and $i \leq m < \omega$ and whenever $p_i \in P_i$, then

$$\varphi_{mi}(p_i) = \bigvee_m \{\varphi_{m0}(x_0) : x_0 \in \eta_i^-(p_i)\}$$

and for all $q_m \in P_m$ such that $q_m <_m \varphi_{mi}(p_i)$, there exists $x_0 \in \eta_i^-(p_i)$ with $q_m <_m \varphi_{m0}(x_0)$.

Lemma 5.9. *Let $\circledast$ hold. Let $p_k \in P_k$. Then*

$$p_k = \bigvee_k \{\varphi_{k0}(x_0) : x_0 \in \eta_k^-(p_k)\}.$$

Further, $\{\varphi_{k0}(x_0) : x_0 \in \eta_k^-(p_k)\}$ is directed. If $q_k \in P_k$ and $q_k <_k p_k$, then there exists $x_0 \in \eta_k^-(p_k)$ such that $q_k <_k \varphi_{k0}(x_0)$.

Proof. By Definition 5.1 and Lemma 3.10,

$$\begin{aligned}
\eta_k^-(p_k) &= \eta^-(\underbrace{\eta^-(\dots \eta^-(\{p_k\}) \dots)}_{k \text{ times}}) \\
&= \eta^-\left(\eta^-\left(\dots \eta^-\left(\bigcup_{k-1 \text{ times}} \{\downarrow_{k-1} y_{k-1} : y_{k-1} \in p_k\}\right) \dots\right)\right) \\
&= \bigcup_{y_{k-1} \in p_k} \eta^-(\underbrace{\eta^-(\dots \eta^-(\{y_{k-1}\}) \dots)}_{k-1 \text{ times}}) \quad (\text{by Lemmas 3.3 and 3.4}) \\
&= \begin{cases} \bigcup_{y_{k-1} \in p_k} \eta_{k-1}^-(y_{k-1}) & \text{if } k > 1 \\ p_k & \text{if } k = 1. \end{cases}
\end{aligned}$$

By Lemma 3.10, $p_k = \bigvee_k \{\varphi_{k,k-1}(y_{k-1}) : y_{k-1} \in p_k\}$, which settles the $k = 1$ case of the first claim. By hypothesis, if $k > 1$, for all $y_{k-1} \in p_k$, $\varphi_{k,k-1}(y_{k-1}) = \bigvee_k \{\varphi_{k0}(x_0) : x_0 \in \eta_{k-1}^-(y_{k-1})\}$. Hence, if $k > 1$, $p_k = \bigvee_{y_{k-1} \in p_k} \bigvee_{x_0 \in \eta_{k-1}^-(y_{k-1})} \varphi_{k0}(x_0) = \bigvee_{x_0 \in \eta_k^-(p_k)} \varphi_{k0}(x_0)$ by the above. By Lemma 5.5, $\{\varphi_{k0}(x_0) : x_0 \in \eta_k^-(p_k)\}$ is directed.

Now let $q_k \in P_k$ be such that $q_k <_k p_k$. By Lemma 3.10,

$$q_k \subsetneq \bigcup \{\downarrow_{k-1} y_{k-1} : y_{k-1} \in p_k\},$$

which is a directed down-set (by Lemma 3.6), as is q_k . By Lemma 4.1, Proposition 4.2 and induction, there exists $y'_{k-1} \in p_k$ such that $q_k \subseteq_{\downarrow_{k-1}}^{\circ} y'_{k-1}$, so $q_k \subsetneq \downarrow_{k-1} y'_{k-1}$, i.e., $q_k <_k \varphi_{k,k-1}(y'_{k-1})$. This is the last statement if $k = 1$. If $k > 1$, then by $\otimes$, there exists $x'_0 \in \eta_{k-1}^-(y'_{k-1})$ such that $q_k <_k \varphi_{k0}(x'_0)$. By the above, $x'_0 \in \eta_k^-(p_k)$. $\square$

Lemma 5.10. *Let $\otimes$ hold. Assume $p_k \in P_k$. Assume $k < n < \omega$. Assume that, for $k < i < n$,*

$$\varphi_{ik}(p_k) = \bigvee_i \{\varphi_{i0}(x_0) : x_0 \in \eta_k^-(p_k)\}$$

and also that, for all $q_i \in P_i$ such that $q_i <_i \varphi_{ik}(p_k)$, there exists $x_0 \in \eta_k^-(p_k)$ with $q_i <_i \varphi_{i0}(x_0)$. Then for all $q_n \in P_n$ such that $q_n <_n \varphi_{nk}(p_k)$, there exists $x_0 \in \eta_k^-(p_k)$ such that $q_n <_n \varphi_{n0}(x_0)$. Further, $\varphi_{nk}(p_k) = \bigvee_n \{\varphi_{n0}(x_0) : x_0 \in \eta_k^-(p_k)\}$.

Proof. By Proposition 4.2 and induction, P_{n-1} does not contain $\omega \times 2$.

First assume $q_n = \iota_{n-1}^+(q_{n-1})$, where $q_{n-1} \in P_{n-1}$. Since ι_{n-1}^+ is an order-embedding and $\varphi_{nk}(p_k) = \iota_{n-1}^+(\varphi_{n-1,k}(p_k))$, $q_{n-1} <_{n-1} \varphi_{n-1,k}(p_k)$, so by hypothesis and Lemma 5.9, there exists $x_0 \in \eta_k^-(p_k)$ with $q_{n-1} <_{n-1} \varphi_{n-1,0}(x_0)$, and hence $q_n <_n \varphi_{n,n-1}(\varphi_{n-1,0}(x_0)) = \varphi_{n0}(x_0)$.

Now suppose $q_n \in \mathcal{I}(P_{n-1})$ such that $\bigvee_{n-1} q_n$ exists and there exists $a_{n-1} \in P_{n-1}$ such that $a_{n-1} <_{n-1} \bigvee_{n-1} q_n$ but $a_{n-1} \not\prec_{n-1} b_{n-1}$ for all $b_{n-1} \in q_n$. For every $b_{n-1} \in q_n$, we have $\downarrow_{n-1} b_{n-1} \subseteq q_n$, so $\iota_{n-1}^+(b_{n-1}) \leq_n q_n < \varphi_{nk}(p_k)$; from what we just showed, there exists $x_0 \in \eta_k^-(p_k)$ such that $\iota_{n-1}^+(b_{n-1}) <_n \varphi_{n0}(x_0) = \iota_{n-1}^+(\varphi_{n-1,0}(x_0))$, so $b_{n-1} <_{n-1} \varphi_{n-1,0}(x_0)$. Hence $q_n \subseteq \bigcup_{x_0 \in \eta_k^-(p_k)} \downarrow_{n-1} \varphi_{n-1,0}(x_0)$.

By Lemma 3.6 and Lemma 5.5, the latter is in $\mathcal{I}(P_{n-1})$.

Case 1. $q_n \subsetneq \bigcup_{x_0 \in \eta_k^-(p_k)} \downarrow_{n-1} \varphi_{n-1,0}(x_0)$. By Lemma 4.1, there exists $x_0 \in \eta_k^-(p_k)$

such that $q_n \subseteq \bigcup_{x_0 \in \eta_k^-(p_k)} \downarrow_{n-1} \varphi_{n-1,0}(x_0)$. Hence $q_n <_n \varphi_{n0}(x_0)$.

Case 2. $q_n = \bigcup_{x_0 \in \eta_k^-(p_k)} \downarrow_{n-1} \varphi_{n-1,0}(x_0)$. Then

$$a_{n-1} <_{n-1} \bigvee_{n-1} q_n = \bigvee_{n-1} \{\varphi_{n-1,0}(x_0) : x_0 \in \eta_k^-(p_k)\} = \varphi_{n-1,k}(p_k)$$

(by Lemma 5.9 and the hypothesis), so there exists $x'_0 \in \eta_k^-(p_k)$ such that $a_{n-1} <_{n-1} \varphi_{n-1,0}(x'_0) \in q_n$ (by Lemma 5.9 and the hypothesis), a contradiction.

Note that a version of the hypothesis holds if we let $i = k$, by Lemma 5.9, and a version of the second part of the hypothesis holds for $i = n$ by the above and Lemma 5.9. We see that $\varphi_{nk}(p_k)$ is an upper bound in P_n of

$$\{\varphi_{n0}(x_0) : x_0 \in \eta_k^-(p_k)\},$$

because $\varphi_{n-1,k}(p_k)$ is an upper bound of $\{\varphi_{n-1,0}(x_0) : x_0 \in \eta_k^-(p_k)\}$ in P_{n-1} .

Let $r_n \in P_n$ be an upper bound in P_n of $\{\varphi_{n0}(x_0) : x_0 \in \eta_k^-(p_k)\}$. We show $r_n \geq_n \varphi_{nk}(p_k)$, completing the proof that

$$\varphi_{nk}(p_k) = \bigvee_n \{\varphi_{n0}(x_0) : x_0 \in \eta_k^-(p_k)\}.$$

Case I. $r_n = \iota_{n-1}^+(r_{n-1})$ for some $r_{n-1} \in P_{n-1}$. By the fact ι_{n-1}^+ is an order-embedding, r_{n-1} is an upper bound in P_{n-1} of $\{\varphi_{n-1,0}(x_0) : x_0 \in \eta_k^-(p_k)\}$. By the hypothesis $\varphi_{n-1,k}(p_k) \leq_{n-1} r_{n-1}$. Applying $\varphi_{n,n-1}$, we get $\varphi_{nk}(p_k) \leq_n r_n$.

Case II. $r_n \in \mathcal{I}(P_{n-1})$ is such that $\bigvee_{n-1} r_n$ exists and there exists $a_{n-1} \in P_{n-1}$ with $a_{n-1} <_{n-1} \bigvee_{n-1} r_n$ but, for all $r_{n-1} \in r_n$, $a_{n-1} \not\prec_{n-1} r_{n-1}$. For $x_0 \in \eta_k^-(p_k)$, $\iota_{n-1}^+(\varphi_{n-1,0}(x_0)) \leq_n r_n$, that is, $\iota_{n-1}^+(\varphi_{n-1,0}(x_0)) \subseteq r_n$, so $\varphi_{n-1,0}(x_0) \in r_n$. Hence $\bigcup_{x_0 \in \eta_k^-(p_k)} \downarrow_{n-1} \varphi_{n-1,0}(x_0) \in \mathcal{I}(P_{n-1})$ (by Lemmas 3.6 and 5.5) and it is a subset of r_n .

Case IIa. $\bigcup_{x_0 \in \eta_k^-(p_k)} \downarrow_{n-1} \varphi_{n-1,0}(x_0) \subsetneq r_n$. By Lemma 4.1, there exists $r'_{n-1} \in r_n$ such that

$$\bigcup_{x_0 \in \eta_k^-(p_k)} \downarrow_{n-1} \varphi_{n-1,0}(x_0) \subseteq \overset{\circ}{\downarrow}_{n-1} r'_{n-1}.$$

Thus, $\varphi_{n-1,0}(x_0) <_{n-1} r'_{n-1}$ for all $x_0 \in \eta_k^-(p_k)$. By the hypothesis and Lemma 5.9, $\varphi_{n-1,k}(p_k) \leq_{n-1} r'_{n-1}$, so $\varphi_{nk}(p_k) \leq_n \downarrow_{n-1} r'_{n-1} \subseteq r_n$.

Case IIb. $\bigcup_{x_0 \in \eta_k^-(p_k)} \downarrow_{n-1} \varphi_{n-1,0}(x_0) = r_n$. Then

$$\bigvee_{n-1} r_n = \bigvee_{n-1} \{\varphi_{n-1,0}(x_0) : x_0 \in \eta_k^-(p_k)\} = \varphi_{n-1,k}(p_k),$$

so, by hypothesis and Lemma 5.9, there exists $x'_0 \in \eta_k^-(p_k)$ such that $a_{n-1} <_{n-1} \varphi_{n-1,0}(x'_0) \in r_n$, a contradiction. $\square$

Lemma 5.11. Assume P_0 does not contain $\omega \times 2$. Then for $0 < k < \omega$, for all $p_k \in P_k$, and for every integer n such that $k \leq n < \omega$,

$$\varphi_{nk}(p_k) = \bigvee_n \{\varphi_{n0}(x_0) : x_0 \in \eta_k^-(p_k)\}$$

and for all $q_n \in P_n$ with $q_n <_n \varphi_{nk}(p_k)$, there exists $x_0 \in \eta_k^-(p_k)$ such that $q_n <_n \varphi_{n0}(x_0)$.

Proof. One proof by induction involves two cases: $k = 1$ and $k > 1$. In each case, do $k = n$, $k = n + 1$, and $k > n + 1$ separately. $\square$

Theorem 5.12. Assume P_0 does not contain $\omega \times 2$. Then for all $p_\omega \in P_\omega$,

$$p_\omega = \bigvee_\omega \{\varphi_0(x_0) : x_0 \in \eta_\omega^-(p_\omega)\}.$$

Proof. The c_ω in the statement of Lemma 5.8 is such that $c_j \geq_j \varphi_{ji}(p_i)$ by Lemma 5.11. Applying φ_j , we get $c_\omega \geq_\omega p_\omega$. $\square$

Corollary 5.13. Assume P_0 does not contain $\omega \times 2$. Let $p_\omega, p'_\omega \in P_\omega$. Then $p_\omega \leq_\omega p'_\omega$ if and only if $\eta_\omega^-(p_\omega) \subseteq \eta_\omega^-(p'_\omega)$.

Proof. One direction is Lemma 5.6. The other follows from Theorem 5.12. $\square$

The following is *not* needed in the sequel.

Observation 5.14. Assume P_0 does not contain $\omega \times 2$. Let $b, a_0, a_1, a_2, \dots \in P_\omega$ be such that $a_0 <_\omega a_1 <_\omega a_2 <_\omega \dots$ and $\eta_\omega^-(b) \subseteq \bigcup_{i=0}^\infty \eta_\omega^-(a_i)$ and $a_i \not\leq_\omega b$ for $i = 0, 1, 2, \dots$. Then for some $i \in \{0, 1, 2, \dots\}$, $b \leq_\omega a_i$.

P r o o f. Assume not, for a contradiction. Pick any $b'_0 \in \eta_{\omega}^-(b)$, which is nonempty by Lemma 5.5. (Directed sets are nonempty.) Let $k_0 \in \mathbb{N}_0$ be such that $b'_0 \in \eta_{\omega}^-(a_{k_0})$. We can pick $\tilde{a}'_0 \in \eta_{\omega}^-(a_{k_0}) \setminus \eta_{\omega}^-(b)$ by Corollary 5.13. By Lemma 5.5, there exists $a'_0 \in \eta_{\omega}^-(a_{k_0})$ such that $\tilde{a}'_0, b'_0 \leq_0 a'_0$. Since $a'_0 \notin \eta_{\omega}^-(b)$ but $b'_0 \in \eta_{\omega}^-(b)$, we have $b'_0 <_0 a'_0$.

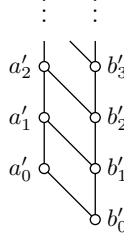


Figure 6.

Let $n \in \mathbb{N}$ (i.e., $n > 0$). Assume there exist $k_0, k_1, \dots, k_{n-1} \in \mathbb{N}_0$ and $a'_0, a'_1, \dots, a'_{n-1} \in P_0$ such that $k_0 < k_1 < \dots < k_{n-1}$ and $a'_0 <_0 a'_1 <_0 a'_2 <_0 \dots <_0 a'_{n-1}$ and $a'_i \in \eta_{\omega}^-(a_{k_i}) \setminus \eta_{\omega}^-(b)$ ($i = 0, 1, \dots, n-1$). Also assume there exist $b'_0, b'_1, \dots, b'_{n-1} \in \eta_{\omega}^-(b)$ such that $b'_0 <_0 \dots <_0 b'_{n-1}$ and $b'_i <_0 a'_i$ ($i = 0, 1, \dots, n-1$) and $b'_{i+1} \not\leq_0 a'_i$ ($i = 0, 1, \dots, n-2$).

Since $b \not\leq_{\omega} a_{k_{n-1}}$, by Corollary 5.13 there exists $\tilde{b}'_n \in \eta_{\omega}^-(b) \setminus \eta_{\omega}^-(a_{k_{n-1}})$; by Lemma 5.5, $\eta_{\omega}^-(b)$ is directed, so there exists $b'_n \in \eta_{\omega}^-(b)$ such that $b'_{n-1}, \tilde{b}'_n \leq_0 b'_n$. As $b'_{n-1} \in \eta_{\omega}^-(a_{k_{n-1}})$ (a down-set by Lemma 5.5), we have $b'_{n-1} <_0 b'_n$. Because $\eta_{\omega}^-(a_{k_{n-1}})$ is a down-set by Lemma 5.5, $b'_n \not\leq_0 a'_{n-1}$.

By the assumption, there exists $k_n \in \mathbb{N}_0$ such that $b'_n \in \eta_{\omega}^-(a_{k_n})$. By Corollary 5.13, we may assume $k_n > k_{n-1}$. By Corollary 5.13, $a'_{n-1} \in \eta_{\omega}^-(a_{k_n})$. By Lemma 5.5, $\eta_{\omega}^-(a_{k_n})$ is directed, so there exists $a'_n \in \eta_{\omega}^-(a_{k_n})$ such that $a'_{n-1}, b'_n \leq_0 a'_n$. Since $b'_n \not\leq_0 a'_{n-1}$, then $a'_{n-1} <_0 a'_n$. Since $a'_{n-1} \notin \eta_{\omega}^-(b)$, we have $b'_n <_0 a'_n$.

Hence, we have $a'_0, a'_1, a'_2, \dots, b'_0, b'_1, b'_2, \dots \in P_0$ such that $a'_0 <_0 a'_1 <_0 \dots$ and $b'_0 <_0 b'_1 <_0 \dots$ and $b'_i <_0 a'_i$ and $b'_{i+1} \not\leq_0 a'_i$ and $a'_i \notin \eta_{\omega}^-(b)$ and $b'_i \in \eta_{\omega}^-(b)$ ($i = 0, 1, 2, \dots$). By Lemma 5.5, $\eta_{\omega}^-(b)$ is a down-set, so we have $a'_i \not\leq_0 b'_j$ ($i, j = 0, 1, 2, \dots$). That is, $\omega \times \mathbf{2}$ embeds in P_0 by Observation 1.3. $\square$

Proposition 5.15. *If P_0 does not contain $\omega \times \mathbf{2}$, then P_{ω} does not contain $\omega \times \mathbf{2}$.*

P r o o f. Assume P_{ω} contains $\omega \times \mathbf{2}$, for a contradiction. Use Corollary 5.13, Lemma 5.5, and Proposition 4.2. $\square$

Proposition 5.16. *If P_0 does not contain $\omega \times \mathbf{2}$, then P_{ω} is sup-pre-regular.*

Proof. Let D be a directed subset of P_ω such that $b := \bigvee^\omega D$ exists. Assume $a \in P_\omega$ is such that $a <_\omega b$. Assume for a contradiction that $a \not\leq_\omega d$ for all $d \in D$. Hence, $b \notin D$ and D has no maximal element. By Lemma 2.4, Proposition 4.2 and induction, we may assume there exist $a_0, b_0 \in P_0$ such that $\varphi_0(a_0) = a$ and $\varphi_0(b_0) = b$. Let $a_i := \varphi_{i0}(a_0)$ and $b_i := \varphi_{i0}(b_0)$ for $i < \omega$.

Claim 1. The element b_0 is an upper bound of $\bigcup_{d \in D} \eta_\omega^-(d)$.

Proof of Claim 1. Take any $d \in D$. There exist a nonzero $i < \omega$ and $d_i \in P_i$ such that $\varphi_i(d_i) = d$ and $\eta_\omega^-(d) = \eta_i^-(d_i)$. Now $\varphi_i(b_i) = b$, so since φ_i is an order-embedding, $b_i >_i d_i$. Hence $b_i \supseteq d_i$ and $\downarrow_i b_i \supseteq \{d_i\}$, so by Remark 3.2 (2),

$$\begin{aligned} \eta_i^-(b_i) &= \eta^-(\underbrace{\eta^-(\dots \eta^-(\{b_i\}) \dots)}_{i \text{ times}}) = \eta^-(\underbrace{\eta^-(\dots \eta^-(\downarrow_i b_i) \dots)}_{i \text{ times}}) \quad (\text{by Lemma 3.4}) \\ &\supseteq \eta^-(\underbrace{\eta^-(\dots \eta^-(\{d_i\}) \dots)}_{i \text{ times}}) = \eta_i^-(d_i) = \eta_\omega^-(d). \end{aligned}$$

By Lemma 5.2, $\eta_i^-(b_i) = \eta_1^-(\downarrow_0 b_0) = \eta^-(\{\downarrow_0 b_0\}) = \bigcup \{\downarrow_0 b_0\} = \downarrow_0 b_0$. Thus, b_0 is an upper bound of $\eta_\omega^-(d)$. $\square$

Claim 2. Assume $c_0 \in P_0$ is an upper bound of $\bigcup_{d \in D} \eta_\omega^-(d)$. Then $b_0 \leq_0 c_0$.

Proof of Claim 2. For $0 < i < \omega$, let $c_i := \varphi_{i0}(c_0)$ and let $c := \varphi_0(c_0)$. Choose $d \in D$. As before, there exist i such that $0 < i < \omega$ and $d_i \in P_i$ such that $\varphi_i(d_i) = d$ and $\eta_\omega^-(d) = \eta_i^-(d_i) = \eta^-(\underbrace{\eta^-(\dots \eta^-(\{d_i\}) \dots)}_{i \text{ times}})$. By Lemma 3.9, c_i is an upper bound of $\{d_i\}$ in P_i . Hence, $c_i \geq_i d_i$, so, applying φ_i , $c \geq_\omega d$. Therefore $c \geq_\omega b$. Since φ_0 is an order-embedding, $c_0 \geq_0 b_0$. $\square$

By Claims 1 and 2, $b_0 = \bigvee_0 \left[\bigcup_{d \in D} \eta_\omega^-(d) \right]$.

Case 1. $a_0 \leq_0 x_0$ for some $x_0 \in \bigcup_{d \in D} \eta_\omega^-(d)$. Say $x_0 \in \eta_\omega^-(d)$, where $d \in D$. As before, there exist i such that $0 < i < \omega$ and $d_i \in P_i$ such that $\varphi_i(d_i) = d$ and $\eta_\omega^-(d) = \eta_i^-(d_i)$. By Lemma 5.5, a_0 belongs to the down-set $\eta_\omega^-(d)$. By Lemmas 3.4, 3.7 and 3.8, $a_i = \varphi_{i0}(a_0) \in \downarrow_i d_i$, so $a_i \leq_i d_i$ and $a \leq_\omega d$. Since D has no maximal element, there exists $d' \in D$ such that $a <_\omega d'$.

Case 2. $a_0 \not\leq_0 x_0$ for all $x_0 \in \bigcup_{d \in D} \eta_\omega^-(d)$. By Corollary 5.7, $\bigcup_{d \in D} \eta_\omega^-(d) \in P_1$.

Claim 3. For all $d \in D$, $d \leq_\omega \varphi_1(\bigcup_{\bar{d} \in D} \eta_\omega^-(\bar{d}))$.

Proof of Claim 3. Let $d \in D$. As before, let i be such that $0 < i < \omega$ and let $d_i \in P_i$ be such that $\varphi_i(d_i) = d$ and $\eta_\omega^-(d) = \eta_i^-(d_i)$. By Lemma 5.4, we may assume $i \geq 2$. Assume for a contradiction that

$$d \not\leq_\omega \varphi_1\left(\bigcup_{\bar{d} \in D} \eta_\omega^-(\bar{d})\right).$$

Since φ_i is an order-embedding,

$$d_i \not\leq_i \varphi_{i1} \left(\bigcup_{\bar{d} \in D} \eta_{\omega}^{-}(\bar{d}) \right).$$

By applying Lemma 3.8 (2) $i - 1$ times (initially with $B = \{d_i\}$), there exists

$$\tilde{a}_1 \in \eta^{-}(\eta^{-}(\dots \eta^{-}(\{d_i\}) \dots))_{i-1 \text{ times}}$$

such that $\tilde{a}_1 \not\leq_1 \bigcup_{\bar{d} \in D} \eta_{\omega}^{-}(\bar{d})$. Thus, there exists $\tilde{p}_0 \in \tilde{a}_1$ such that $\tilde{p}_0 \notin \bigcup_{\bar{d} \in D} \eta_{\omega}^{-}(\bar{d})$. But

$$\tilde{p}_0 \in \bigcup_{i-1 \text{ times}} \eta^{-}(\eta^{-}(\dots \eta^{-}(\{d_i\}) \dots)) = \eta^{-}(\eta^{-}(\dots \eta^{-}(\{d_i\}) \dots)) = \eta_{\omega}^{-}(d),$$

a contradiction. □

By Claim 3, $b \leq_{\omega} \varphi_1 \left(\bigcup_{d \in D} \eta_{\omega}^{-}(d) \right)$. Thus $a <_{\omega} \varphi_1 \left(\bigcup_{d \in D} \eta_{\omega}^{-}(d) \right)$. By Proposition 2.1 (1) there exists h such that $0 < h < \omega$ and $a_h \leq_h \varphi_{h1} \left(\bigcup_{d \in D} \eta_{\omega}^{-}(d) \right)$ and the inequality is strict. Since φ_{h1} is an order-embedding, $a_1 <_1 \bigcup_{d \in D} \eta_{\omega}^{-}(d)$. Thus $a_0 \in \bigcup_{d \in D} \eta_{\omega}^{-}(d)$, violating the assumption behind Case 2. □

Proposition 5.17. *If P_0 does not contain $\omega \times \mathbf{2}$ and does not contain $\omega^{\partial} \times \mathbf{2}$, then P_{ω} does not contain $\omega^{\partial} \times \mathbf{2}$.*

Proof. Use Lemma 5.5, Corollary 5.13, and Lemma 4.3. □

Proposition 5.18. *Assume P_0 does not contain $\omega \times \mathbf{2}$.*

- (1) *If every nonempty chain of P_0 has an infimum, then every nonempty chain of P_{ω} has an infimum.*
- (2) *If P_0 is inf-regular, then P_{ω} is inf-regular.*

Proof. (1) Let C be a nonempty chain in P_{ω} with no least element. By Lemma 5.5 and Corollary 5.13, $\{\eta_{\omega}^{-}(c) : c \in C\}$ satisfies the conditions of Lemma 4.5. Hence, there exists $e_0 \in P_0$ such that $e_0 \in \bigcap_{c \in C} \eta_{\omega}^{-}(c)$. By Theorem 5.12, $\varphi_0(e_0) \leq_{\omega} c$ for all $c \in C$.

Further, if $f \in P_{\omega}$ is a lower bound of C in P_{ω} , then by Corollary 5.13, $\eta_{\omega}^{-}(f) \subseteq \eta_{\omega}^{-}(c)$ for all $c \in C$. By the conditions of e_0 given by Lemma 4.5, $\eta_{\omega}^{-}(f) \subseteq \downarrow_0 e_0$. Thus $x_0 \leq_0 e_0$ for all $x_0 \in \eta_{\omega}^{-}(f)$, so $\varphi_0(x_0) \leq_{\omega} \varphi_0(e_0)$ for all $x_0 \in \eta_{\omega}^{-}(f)$, and hence by Theorem 5.12, $f \leq_{\omega} \varphi_0(e_0)$. Thus $\varphi_0(e_0) = \bigwedge_{\omega} C$.

(2) Let $a \in P_\omega$ be such that $a >_\omega \varphi_0(e_0) = \varphi_1(\varphi_{10}(e_0))$. By Lemma 5.4, $\eta_\omega^-(\varphi_0(e_0)) = \eta_1^-(\varphi_{10}(e_0)) = \eta^-(\{\downarrow_0 e_0\}) = \downarrow_0 e_0$. By Corollary 5.13 and Lemma 5.5, $\eta_\omega^-(a) \not\supseteq \downarrow_0 e_0$, so by Lemma 4.5, there exists $c \in C$ such that $\eta_\omega^-(a) \not\supseteq \eta_\omega^-(c)$. By Corollary 5.13, $a >_\omega c$. $\square$

Corollary 5.19. *Let P_0 be a poset not containing $\omega \times 2$. Let R be a sup-regular poset. Let $f_0: P_0 \rightarrow R$ be an order-embedding. Then there is an order-embedding $f_\omega: P_\omega \rightarrow R$ such that $f_\omega \circ \varphi_0 = f_0$.*

Proof. Use Lemma 3.11 and Lemma 2.3 and Lemma 2.2(1). $\square$

6. “CHAIN-COMPLETING” A SUP-PRE-REGULAR POSET

The articles [11] and [15] also discuss posets that could be called “chain-completions.” In fact, the construction below may be equivalent to a special case of a version of [11], Definition 6, although we have not checked this.

Definition 6.1. Let P be a sup-pre-regular poset. Let $\tilde{\mathcal{D}} = \{D \in \mathcal{I}(P) : \bigvee_P D \text{ does not exist}\}$. Let $P^\# = \{\downarrow p : p \in P\} \cup \tilde{\mathcal{D}}$. Define a binary relation $\sqsubseteq$ on $P^\#$ as follows. (Let $p, p' \in P$ and let $D, D' \in \tilde{\mathcal{D}}$.)

- $\downarrow p \sqsubseteq \downarrow p'$ if $p \leq_P p'$,
- $\downarrow p \sqsubseteq D'$ if there exists a directed subset $E \subseteq D'$ such that $p \leq_P \bigvee E$,
- $D \sqsubseteq \downarrow p'$ if $D \subseteq \downarrow p'$,
- $D \sqsubseteq D'$ if $D \subseteq D'$.

Let $\iota_P^\# : P \rightarrow P^\#$ be the map $p \mapsto \downarrow p$ ($p \in P$).

Remark 6.2. Note that for all $p \in P$, $\downarrow p \notin \tilde{\mathcal{D}}$.

In the following theorem, if W is a poset and $w \in W$, we write $\downarrow_W w$ to mean $\{v \in W : v \leq w\}$; we use the “ W ” subscript merely to remind the reader what poset we are referring to, since the theorem deals with several posets.

Theorem 6.3. *Let P be a sup-pre-regular poset. Then $(P^\#, \sqsubseteq)$ is a poset and $\iota_P^\#$ is an order-embedding.*

Let $\mathcal{E}$ be a directed subset of $P^\#$. Then $\bigvee_{P^\#} \mathcal{E}$ exists. Indeed:

- (1) *If for each $e \in \mathcal{E}$, there exists $q_e \in P$ such that $e \sqsubseteq \downarrow q_e$ and $\downarrow q_e \in \mathcal{E}$, let $F := \{q \in P : \downarrow q \in \mathcal{E}\}$.*
 - (1a) *The set F is directed.*
 - (1b) *If $\bigvee_P F$ exists, then $\downarrow_P (\bigvee_P F) = \bigvee_{P^\#} \mathcal{E}$.*
 - (1c) *If $\bigvee_P F$ does not exist, then $\downarrow F = \bigvee_{P^\#} \mathcal{E}$.*

(2) If there exists $\tilde{e} \in \mathcal{E}$ such that $\tilde{e} \not\sqsubseteq \downarrow q$ for all $q \in P$ with $\downarrow q \in \mathcal{E}$, let $G := \bigcup \{D \in \mathcal{E} : \tilde{e} \sqsubseteq D\}$.

(2a) We have $\tilde{e} \in \tilde{\mathcal{D}}$ and $G \in \mathcal{I}(P)$.

(2b) If $\bigvee_P G$ exists, then $\downarrow_P (\bigvee_P G) = \bigvee_{P^\#} \mathcal{E}$.

(2c) If $\bigvee_P G$ does not exist, $G = \bigvee_{P^\#} \mathcal{E}$.

Proof. To avoid having to repeat this, assume $p, p', p'' \in P$ and $D, D', D'' \in \tilde{\mathcal{D}}$. We show $\sqsubseteq$ is a partial ordering.

Reflexivity: Obviously $\downarrow p \sqsubseteq \downarrow p$ and $D \subseteq D$.

Antisymmetry: Assume $\downarrow p \sqsubseteq \downarrow p'$ and $\downarrow p' \sqsubseteq \downarrow p$. Then $p = p'$ and hence $\downarrow p = \downarrow p'$.

Now assume $\downarrow p \sqsubseteq D$ and $D \sqsubseteq \downarrow p$. Then there exists a directed subset $E \subseteq D$ such that $p \leq \bigvee E$ and also $D \sqsubseteq \downarrow p$. Thus, $p = \bigvee E$ and hence $\bigvee D$ exists (and equals p), contradicting the fact that $D \in \tilde{\mathcal{D}}$. Finally, assume $D \sqsubseteq D'$ and $D' \sqsubseteq D$. Then $D = D'$.

Transitivity: Case 1. $\downarrow p \sqsubseteq \downarrow p'$ and $\downarrow p' \sqsubseteq \downarrow p''$. Then $p \leq p''$, so $\downarrow p \sqsubseteq \downarrow p''$.

Case 2. $\downarrow p \sqsubseteq \downarrow p'$ and $\downarrow p' \sqsubseteq D''$. Then $p \leq p'$ and there exists a directed subset $E' \subseteq D''$ such that $p' \leq \bigvee E'$. Then $\downarrow p \sqsubseteq D''$.

Case 3. $\downarrow p \sqsubseteq D'$ and $D' \sqsubseteq \downarrow p''$. Then there exists a directed set $E \subseteq D'$ such that $p \leq \bigvee E$ and $D' \sqsubseteq \downarrow p''$. As $\bigvee E \leq p''$, we have $p \leq p''$, so $\downarrow p \sqsubseteq \downarrow p''$.

Case 4. $\downarrow p \sqsubseteq D'$ and $D' \sqsubseteq D''$. Then there exists a directed set $E \subseteq D'$ such that $p \leq \bigvee E$, and $D' \subseteq D''$. Hence $\downarrow p \sqsubseteq D''$.

Case 5. $D \sqsubseteq \downarrow p'$ and $\downarrow p' \sqsubseteq \downarrow p''$. Then $D \sqsubseteq \downarrow p' \sqsubseteq \downarrow p''$, so $D \sqsubseteq \downarrow p''$.

Case 6. $D \sqsubseteq \downarrow p'$ and $\downarrow p' \sqsubseteq D''$. Then $D \sqsubseteq \downarrow p'$ and there exists a directed set $E' \subseteq D''$ such that $p' \leq \bigvee E'$. As D has no greatest element, $D \sqsubseteq \bigcirc \downarrow p'$. Thus, for all $d \in D$, $d < \bigvee E'$, so by sup-pre-regularity, there exists $e'_d \in E'$ such that $d < e'_d$. Hence $D \subseteq D''$, so $D \sqsubseteq D''$.

Case 7. $D \sqsubseteq D'$ and $D' \sqsubseteq \downarrow p''$. Then $D \subseteq D' \sqsubseteq \downarrow p''$, so $D \sqsubseteq \downarrow p''$.

Case 8. $D \sqsubseteq D'$ and $D' \sqsubseteq D''$. Then $D \subseteq D' \subseteq D''$, so $D \sqsubseteq D''$.

Let $\mathcal{E}$ be a directed subset of $P^\#$.

(1) Let $F = \{q \in P : \downarrow q \in \mathcal{E}\}$. Assume that for all $e \in \mathcal{E}$, there exists $q \in F$ such that $e \sqsubseteq \downarrow q$.

(1a) *Claim 1. The set F is directed.*

Proof of Claim 1. Since $\mathcal{E} \neq \emptyset$, being directed, $F \neq \emptyset$. Now let $q, q' \in F$. Then $\downarrow q, \downarrow q' \in \mathcal{E}$, so there exists $e \in \mathcal{E}$ such that $\downarrow q, \downarrow q' \sqsubseteq e$. By assumption, there exists $q'' \in F$ such that $e \sqsubseteq \downarrow q''$, so $\downarrow q, \downarrow q' \sqsubseteq \downarrow q''$, so $q, q' \leq q''$. $\square$

(1b) Assume $\bigvee_P F$ exists.

Claim 2. $\downarrow_P (\bigvee_P F) = \bigvee_{P^\#} \mathcal{E}$.

Proof of Claim 2. First we show $\downarrow_P \left(\bigvee_P F \right)$ is an upper bound of $\mathcal{E}$ in $P^\#$. Pick $e \in \mathcal{E}$. If $e = \downarrow q$ for some $q \in P$, then $q \in F$, so $q \leq \bigvee_P F$ and hence $e = \downarrow q \sqsubseteq \downarrow_P \left(\bigvee_P F \right)$. If $e = D \in \tilde{\mathcal{D}}$, then by assumption there exists $q' \in F$ such that $e \sqsubseteq \downarrow q'$. Hence, by transitivity and the above sentence, we have $e \sqsubseteq \downarrow \left(\bigvee_P F \right)$.

Now suppose $\downarrow r$ is an upper bound of $\mathcal{E}$ in $P^\#$, where $r \in P$. Then for all $q \in F$, $\downarrow q \sqsubseteq \downarrow r$, so $q \leq r$. Thus $\bigvee_P F \leq r$ and hence $\downarrow \left(\bigvee_P F \right) \sqsubseteq \downarrow r$.

Finally, suppose $D \in \tilde{\mathcal{D}}$ is an upper bound of $\mathcal{E}$. Then for all $q \in F$, $\downarrow q \sqsubseteq D$, so there exists a directed set $E_q \subseteq D$ such that $q \leq \bigvee E_q$. If F has a maximal element q^* , then $q^* = \bigvee_P F$ and $\downarrow \left(\bigvee_P F \right) = \downarrow q^* \sqsubseteq D$.

Otherwise, given $q \in F$, there exists $\hat{q} \in F$ such that $q < \hat{q}$, so $q < \bigvee E_{\hat{q}}$. By sup-pre-regularity, there exists $g \in E_{\hat{q}}$ such that $q < g$ and thus $q \in D$. Hence $F \subseteq D$ is a directed set (by Claim 1) and $\bigvee_P F \leq \bigvee F$, so $\downarrow_P \left(\bigvee_P F \right) \sqsubseteq D$. $\square$

(1c) Assume $\bigvee_P F$ does not exist. Then $\downarrow_P F \in \tilde{\mathcal{D}}$.

First, we show $\downarrow_P F$ is an upper bound of $\mathcal{E}$ in $P^\#$. Pick $q \in F$. Then $\{q\}$ is a directed set, $q = \bigvee \{q\}$, and $\{q\} \sqsubseteq \downarrow F$, so $\downarrow q \sqsubseteq \downarrow_P F$.

Now let $D \in \tilde{\mathcal{D}} \cap \mathcal{E}$. By assumption, there exists $q \in F$ such that $D \sqsubseteq \downarrow q$, and by the above, $\downarrow q \sqsubseteq \downarrow_P F$, so by transitivity $D \sqsubseteq \downarrow_P F$.

Let $q \in P$ be such that $\downarrow q$ is an upper bound of $\mathcal{E}$. Thus, for all $p \in F$, $\downarrow p \sqsubseteq \downarrow q$, i.e., $\downarrow p \sqsubseteq \downarrow q$, so $\downarrow F \sqsubseteq \downarrow q$ and hence $\downarrow F \sqsubseteq \downarrow q$.

Finally, let $D \in \tilde{\mathcal{D}}$ be an upper bound of $\mathcal{E}$. Then for all $q \in F$, $\downarrow q \sqsubseteq D$, so there exists a directed set $E_q \subseteq D$ such that $q \leq \bigvee E_q$. Since $\bigvee_P F$ does not exist, F does not have a maximal element, so for any $q \in F$ there exists $\hat{q} \in F$ such that $q < \bigvee E_{\hat{q}}$; by sup-pre-regularity, there exists $g \in E_{\hat{q}}$ such that $q < g$, and hence $q \in D$. That is, $\downarrow F \subseteq D$, so $\downarrow F \sqsubseteq D$.

(2) Assume there exists $\tilde{e} \in \mathcal{E}$ such that $\tilde{e} \not\sqsubseteq \downarrow q$ for all $q \in P$ such that $\downarrow q \in \mathcal{E}$. Let $G := \bigcup \{D \in \mathcal{E} : \tilde{e} \sqsubseteq D\}$.

(2a) Obviously $\tilde{e} \in \tilde{\mathcal{D}}$ and $G \in \mathcal{I}(P)$ by Lemma 3.6.

(2b) Assume $\bigvee_P G$ exists. First, we show $\downarrow_P \left(\bigvee_P G \right)$ is an upper bound of $\mathcal{E}$ in $P^\#$. Let $e \in \mathcal{E}$. Since $\mathcal{E}$ is directed, we may assume by transitivity that $\tilde{e} \sqsubseteq e$. Then $e \subseteq G$, so $p \leq \bigvee_P G$ for all $p \in e$, and hence $e \sqsubseteq \downarrow \left(\bigvee_P G \right)$, so $e \sqsubseteq \downarrow_P \left(\bigvee_P G \right)$.

Now let $q \in P$ be such that $\downarrow q$ is an upper bound of $\mathcal{E}$ in $P^\#$. Then for all $D \in \mathcal{E}$ such that $\tilde{e} \sqsubseteq D$, we have $D \sqsubseteq \downarrow q$, so $D \sqsubseteq \downarrow q$, and hence $G \sqsubseteq \downarrow q$, and thus $\bigvee_P G \leq q$ and $\downarrow \bigvee_P G \sqsubseteq \downarrow q$, or $\downarrow \bigvee_P G \sqsubseteq \downarrow q$. If $D' \in \tilde{\mathcal{D}}$ is an upper bound of $\mathcal{E}$ in $P^\#$, then for all $D \in \mathcal{E}$ such that $\tilde{e} \sqsubseteq D$, we have $D \sqsubseteq D'$, or $D \subseteq D'$, so $G \subseteq D'$. By (2a), G is directed, and $\bigvee_P G \leq \bigvee G$, so $\downarrow \bigvee_P G \sqsubseteq \downarrow \bigvee G \sqsubseteq D'$.

We conclude that $\downarrow \bigvee G = \bigvee_{P^\#} \mathcal{E}$.

(2c) Assume $\bigvee_P G$ does not exist. In this case, $G \in \tilde{\mathcal{D}}$. First, we show G is an upper bound of $\mathcal{E}$ in $P^\#$. Take $e \in \mathcal{E}$. Since $\mathcal{E}$ is directed, we may assume (since we then use transitivity) that $\tilde{e} \sqsubseteq e$, so $e \subseteq G$ and hence $e \sqsubseteq G$.

Now let $Q \in P^\#$ be an upper bound of $\mathcal{E}$ in $P^\#$. Then for all $D \in \mathcal{E}$ such that $\tilde{e} \sqsubseteq D$, we have $D \sqsubseteq Q$, so $D \subseteq Q$. Hence, $G \subseteq Q$ and thus $G \sqsubseteq Q$.

We conclude that $G = \bigvee_{P^\#} \mathcal{E}$. □

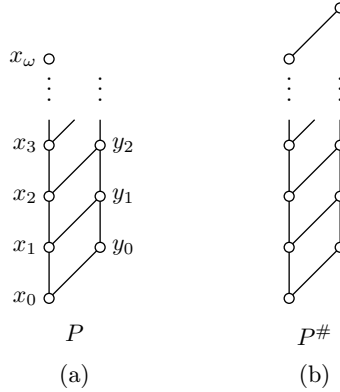


Figure 7.

Lemma 6.4. *Let P be a sup-pre-regular poset not containing $\omega \times 2$. Then for all $e, e' \in P^\#$, $e \sqsubseteq e'$ if and only if $e \subseteq e'$.*

Proof. We only need to consider the case in which $e = \downarrow_P p$ and $e' = D' \in \tilde{\mathcal{D}}$. In particular, D' has no maximal element. If $e \subseteq e'$, then let $E = \{p\}$, a directed subset of D' such that $p \leq p = \bigvee E$. Hence $e \sqsubseteq e'$.

Now assume $e \sqsubseteq e'$ and let $E \subseteq D'$ be directed and such that $p \leq \bigvee_P E$. If $p \in D'$, we are done, so assume $p \notin D'$. Hence, for all $d' \in D'$, $p \not\leq d'$, so there exists $g \in E$ such that $g \not\leq d'$. Since $\bigvee E = \bigvee(\downarrow E)$ exists but $\bigvee D'$ does not, $\downarrow E \subsetneq D'$. By Lemma 4.1, there exists $d' \in D'$ such that $\downarrow E \subseteq \downarrow d'$, a contradiction. □

Corollary 6.5. *Let P be a sup-pre-regular poset not containing $\omega \times 2$. Then $P^\#$ is sup-regular.*

Proof. By Theorem 6.3, every nonempty chain $\mathcal{E}$ has a supremum. Let $a \in P^\#$ be such that $a \sqsubset \bigvee_{P^\#} \mathcal{E}$. By Lemmas 4.1 and 6.4, there exists $d' \in \bigvee_{P^\#} \mathcal{E}$ such that $a \subseteq \downarrow_P d'$. We consider the four possibilities of Theorem 6.3, (1b), (1c), (2b), and (2c).

(1b) If $a = \downarrow_P p$ for some $p \in P$, then $p <_P \bigvee_P F$, so by sup-pre-regularity, there exists $f \in F$ such that $p <_P f$, and hence $\downarrow p \subsetneq \downarrow f \in \mathcal{E}$, so $\downarrow p \subsetneq \downarrow f$, by Lemma 6.4.

If $a = D \in \tilde{\mathcal{D}}$, the previous paragraph tells us $D \subseteq \downarrow F$. We cannot have $D = \downarrow F$ since $\bigvee(\downarrow F)$ exists but $\bigvee D$ does not. Hence, $D \subsetneq \downarrow F$, so by Lemma 4.1, there exists $g'' \in \downarrow F$ such that $D \subseteq \downarrow g''$. Hence, there exists $f \in F$ such that $D \subsetneq \downarrow f$, i.e., by Lemma 6.4, $D \subsetneq e$ for some $e \in \mathcal{E}$.

(1c) By Lemmas 4.1 and 6.4, there exists $f \in F$ such that $a \subseteq \downarrow f$, so $a \subsetneq \downarrow f \in \mathcal{E}$.

(2b) If $a = \downarrow_P p$ for some $p \in P$, then by sup-pre-regularity, $p <_P g$ for some $g \in G$, so $a = \downarrow p \subsetneq D''$ for some $D'' \in \mathcal{E}$. If $a = D \in \tilde{\mathcal{D}}$, the previous sentence tells us that $D \subseteq G$. We cannot have $D = G$, since $\bigvee G$ exists but $\bigvee D$ does not, so $D \subsetneq G$, and hence, by Lemma 4.1 there exists $g'' \in G$ such that $D \subseteq \downarrow g''$ —thus $D \subsetneq \downarrow g'' \subseteq D''$ for some $D'' \in \mathcal{E}$.

(2c) By Lemmas 4.1 and 6.4, there exists $g \in G$ such that $a \subseteq \downarrow g$, so $a \subsetneq \downarrow g \subseteq D''$ for some $D'' \in \mathcal{E}$. $\square$

Corollary 6.6. *Let P be a sup-pre-regular poset not containing $\omega \times 2$.*

- (1) *The poset $P^\#$ does not contain $\omega \times 2$.*
- (2) *If P does not contain $\omega^\partial \times 2$, then $P^\#$ does not contain $\omega^\partial \times 2$.*
- (3) *If every nonempty chain of P has an infimum, every nonempty chain of $P^\#$ has an infimum.*
- (4) *If P is inf-regular, then $P^\#$ is inf-regular.*

Proof. (1) This follows from Corollary 6.5 and Observation 1.2.

(2) This follows from Lemmas 4.3 and 6.4.

(3) and (4) These follow from Lemmas 4.5 and 6.4. $\square$

Proposition 6.7. *Let P be a sup-pre-regular poset. Let $f: P \rightarrow R$ be an order-embedding into a sup-regular poset. Then there exists an order-embedding $f^\#: P^\# \rightarrow R$ such that $f^\# \circ \iota_P^\# = f$.*

Proof. Since f is an embedding, $\omega \times 2$ does not embed in P by Observation 1.2. For $D \in P^\#$, define $f^\#(D) := \bigvee_R \{f(d) : d \in D\}$.

Let $q, q' \in P^\#$. If $q \sqsubseteq q'$, then by Lemma 6.4 $q \subseteq q'$, so $f^\#(q) \leq_R f^\#(q')$. Now assume $q \not\sqsubseteq q'$. By Lemma 6.4, $q \not\subseteq q'$, so there exists $\bar{p} \in q \setminus q'$.

Assume for a contradiction that $f^\#(q) \leq_R f^\#(q')$. If $f^\#(q) <_R f^\#(q')$, then $f(\bar{p}) <_R \bigvee_R \{f(d') : d' \in q'\}$. By sup-regularity, there exists $d' \in q'$ such that $f(\bar{p}) <_R f(d')$, so $\bar{p} <_P d'$ and hence $\bar{p} \in q'$, a contradiction. Thus $f^\#(q) = f^\#(q')$.

Case 1. $q = \downarrow p$ and $q' = \downarrow p'$ for some $p, p' \in P$. Then $f(p) = f(p')$, so $p = p'$, and hence $q = q'$, a contradiction.

Case 2. $q = \downarrow p$ for some $p \in P$ and $q' = D' \in \tilde{\mathcal{D}}$. In particular, D' has no maximal element. Hence, $f(d') <_R f(p)$ for all $d \in D'$, so $d' <_P p$ for all $d' \in D'$, i.e., $D' \subsetneq \downarrow_P p$. Thus p is an upper bound of D' . Since $\bigvee_P D'$ does not exist, there exists $\tilde{p} \in P$ such that $D' \subseteq \downarrow_P \tilde{p}$ but $p \not\leq_P \tilde{p}$. Hence, $f(p) = f^\#(q) = f^\#(q') = \bigvee_R \{f(d') : d' \in D'\} \leq_R f^\#(\downarrow_P \tilde{p}) = f(\tilde{p})$, so $p \leq_P \tilde{p}$, a contradiction.

Case 3. $q = D \in \tilde{\mathcal{D}}$. Thus, D does not have a maximal element, so there exists $\bar{p} \in q \setminus q'$ such that $\bar{p} <_P \bar{\bar{p}}$. Hence, $f(\bar{p}) <_R \bigvee_R \{f(d') : d' \in q'\}$. By sup-regularity, there exists $d' \in q'$ such that $f(\bar{p}) < f(d')$, so $\bar{p} <_P d'$ and hence $\bar{p} \in q'$, a contradiction. $\square$

Theorem 6.8. *Let P be a poset not containing $\omega \times \mathbf{2}$. Then there is a sup-regular poset $P^\sim$ and there is an order-embedding $\varphi_P : P \rightarrow P^\sim$ such that:*

- (1) *If $f : P \rightarrow R$ is an order-embedding into a sup-regular poset, there is an order-embedding $f^\sim : P^\sim \rightarrow R$ such that $f^\sim \circ \varphi_P = f$.*
- (2) *If P does not contain $\omega^\partial \times \mathbf{2}$, then $P^\sim$ does not contain $\omega^\partial \times \mathbf{2}$.*
- (3) *If P is inf-regular, then $P^\sim$ is inf-regular.*

Proof. Let $P_0 = P$. Let $P^\sim := (P_\omega)^\#$. Let $\varphi_P := \iota_\omega^\# \circ \varphi_0 : P_0 \rightarrow P^\sim$. The poset $P^\sim$ is well-defined by Proposition 5.16. By Proposition 5.15, Proposition 5.16, and Corollary 6.5, P_ω is sup-pre-regular and does not contain $\omega \times \mathbf{2}$; $P^\sim$ is sup-regular. By Theorem 6.3, φ_P is an order-embedding.

(1) Let R be a sup-regular poset and $f : P \rightarrow R$ an order-embedding. By Corollary 5.19 with $f_0 = f$, there exists an order-embedding $f_\omega : P_\omega \rightarrow R$ such that $f_\omega \circ \varphi_0 = f$. By Proposition 6.7, there is an order-embedding $f^\sim := (f_\omega)^\# : P^\sim \rightarrow R$ such that $(f_\omega)^\# \circ \iota_\omega^\# = f_\omega$. Hence, $f^\sim \circ \varphi_P = (f_\omega)^\# \circ \iota_\omega^\# \circ \varphi_0 = f_\omega \circ \varphi_0 = f$.

(2) If P does not contain $\omega^\partial \times \mathbf{2}$, then P_ω does not contain $\omega^\partial \times \mathbf{2}$ by Proposition 5.17. By Corollary 6.6 (2), $P^\sim$ does not contain $\omega^\partial \times \mathbf{2}$.

(3) If P is inf-regular, then, by Proposition 5.18 (2), P_ω is inf-regular. By Corollary 6.6 (4), $P^\sim$ is inf-regular. $\square$

Corollary 6.9. *Let P be a poset that does not contain $\omega \times \mathbf{2}$ and does not contain $\omega^\partial \times \mathbf{2}$. Then there is a regular poset P^{reg} and there is an order-embedding $\Psi_P : P \rightarrow P^{\text{reg}}$ such that: If $f : P \rightarrow R$ is an order-embedding into a regular poset, there is an order-embedding $f^{\text{reg}} : P^{\text{reg}} \rightarrow R$ such that $f^{\text{reg}} \circ \Psi_P = f$.*

P r o o f. By Theorem 6.8, especially part (2), $(P^\sim)^\partial$ is inf-regular and does not contain $\omega \times 2$. By Theorem 6.8, especially part (3), $[(P^\sim)^\partial]^\sim$ is regular, so $P^{\text{reg}} := \{[(P^\sim)^\partial]^\sim\}^\partial$ is regular. The map

$$p \mapsto (\partial_{[(P^\sim)^\partial]^\sim} \circ \varphi_{(P^\sim)^\partial} \circ \partial_{P^\sim} \circ \varphi_P)(p) \quad (p \in P)$$

is an order-embedding Ψ_P from P to P^{reg} .

If $f: P \rightarrow R$ is an order-embedding to a regular poset, then by Theorem 6.8(1), there is an order-embedding $f^\sim: P^\sim \rightarrow R$ such that

$$f^\sim \circ \varphi_P = f.$$

Thus, we get an order-embedding $g := \partial_R \circ f^\sim \circ \partial_{P^\sim}^{-1}: (P^\sim)^\partial \rightarrow R^\partial$. Since R^∂ is a regular poset, by Theorem 6.8(1) there is an order-embedding

$$g^\sim: [(P^\sim)^\partial]^\sim \rightarrow R^\partial$$

such that

$$g^\sim \circ \varphi_{(P^\sim)^\partial} = g,$$

so

$$f^{\text{reg}} = \partial_R^{-1} \circ g^\sim \circ \partial_{[(P^\sim)^\partial]^\sim}^{-1}$$

is an order-embedding from P^{reg} to R and

$$\begin{aligned} f^{\text{reg}} \circ \Psi_P &= f^{\text{reg}} \circ \partial_{[(P^\sim)^\partial]^\sim} \circ \varphi_{(P^\sim)^\partial} \circ \partial_{P^\sim} \circ \varphi_P \\ &= \partial_R^{-1} \circ g^\sim \circ \partial_{[(P^\sim)^\partial]^\sim}^{-1} \circ \partial_{[(P^\sim)^\partial]^\sim} \circ \varphi_{(P^\sim)^\partial} \circ \partial_{P^\sim} \circ \varphi_P \\ &= \partial_R^{-1} \circ g^\sim \circ \varphi_{(P^\sim)^\partial} \circ \partial_{P^\sim} \circ \varphi_P = \partial_R^{-1} \circ g \circ \partial_{P^\sim} \circ \varphi_P \\ &= \partial_R^{-1} \circ \partial_R \circ f^\sim \circ \partial_{P^\sim}^{-1} \circ \partial_{P^\sim} \circ \varphi_P = f^\sim \circ \varphi_P = f. \end{aligned}$$

□

Obviously some work could be done regarding the uniqueness of these extensions, or regarding extensions of order-preserving maps that are not necessarily order-embeddings. And perhaps these techniques can be used to address Grillet's conjecture [7], page 157, mentioned in Section 1.

7. AN EXAMPLE SOUGHT BY GINSBURG IN 1984

Let P be a poset. We write $p < q$ if p is a lower cover of q in P . An element $m \in P$ is *completely meet-irreducible* if $\{p \in P: p > m\}$ has a least element m^* , which will be the unique upper cover of m . For $p, p' \in P$, we write $p \sim p'$ if $p \leq p'$ or $p \geq p'$; we write $p \parallel p'$ if $p \not\sim p'$; we write $p \trianglelefteq p'$ if every maximal chain of P containing p also contains p' (equivalently, if $p \sim q \in P$, then $p' \sim q$); we write $p \diamond p'$ if $p \trianglelefteq p'$ and $p' \trianglelefteq p$; see [9], [4].

Observation 7.1. Let C be a minimal cutset of a poset P . If $c_1, c_2 \in C$ and $c_1 \leq c_2$, then $c_1 = c_2$.

Let T be a poset. For each $t \in T$, let P_t be a poset and assume that $P_t \cap P_{t'} = \emptyset$ if $t, t' \in T$ and $t \neq t'$. For $p \in P_t$, let $\tau(p) = t$ ($t \in T$). The *lexicographic sum* of the family $(P_t)_{t \in T}$ is the poset on $\bigcup_{t \in T} P_t$ defined by:

$$x \leq y \quad \text{if} \quad \begin{cases} \tau(x) = \tau(y) \text{ and } x \leq_{P_{\tau(x)}} y, \text{ or} \\ \tau(x) <_T \tau(y) \end{cases}$$

$(x, y \in \bigcup_{t \in T} P_t)$. Two special cases are the disjoint sum of posets P_t and P_u , $P_t + P_u$ (see [2], Section 1.24, if $\{t, u\}$ is a two-element antichain) and the ordinal sum $P_0 \oplus P_1$ (see [2], Section 1.24, where $0 < 1$). Note that if $x, y \in \bigcup_{t \in T} P_t$ and $x \leq y$, then $\tau(x) \leq_T \tau(y)$, so that each P_t ($t \in T$) is a convex subset of the lexicographic sum.

Slatinský in [14], 3.9 Satz looked at when a lexicographic sum was a semilattice, so the following is but a trivial extension.

Proposition 7.2. Let T be a nonempty poset and $(P_t)_{t \in T}$ a family of pairwise disjoint nonempty posets. Let $X \subseteq \bigcup_{t \in T} P_t$. Then $\bigvee X$ exists in the lexicographic sum P if and only if $\bigvee_T \{\tau(x) : x \in X\}$ exists—call it t_0 —and:

- (1) If $t_0 \notin \tau[X]$, then P_{t_0} has a least element.
- (2) If $t_0 \in \tau[X]$ and the set $\{y \in X : \tau(y) = t_0\}$ has an upper bound in P_{t_0} , then that set has a least upper bound with respect to P_{t_0} .
- (3) If $t_0 \in \tau[X]$ and the set $\{y \in X : \tau(y) = t_0\}$ has no upper bound in P_{t_0} , then t_0 is completely meet-irreducible and $P_{t_0^*}$ has a least element.

Remark 7.3. Note that in

- (1) $\bigvee X = 0_{P_{t_0}}$;
- (2) $\bigvee X = \bigvee_{P_{t_0}} \{y \in X : \tau(y) = t_0\}$;
- (3) $\bigvee X = 0_{P_{t_0^*}}$.

Proof. *Case (1):* $\tau[X]$ has no greatest element. First, assume $z := \bigvee X$ exists. Then for all $x \in X$, $\tau(x) \leq \tau(z)$, so $\tau(z)$ is an upper bound of $\tau[X]$. If $w \in T$ is an upper bound of $\tau[X]$, then $w \notin \tau[X]$. Hence, if $p \in P_w$, then p is an upper bound of X , so $z \leq p$ and thus $\tau(z) \leq \tau(p) = w$. Therefore $\bigvee \tau[X]$ exists and equals $\tau(z)$. Since every $p \in P_{\tau(z)}$ is an upper bound of X , $P_{\tau(z)}$ has a least element (which is z). This is (1).

Now assume $t_0 = \bigvee_T \tau[X]$ exists and (1) holds. Then the least element of P_{t_0} is an upper bound of X . If $q \in P$ is an upper bound of X in P , then $\tau(q)$ is an upper bound of $\tau[X]$ in T , so $t_0 \leq \tau(q)$, and either $t_0 < \tau(q)$, in which case $0_{P_{t_0}} \leq q$, or $t_0 = \tau(q)$, in which case $q \in P_{t_0}$ and again $0_{P_{t_0}} \leq q$. Hence $0_{P_{t_0}} = \bigvee X$.

Case (2): $\tau[X]$ has a greatest element t_0 .

Case (2a): $\{x \in X : \tau(x) = t_0\}$ has an upper bound p' in P_{t_0} . First assume $z := \bigvee X$ exists. Since p' is an upper bound of X , then $z \leq p'$, and since $x \leq z$ for some $x \in P_{t_0}$, we know $\tau(z) = t_0$.

Let $u \in P_{t_0}$ be an upper bound of $\{y \in X : \tau(y) = t_0\}$ in P_{t_0} . Then u is an upper bound of X in P . Hence $z \leq u$. Thus $z = \bigvee_{P_{t_0}} \{y \in X : \tau(y) = t_0\}$. Hence (2) holds.

Now assume $\zeta := \bigvee_{P_{t_0}} \{y \in X : \tau(y) = t_0\}$ exists. Then ζ is an upper bound of X in P . If u is an upper bound of X in P , then in Case 2, $t_0 \leq \tau(u)$. If $\tau(u) > t_0$, then $\zeta \leq u$. If $\tau(u) = t_0$, then u is an upper bound in P_{t_0} of $\{y \in X : \tau(y) = t_0\}$, so $\zeta \leq u$. Thus $\zeta = \bigvee X$.

Case (2b): $\{x \in X : \tau(x) = t_0\}$ has no upper bound in P_{t_0} . First assume $z := \bigvee X$ exists. Then $\tau(z) > t_0$. But any $u \in P$ such that $\tau(u) > t_0$ is an upper bound of X , so $\tau(u) \geq \tau(z)$. Hence, $t_0^* = \tau(z)$ is the unique upper cover of the completely meet-irreducible element t_0 . Since any $u \in P_{t_0^*}$ is an upper bound of X , we must have $z = 0_{P_{t_0^*}}$. Thus (3) holds.

Now assume (3) holds. Then $0_{P_{t_0^*}}$ is an upper bound of X . If u is an upper bound of X , then $t_0 \leq \tau(u)$, but since in Case 2b, $t_0 \neq \tau(u)$, we have $t_0 < \tau(u)$. Hence $t_0^* \leq \tau(u)$, so either $t_0^* < \tau(u)$ and $0_{P_{t_0^*}} \leq u$ or $t_0^* = \tau(u)$ and again $0_{P_{t_0^*}} \leq u$. Thus $0_{P_{t_0^*}} = \bigvee X$. $\square$

Grillet [7], pages 160–161 took a poset E and defined E^* to be

$$\{(u, v) \in E \times E : u < v\}.$$

For some $S \subseteq E^*$, he defined a partial ordering on $\overline{E} := E \cup S$ as follows (where $x, y \in E$ and $(u, v), (u', v') \in S$):

$$\begin{aligned} x <_{\overline{E}} y &\Leftrightarrow x <_E y, & x <_{\overline{E}} (u, v) &\Leftrightarrow x \leq_E u, \\ (u, v) <_{\overline{E}} x &\Leftrightarrow v \leq_E x, & (u, v) <_{\overline{E}} (u', v') &\Leftrightarrow v \leq_E u'. \end{aligned}$$

Grillet had the hypothesis that E was regular, and proved that $\overline{E}$ was regular (in particular, he showed what the supremum of a nonempty chain was); but we will not assume E is regular.

The above definition makes it clear that $(u, v) \in S$ is completely meet-irreducible in $\overline{E}$ with unique upper cover v , and dually.

Lemma 7.4. *Let $X \subseteq E$. Then $\bigvee_{\overline{E}} X$ exists if and only if $\bigvee_E X$ exists. In this case, the two are equal.*

Proof. Assume $\bigvee_{\overline{E}} X$ exists. Since every $(u, v) \in S$ is completely join-irreducible, $\bigvee_{\overline{E}} X \in E$. Obviously this means $\bigvee_{\overline{E}} X = \bigvee_E X$.

Now assume $\bigvee_E X$ exists. It is an upper bound for X with respect to $\overline{E}$. If $(u, v) \in S$ is an upper bound for X , then $x \leq u$ for all $x \in X$; hence $\bigvee_E X \leq u$ and thus $\bigvee_E X \leq (u, v)$. We conclude that $\bigvee_E X = \bigvee_{\overline{E}} X$. $\square$

Let $\Upsilon(x) = x$ for $x \in E$ and let $\Upsilon(u, v) = v$ for $(u, v) \in S$. The following is a trivial modification of Grillet's [7], Lemma 9.

Proposition 7.5. *Let $Y \subseteq \overline{E}$. If Y does not have a greatest element, then $\bigvee_{\overline{E}} Y$ exists if and only if $\bigvee_E \Upsilon[Y]$ exists. In this case, the two are equal.*

Proof. Assume $\bigvee_{\overline{E}} Y$ exists. Since Y has no greatest element, then $\bigvee_{\overline{E}} Y \notin E^*$. Thus $\bigvee_{\overline{E}} Y \in E$ and for every $y \in Y$, $\Upsilon(y) \leq \bigvee_{\overline{E}} Y$. Obviously any upper bound e of $\Upsilon[Y]$ in E is an upper bound of Y , so $\bigvee_{\overline{E}} Y \leq e$. Thus $\bigvee_{\overline{E}} Y = \bigvee_E \Upsilon[Y]$.

Now assume $\bigvee_E \Upsilon[Y]$ exists. Then it is an upper bound of Y . Let $e \in E$ be an upper bound of Y in $\overline{E}$. For all $y \in Y$, $\Upsilon(y) \leq e$, so $\bigvee_E \Upsilon[Y] \leq e$. Let $(u, v) \in S$ be an upper bound of Y . Then for all $y \in Y$, $\Upsilon(y) \leq u$, so $\bigvee_E \Upsilon[Y] \leq_E u$ and $u \leq_{\overline{E}} (u, v)$. Hence $\bigvee_{\overline{E}} Y = \bigvee_E \Upsilon[Y]$. $\square$

Definition 7.6. Let E be a poset. Define $E\mathbb{R}$ to be the lexicographic sum of $(P_{\bar{e}})_{\bar{e} \in \overline{E}}$, where

$$P_{\bar{e}} = \begin{cases} \{\bar{e}\} & \text{if } \bar{e} \in E, \\ \mathbb{R} \times \{\bar{e}\} & \text{if } \bar{e} \in E^*. \end{cases}$$

For $(u, v) \in E^*$ and $r \in \mathbb{R}$, we will denote $(r, (u, v))$ by $r_{(u, v)}$.

Corollary 7.7. *If E is a complete lattice, then $E\mathbb{R}$ is a complete lattice.*

Proof. By Proposition 7.5, $\overline{E}$ is a complete lattice and in fact for all $Y \subseteq \overline{E}$, $\bigvee_{\overline{E}} Y \in E$ (with $\bigvee_{\overline{E}} Y = \bigvee_E X$ for some $X \subseteq E$) unless Y has a greatest element.

Let $W \subseteq E\mathbb{R}$ and let $Y = \tau[W] \subseteq \overline{E}$. If $\bigvee Y \in E$, then $|P_{\bigvee Y}| = 1$, so (3) cannot occur and in case (1) or (2), we have what we need to guarantee the existence of $\bigvee_{E\mathbb{R}} W$.

If Y does have a greatest element and $\bigvee Y \in E^*$, then the Least Upper Bound Property for $\mathbb{R}$ tells us $\bigvee_{E\mathbb{R}} W$ exists in situation (2); in situation (3), the fact that $\bigvee Y$ is completely meet-irreducible in $\overline{E}$ with upper cover $(\bigvee Y)^*$ such that $|P_{(\bigvee Y)^*}| = 1$ tells us $\bigvee_{E\mathbb{R}} W$ exists. $\square$

Lemma 7.8. *If E is countable, then every antichain of $E\mathbb{R}$ is countable. Also, $E\mathbb{R}$ is densely ordered.*

Proof. Let A be an uncountable antichain of $E\mathbb{R}$. Let $A_1 = A \cap \bigcup_{e \in E} P_e$ and let $A_2 = A \setminus A_1$. Then A_2 is uncountable. Since $E^* \subseteq E \times E$ is countable, there exist distinct $a, a' \in A_2$ such that $\tau(a) = \tau(a')$ —say, $(u, v) \in E \times E$ —so $a = r_{(u,v)}$ and $a' = r'_{(u,v)}$. But $r < r'$ or $r' < r$, so $a < a'$ or $a > a'$, a contradiction.

Now let $q, q' \in E\mathbb{R}$ be such that $q < q'$. We show there exists $q'' \in E\mathbb{R}$ such that $q < q'' < q'$.

Case 1. $q \in \bigcup_{e \in E} P_e$.

Case 1a. $q' \in \bigcup_{e \in E} P_e$. Then $q <_E q'$. If there exists $q'' \in E$ such that $q <_E q'' <_E q'$, then $q <_{E\mathbb{R}} q'' <_{E\mathbb{R}} q'$. Otherwise, $(q, q') \in E^*$, so $q <_{E\mathbb{R}} 0.5_{(q,q')} <_{E\mathbb{R}} q'$.

Case 1b. $q' = r'_{(u',v')}$ for some $r' \in \mathbb{R}$, $(u', v') \in E^*$. Then

$$q <_{E\mathbb{R}} (r' - 1)_{(u',v')} <_{E\mathbb{R}} r'_{(u',v')}.$$

Case 2. $q = r_{(u,v)}$ for some $r \in \mathbb{R}$, $(u, v) \in E^*$.

Case 2a. $q' \in \bigcup_{e \in E} P_e$. Then $r_{(u,v)} <_{E\mathbb{R}} (r + 1)_{(u,v)} <_{E\mathbb{R}} q'$.

Case 2b. $q' = r'_{(u',v')}$ for some $r' \in \mathbb{R}$, $(u', v') \in E^*$. If $v \leq_E u'$, then $r_{(u,v)} <_{E\mathbb{R}} r'_{(u',v')}$.

Otherwise, $u = u'$, $v = v'$, and $r <_{\mathbb{R}} r'$, so $r_{(u,v)} <_{E\mathbb{R}} (r + r')/2_{(u,v)} <_{E\mathbb{R}} r'_{(u,v)}$. $\square$

Higgs in [8], page 373 defined a poset $M = \{(a_0, \dots, a_m) \in \mathbb{Z}^{m+1} : m \in \mathbb{N}_0\}$, where $(x_0, \dots, x_m) \geq (y_0, \dots, y_n)$ if $m \leq n$, $x_0 = y_0, \dots, x_{m-1} = y_{m-1}$, and $x_m \geq y_m$. We say that the *length* of $(a_0, \dots, a_m)$ is $m + 1$.

Higgs showed that M has no minimal cutset, he stated that every principal up-set is a chain, and he said that $\mathbf{1} \oplus (M + F_4) \oplus \mathbf{1}$ is a complete lattice, where F_4 is the four-element fence [2], page 28 and $\mathbf{1}$ is a one-element poset.

Lemma 7.9 (Higgs [8], page 373). *Every principal up-set of M is a chain.*

Proof. Let $\bar{a} = (a_0, \dots, a_m)$, $\bar{a}' = (a'_0, \dots, a'_{m'}) \geq (b_0, \dots, b_n)$, where $m \leq m' \leq n$. Then $a_0 = b_0 = a'_0$; $a_1 = b_1 = a'_1$; $\dots$; $a_{m-1} = b_{m-1} = a'_{m-1}$; and $a_m \geq b_m$; $b_m = a'_m$; $b_{m+1} = a'_{m+1}$; $\dots$; $b'_{m'-1} = a'_{m'-1}$; and $a'_{m'} \geq b_{m'}$.

Case 1. $m' < n$.

Case 1a. $m < m'$. Then $a_0 = a'_0; a_1 = a'_1; \dots; a_{m-1} = a'_{m-1}; a'_m = b_m$, so $a_m \geq a'_m$ and $\bar{a} \geq \bar{a}'$.

Case 1b. $m = m'$. Then $a_0 = a'_0; a_1 = a'_1; \dots; a_{m-1} = a'_{m-1}$; and $a_m \sim a'_m$, so $\bar{a} \sim \bar{a}'$.

Case 2. $m' = n$.

Case 2a. $m < m'$. Then $a_0 = a'_0; a_1 = a'_1; \dots; a_{m-1} = a'_{m-1}; a_m \geq b_m = a'_m$. Hence $\bar{a} \geq \bar{a}'$.

Case 2b. $m = m'$. Then $a_0 = a'_0; a_1 = a'_1; \dots; a_{m-1} = a'_{m-1}$; and $a_m \sim a'_m$, so $\bar{a} \sim \bar{a}'$. $\square$

Lemma 7.10.

- (1) Let E be a poset such that $\uparrow_E e$ is a chain for all $e \in E$. Then every principal up-set of $\bar{E}$ is a chain.
- (2) Let T be a poset such that $\uparrow_T t$ is a chain and P_t is a chain for all $t \in T$. Then the lexicographic sum of the family $(P_t)_{t \in T}$ is such that every principal up-set is a chain.

Proof. (1) Let $\alpha, \beta, \gamma \in \bar{E}$ be such that $\alpha < \beta, \gamma$.

Case 1. $\alpha \in E$.

Case 1a. $\beta, \gamma \in E$. Then $\beta \sim \gamma$.

Case 1b. $\beta \in E$ and $\gamma = (u, v) \in E^*$. Then $\alpha \leq u, v$, so $\beta \sim u, v$, and either $\beta \leq u$ or $v \leq \beta$, since $u \leq_E v$. Hence $\beta \sim \gamma$.

Case 1c. $\beta, \gamma \in E^*$. Say $\beta = (u, v)$ and $\gamma = (u', v')$. We have $\alpha \leq u, u', v, v'$. Without loss of generality, $u \leq u'$. If $u = u'$, then as $u \leq_E v$ and $u' \leq_E v'$ but $v \sim v'$, we have $v = v'$. Hence $\beta = \gamma$.

If $u < u'$, then since $u \leq_E v$ and $v \sim u'$, we cannot have $u' < v$, so $v \leq u'$ and $\beta \leq \gamma$.

Case 2. $\alpha \in E^*$. Then $\Upsilon(\alpha) \leq \beta, \gamma$. By Case 1, $\beta \sim \gamma$.

(2) Let $p, q, r \in \bigcup_{t \in T} P_t$ be such that $p \leq q, r$. So $\tau(p) \leq_T \tau(q), \tau(r)$. Without loss of generality, $\tau(q) \leq \tau(r)$. If $\tau(q) < \tau(r)$, then $q \leq r$. If $\tau(q) = \tau(r)$, then $q \sim r$. $\square$

Corollary 7.11. Every principal up-set of $M\mathbb{R}$ is a chain.

Observation 7.12. Let P be a poset such that every principal up-set is a chain. Then $p \leq q$ for all $p, q \in P$ such that $p \leq q$.

Let $H := \{0\} \oplus M \oplus \{1\}$.

Proposition 7.13. The poset H is a complete lattice.

Proof. Let $D \subseteq H$. If $D = \emptyset$ or $D = \{1_H\}$, then $\bigwedge D = 1_H$. If $0_H \in D$, then $\bigwedge D = 0_H$. So assume $D \subseteq M$ and $D \neq \emptyset$. If there exist $x, y \in D$ such that $x \parallel y$, then $\bigwedge D = 0_H$ by Lemma 7.9, so assume D is a chain.

Case 1. $\sup\{\text{length of } \bar{a} : \bar{a} \in D\} = \infty$. Then $\bigwedge D = 0_H$.

Case 2. $\sup\{\text{length of } \bar{a} : \bar{a} \in D\} = n + 1$.

Case 2a. $\inf\{k \in \mathbb{Z} : (a_0, \dots, a_{n-1}, k) \in D\} = -\infty$. Then $\bigwedge D = 0_H$.

Case 2b. $\inf\{k \in \mathbb{Z} : (a_0, \dots, a_{n-1}, k) \in D\} = m \in \mathbb{Z}$. Then

$$\bigwedge D = (a_0, \dots, a_{n-1}, m).$$

□

Lemma 7.14. *Let $h_1, h_2 \in H$. Then $h_1 <_H h_2$ if and only if $h_1, h_2 \in M$ and for some $n \in \mathbb{N}_0$, $a_0, \dots, a_n \in \mathbb{Z}$, $h_1 = (a_0, \dots, a_n)$ and $h_2 = (a_0, \dots, a_{n-1}, a_n + 1)$.*

Proof. The element 0_H has no upper covers since for any $(a_0, \dots, a_n) \in M$, $0_H < (a_0, \dots, a_{n-1}, a_n - 1) < (a_0, \dots, a_n)$. Also, 1_H has no lower covers, since $(a_0, a_1, \dots, a_n) < (a_0, \dots, a_{n-1}, a_n + 1) < 1_H$.

Now suppose $h_1, h_2 \in H$ and $h_1 < h_2$. Then $h_1, h_2 \in M$. Say $h_1 = (b_0, \dots, b_n)$ and $h_2 = (a_0, \dots, a_m)$, where $m, n \in \mathbb{N}_0$. If $m < n$, then $h_2 > (b_0, b_1, \dots, b_{n-1}, b_n + 1) > h_1$. Thus $m = n$, and hence $h_2 = (b_0, \dots, b_{n-1}, b_n + 1)$.

Conversely, $(a_0, \dots, a_{m-1}, a_m + 1) > (a_0, \dots, a_{m-1}, a_m)$, and if $(a_0, \dots, a_{m-1}, a_m + 1) > (b_0, \dots, b_n) > (a_0, \dots, a_{m-1}, a_m)$, then $m \leq n \leq m$, so $m = n$ and $a_m + 1 > b_m > a_m$, which is impossible. □

Corollary 7.15. *The set $H\mathbb{R} = \{0_H\} \oplus M\mathbb{R} \oplus \{1_H\}$.*

Lemma 7.16. *Let $\bar{a}, \bar{b} \in M \cap M\mathbb{R}$ be such that $\bar{a} \leq_M \bar{b}$. Then for any $r \in \mathbb{R}$, $r_{(\bar{a}, \bar{b})} \diamond_{H\mathbb{R}} \bar{a}$.*

Remark 7.17. By Lemma 7.14, the hypothesis means that $\bar{a} = (a_0, \dots, a_m)$ and $\bar{b} = (a_0, \dots, a_{m-1}, a_m + 1)$.

Proof. If $\bar{c} \in M$ and $\bar{c} \leq r_{(\bar{a}, \bar{b})}$, then $\bar{c} \leq \bar{a}$. Now let $(\bar{x}, \bar{y}) \in M^*$ and $s \in \mathbb{R}$. If $s_{(\bar{x}, \bar{y})} \leq r_{(\bar{a}, \bar{b})}$, then either $\bar{y} \leq \bar{a}$, so $s_{(\bar{x}, \bar{y})} \leq \bar{a}$, or else $(\bar{x}, \bar{y}) = (\bar{a}, \bar{b})$, so $\bar{a} \leq s_{(\bar{a}, \bar{b})}$. This shows that $r_{(\bar{a}, \bar{b})} \trianglelefteq_{H\mathbb{R}} \bar{a}$. By Corollary 7.11 and Observation 7.12, $\bar{a} \trianglelefteq_{H\mathbb{R}} r_{(\bar{a}, \bar{b})}$. □

The following is not used but may help the reader understand the structure of $M\mathbb{R}$.

Observation 7.18. *Let $\alpha, \beta \in M\mathbb{R}$ be such that $\alpha \diamond \beta$ but $\alpha \neq \beta$. Then there exist $\bar{a}, \bar{b} \in M$ and $r, s \in \mathbb{R}$ such that $\bar{a} \leq_M \bar{b}$ and $\{\alpha, \beta\} = \{\bar{a}, r_{(\bar{a}, \bar{b})}\}$ or $\{\alpha, \beta\} = \{r_{(\bar{a}, \bar{b})}, s_{(\bar{a}, \bar{b})}\}$.*

Proof. Say $\alpha >_{M\mathbb{R}} \beta$. We will show using only this assumption that there is an element $\gamma \in M$ such that $\alpha \sim \gamma$ but $\gamma \parallel \beta$, unless one of the two situations in the statement of the lemma holds.

Case 1. $\alpha = (a_0, \dots, a_m) \in M$.

Case 1a. $\beta = (b_0, \dots, b_n) \in M$. If $m < n$, then $a_0 = b_0$; $a_1 = b_1$; $\dots$; $a_{m-1} = b_{m-1}$; and $a_m \geq b_m$, so $\alpha > (a_0, a_1, \dots, a_{m-1}, b_m - 1) \parallel \beta$.

If $m = n$, then $a_0 = b_0$; $a_1 = b_1$; $\dots$; $a_{m-1} = b_{m-1}$; $a_m > b_m$; $\alpha > (a_0, \dots, a_m, 0) \parallel \beta$.

Case 1b. $\beta = s_{(\bar{x}, \bar{y})}$ where $\bar{x} \leq_M \bar{y}$. Then $\alpha \geq \bar{y}$. If $\alpha > \bar{y}$, then by the above, there exists $\gamma \in M$ such that $\alpha \sim \gamma$ and $\gamma \parallel \bar{y}$, and hence $\gamma \parallel \beta \leq \bar{y}$ (by Corollary 7.11 and Observation 7.12).

If $\alpha = \bar{y}$, then $\alpha > (a_0, \dots, a_m, 0) \parallel \bar{x}$ by Lemma 7.14. Hence, by Lemma 7.16, $(a_0, \dots, a_m, 0) \parallel \beta$.

Case 2. $\alpha = r_{(\bar{u}, \bar{v})}$ where $\bar{u} \leq_M \bar{v}$. Use the complete join-irreducibility of $(\bar{u}, \bar{v})$ in $\overline{M}$. If $\beta < \bar{u}$, then by the above and Lemma 7.16, there is an element comparable to $\bar{u}$ and α but not β . If $\beta = \bar{u}$, we are done. Otherwise, $\beta = s_{(\bar{u}, \bar{v})}$, and we are done. $\square$

Lemma 7.19. *Let C be a maximal chain in $M\mathbb{R}$. Then $C \cap M$ is a maximal chain in M . The map $C \mapsto C \cap M$ is a bijection.*

Proof. Let $m \in M$ be such that $m \sim c$ for all $c \in C \cap M$. Let $c \in C \setminus M$ —say $c = r_{(\bar{a}, \bar{b})}$. By Lemmas 7.14 and 7.16, $\bar{a} \in C$ so $m \sim \bar{a}$, and hence $m \sim c$. Thus $m \in C$ and hence $m \in C \cap M$. Obviously, any maximal chain D of M extends to a maximal chain C of $M\mathbb{R}$, so $D = C \cap M$.

Now suppose $C_1 \cap M = C_2 \cap M$. Let $r_{(\bar{a}, \bar{b})} \in C_1$. Then $\bar{a} \in C_1$ and hence $\bar{a} \in C_2$, so $r_{(\bar{a}, \bar{b})} \in C_2$, by Lemma 7.16. By symmetry, $C_1 = C_2$. $\square$

Notation 7.20. *Let $D \subseteq M\mathbb{R}$. Define $\Phi: D \rightarrow M$ as follows: For $d \in D$,*

$$\Phi(d) = \begin{cases} d & \text{if } d \in M, \\ \bar{a} & \text{if } d = r_{(\bar{a}, \bar{b})}. \end{cases}$$

Note that $d \diamond \Phi(d)$ by Lemma 7.16.

Observation 7.21. *Let Y be a maximal chain of $M\mathbb{R}$. Then $\Phi[Y] = Y \cap M$.*

Proof. Let $y \in Y$. Then $y \diamond \Phi(y)$, so $\Phi(y) \in Y$ and $\Phi(y) \in M$. Thus $\Phi[Y] \subseteq Y \cap M$. Also, if $y \in Y \cap M$, then $\Phi(y) = y$. Hence $Y \cap M \subseteq \Phi[Y]$. $\square$

Lemma 7.22. *Let D be a minimal cutset of $M\mathbb{R}$. Then Φ is one-to-one on D .*

Proof. Use Observation 7.1. $\square$

Lemma 7.23. *Let D be a minimal cutset of $M\mathbb{R}$. Then $\Phi[D]$ is a minimal cutset of M .*

Proof. Let X be a maximal chain of M . Extend it to a maximal chain Y of $M\mathbb{R}$. Then there exists $d \in D \cap Y$. Thus $d \diamond \Phi(d)$, so $\Phi(d) \in Y \cap M = X$. Thus $\Phi[D]$ is a cutset of M .

Now pick any $d \in D$. There exists a maximal chain Y of $M\mathbb{R}$ such that $Y \cap D = \{d\}$. By Observation 7.21 and Lemma 7.19, $\Phi[Y]$ is a maximal chain of M . Suppose $d' \in D$ and $\Phi(d') \in \Phi[Y] = Y \cap M$. Then $d' \diamond \Phi(d') \in Y$, so $d' \in Y$. Thus $d' \in Y \cap D = \{d\}$, so $d = d'$. Hence $\Phi[Y] \cap \Phi[D] = \{\Phi(d)\}$. $\square$

Theorem 7.24. *There is a complete densely ordered lattice $H\mathbb{R}$ with no uncountable antichains such that the only antichain cutsets are $\{0\}$ and $\{1\}$.*

Proof. By Corollary 7.15, $H\mathbb{R} \setminus \{0, 1\} = M\mathbb{R}$. We show $M\mathbb{R}$ has no antichain cutsets. If it had one, it would be a minimal cutset, as Maltby observes [10], page 391. By Lemma 7.23, M would have a minimal cutset, contradicting Higgs [8], page 373. The rest follows from Corollary 7.7, Proposition 7.13 and Lemma 7.8. $\square$

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