Ivan Chajda; Helmut Länger
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INTUITIONISTIC-LIKE UNSHARP IMPLICATION AND NEGATION DEFINED ON A POSET

Ivan Chajda, Helmut Länger

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Abstract. The aim of the present paper is to show that the concepts of the intuitionistic implication and negation formalized by means of a Heyting algebra can be generalized in such a way that these concepts are formalized by means of a bounded poset. In this case it is not assumed that the poset is relatively pseudocomplemented. The considered logical connectives negation, implication or even conjunction are not operations in this poset but so-called operators since they assign to given entries not necessarily an element of the poset as a result but a subset of mutually incomparable elements. We show that these operators for negation and implication can be characterized by several simple conditions formulated in the language of posets together with the operator of taking the lower cone. Moreover, our implication and conjunction form an adjoint pair. We call these connectives "unsharp" or "inexact" in accordance with the existing literature. We also introduce the concept of a deductive system of a bounded poset with implication and prove that it induces an equivalence relation satisfying a certain substitution property with respect to implication. Moreover, the restriction of this equivalence to the base set is uniquely determined by its kernel, i.e., the class containing the top element.

Keywords: bounded poset; logical connectives defined on a poset; unsharp negation; unsharp implication; adjoint operator; Modus Ponens; deductive system; equivalence relation induced by a deductive system

MSC 2020: 06A11, 06D15, 06D20, 03G25

1. Introduction

Intuitionistic logic was algebraically formalized by means of relatively pseudocomplemented semilattices, see [1], [2] and [14]–[17]. If such a semilattice is even a lattice,

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it is called a Heyting algebra, see [14] and [17]. It is well known that every relatively pseudocomplemented lattice is distributive. The concept of relative pseudocomplementation was extended by the first author to non-distributive lattices in [3] under the name sectional pseudocomplementation. This concept was further extended also for posets in [10]. Thus, some kind of intuitionistic connectives defined on sectionally pseudocomplemented lattices were realized.

However, there exist logics defined on bounded posets, e.g. the logic of quantum mechanics, see e.g. [7], [8], [13] and [18]. In particular, orthomodular posets on which the logic of quantum mechanics is defined are thoroughly studied in [7], [8], [11] and [18].

In order to formalize logical connectives defined on a poset, there are two possible ways how to solve the problem that the operations meet and join need not be defined everywhere. One method is to consider these operations as partial only and then the logical connectives formalized by them and by partial operations derived from them are also only partial. The disadvantage of this approach is that in some cases one does not know the result of a given implication. Hence, we prefer another approach, namely we consider so-called operators instead of operations which assign to given entries not necessarily an element as the result, but a certain subset of elements. We work with such results of logical connectives which are subsets, but their elements are mutually incomparable. It means that one cannot prefer one element of this set with respect to the other elements. Hence, such an approach defined on a poset gets a result to any implication, but this result may be "unsharp", see e.g. [9] or [13]. We suppose that if an exact logical derivation is impossible from the reasons mentioned above, it is better to have an unsharp result than none. In our opinion unsharp reasoning is an alternative to multiple-valued reasoning, which is now generally accepted though it was not accepted by all specialists at first.

In their recent paper [9] the authors showed that certain intuitionistic-like logical connectives implication and negation can be defined on arbitrary meet-semilattices satisfying the Ascending Chain Condition (ACC) regardless whether there exist (relative) pseudocomplements or not. We can ask if similar logical connectives may be derived also by means of arbitrary posets with 0 satisfying the ACC. Within meet-semilattices the logical connective conjunction is usually formalized by the meet operation. Then the unsharp implication as introduced in [9] forms an adjoint operator to conjunction. In the case of a poset we must find another operator formalizing conjunction. In [13] so-called unsharp properties of formal logics were used. In [9] we considered unsharpness of implication as well as of negation. This means that for given propositions p and q the results of the implication $p \to q$ and of the negation $\neg p$ need not be elements of the corresponding meet-semilattice $\mathbf{L}$, but may be nonempty subsets of it consisting of mutually incomparable elements. When proceeding from meet-semilattices to posets, we will apply this principle again for the connective conjunction in such a way that implication and conjunction will still be connected by a certain kind of adjointness, see e.g. [6]. Due to this fact, in such logics we still have the derivation rule Modus Ponens.

The aim of the present paper is to get a formal description of the logical connectives defined on a poset but not to constitute such a logic; it means we do not construct axioms and derivation rules or relations for possible Kripke frames.

2. Preliminaries

In the following we identify singletons with their unique element, i.e., we write x instead of $\{x\}$. Moreover, all posets considered in the sequel are assumed to satisfy the Ascending Chain Condition which we will abbreviate by ACC. This condition means that in the considered poset there do not exist infinite ascending chains. This implies that every element of a subset lies under a maximal element of this subset. Of course, every finite poset satisfies the ACC.

In the sequel we will use the following notation: Let $\mathbf{P} = (P, \leqslant)$ be a poset, $a, b, c \in P$ and A, B be nonempty subsets of P.

$$\begin{aligned}
\operatorname{Max} A &:= \text{ set of all maximal elements of } A,\\
L(a,b) &:= \{x \in P\colon\ x \leqslant a \text{ and } x \leqslant b\},\\
\Lambda(A,B) &:= \bigcup_{x\in A, y\in B} L(x,y),\\
A &\leqslant B \text{ if } x \leqslant y \text{ for all } x \in A \text{ and all } y \in B,\\
A &\leqslant_1 B \text{ if for every } x \in A \text{ there exists some } y \in B \text{ with } x \leqslant y,\\
A &=_1 B \text{ if both } A \leqslant_1 B \text{ and } B \leqslant_1 A.
\end{aligned}$$

The relations $\leqslant_1$ and $=_1$ are a quasiorder relation on 2^P and an equivalence relation on 2^P, respectively. (Here and in the following, 2^P denotes the power set of P.) It is easy to see that $A \leqslant_1 \operatorname{Max} B$ provided $A \subseteq B$ and that $A \leqslant_1 b$ is equivalent to $A \leqslant b$.

If a poset $\mathbf{P}$ has a bottom and a top element, we will denote them by 0 and 1, respectively, and we will express this fact by writing $\mathbf{P} = (P, \leqslant, 0, 1)$. Such a poset is called *bounded*.

The element c is called the *relative pseudocomplement* of a with respect to b, formally $c = a * b$, if c is the greatest element x of P satisfying $L(a,x) \leqslant b$. If $\mathbf{P} = (P, \leqslant, 0)$ then the relative pseudocomplement $a * 0$ of a with respect to 0 is denoted by a^* and called the *pseudocomplement* of a, see e.g. [4] or [12]. Especially, we have $L(a, a^*) = 0$.

3. Negation derived in posets

Consider a bounded poset $\mathbf{P} = (P, \leqslant, 0, 1)$ satisfying the ACC. For $a \in P$ we define

$$a^0 := \operatorname{Max}\{x \in P\colon\ L(a, x) = 0\}.$$

Clearly a^0 need not be a singleton, but it is a nonempty subset of P since $L(a, 0) = 0$. From now on, we will call a^0 the *unsharp negation* of a. Of course, if the pseudocomplement a^* of a exists, then a^0 is a singleton and hence $a^0 = \{a^*\}$, or shortly $a^0 = a^*$.

We extend this concept to subsets of P as follows: If A is a nonempty subset of P then

$$A^0 := \operatorname{Max}\{x \in P\colon\ L(x, y) = 0 \text{ for all } y \in A\}.$$

We are going to show that the negation ^{0} defined in this way shares several properties with the negation in intuitionistic logic.

Proposition 3.1. *Let* $\mathbf{P} = (P, \leqslant, 0, 1)$ *be a bounded poset satisfying the* ACC, $a, b \in P$ *and* A, B *nonempty subsets of* P. *Then the following holds:*

(i) A^0 *is an antichain, in particular,* a^0 *is an antichain,*
(ii) $0^0 = 1$ *and* $1^0 = 0$,
(iii) $L(x, y) = 0$ *for all* $x \in A$ *and all* $y \in A^0$, *in particular,* $L(a, y) = 0$ *for all* $y \in a^0$,
(iv) $A \leqslant_1 B$ *implies* $B^0 \leqslant_1 A^0$, *in particular,* $a \leqslant b$ *implies* $b^0 \leqslant_1 a^0$,
(v) *if* A *and* B *are antichains and* $A =_1 B$, *then* $A = B$,
(vi) $A \leqslant_1 A^{00}$, *in particular,* $a \leqslant_1 a^{00}$; $A^{000} = A^0$, *in particular,* $a^{000} = a^0$,
(vii) $\Lambda(a, (L(a, b))^0) =_1 \Lambda(a, b^0)$.

P r o o f. (i)–(iii) follow directly from the definition of ^{0}.

(iv) If $A \leqslant_1 B$, then

$$\{x \in P\colon\ L(x, y) = 0 \text{ for all } y \in B\} \subseteq \{x \in P\colon\ L(x, y) = 0 \text{ for all } y \in A\}.$$

(v) Assume $a \in A =_1 B$. Because $A \leqslant_1 B$ there exists some $b \in B$ with $a \leqslant b$, and because $B \leqslant_1 A$ there exists some $c \in A$ with $b \leqslant c$. Together we obtain $a \leqslant b \leqslant c$ and hence $a \leqslant c$. Since a and c belong to the antichain A, we conclude $a = c$ and therefore $a = b \in B$. This shows $A \subseteq B$. Interchanging the roles of A and B yields $B \subseteq A$. Together we obtain $A = B$.

(vi) We have $A \subseteq \{x \in P\colon L(x, y) = 0$ for all $y \in A^0\}$ and hence $A \leqslant_1 A^{00}$. Replacing A by A^0 we get $A^0 \leqslant_1 A^{000}$. From $A \leqslant_1 A^{00}$ and (iv) we obtain $A^{000} \leqslant_1 A^0$. Together we have $A^{000} =_1 A^0$, whence $A^{000} = A^0$ according to (v).

(vii) Any of the following statements implies the next one:

(1) $L(x, y) = 0$ for all $x \in (L(a, b))^0$ and all $y \in L(a, b)$,
(2) $L(b, y) = 0$ for all $x \in (L(a, b))^0$ and all $y \in L(a, x)$,
(3) $\Lambda(a, (L(a, b))^0) \leqslant_1 b^0$,
(4) $\Lambda(a, (L(a, b))^0) \leqslant_1 \Lambda(a, b^0)$.

The fact that (1) implies (2) can be seen as follows: If $x \in (L(a, b))^0$, $y \in L(a, x)$ and $c \in L(b, y)$, then $c \in L(a, b)$ and $c \leqslant x$ and therefore $L(x, c) = 0$ by (1), whence $L(c) = L(x, c) = 0$, i.e., $c = 0$. From $L(a, b) \leqslant b$ we conclude $b^0 \leqslant_1 (L(a, b))^0$ according to (iv) and hence

$$\Lambda(a, b^0) \leqslant_1 \Lambda(a, (L(a, b))^0).$$

□

Example 3.2. Consider the bounded poset $\mathbf{P} = (P, \leqslant, 0, 1)$ visualized in Fig. 1:

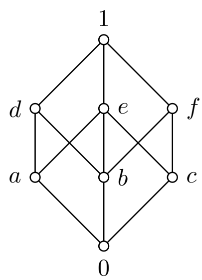

Figure 1. Bounded non-lattice poset with unsharp negation.

We have

x	0	a	b	c	d	e	f	1
x^0	1	f	ac	d	c	0	a	0
x^{00}	0	a	b	c	d	1	f	1

(Here and in the following we write $a_1 \ldots a_n$ instead of $\{a_1, \ldots, a_n\}$.) One can see that $x^{00} = x$ for all $x \in P \setminus \{e\}$ and $e^{00} = 1 \neq e$. But $e^{000} = 1^0 = 0 = e^0$ and hence $\mathbf{P}$ satisfies the identity $x^{000} \approx x^0$ in accordance with (vi) of Proposition 3.1.

The following example shows that (iv) of Proposition 3.1 does not hold for $\leqslant$ instead of $\leqslant_1$.

E x a m p l e 3.3. Consider the bounded poset $\mathbf{P}=(P,\leqslant,0,1)$ depicted in Fig. 2:

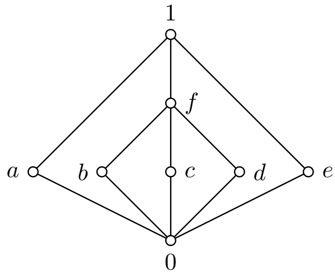

Figure 2. Bounded poset with unsharp negation.

We have

x	0	a	b	c	d	e	f	1
x^0	1	ef	$acde$	$abde$	$abce$	af	ae	0
x^{00}	0	a	b	c	d	e	f	1

and hence, $c\leqslant f$ and $f^0\leqslant_1 c^0$, but $f^0=\{a,e\}\nleqslant\{a,b,d,e\}=c^0$.

The next question is whether the unary operator ^{0} can be characterized by means of simple conditions formulated in the language of posets. This is possible, see the following theorem.

Theorem 3.4. *Let* $(P,\leqslant,0,1)$ *be a bounded poset satisfying the* ACC *and* ^{0} *a unary operator on* P*. Then the following conditions* (1) *and* (2) *are equivalent:*

(1) $x^0=\operatorname{Max}\{y\in P\colon\ L(x,y)=0\}$ *for all* $x\in P$*,*

(2) *the operator* $^0\colon\ 2^P\to 2^P$ *satisfies the following conditions for all* $x\in P$*:*

 (P1) x^0 *is an antichain,*

 (P2) $L(x,y)=0$ *for all* $y\in x^0$*,*

 (P3) $\Lambda(x,(L(x,y))^0)=_1\Lambda(x,y^0)$*.*

P r o o f. (1) $\Rightarrow$ (2): This follows from Proposition 3.1.

(2) $\Rightarrow$ (1): If $L(x,y)=0$, then according to (P3) we have

$$y=_1 L(y,1)=\Lambda(y,(L(x,y))^0)=\Lambda(y,(L(y,x))^0)=_1\Lambda(y,x^0)\leqslant_1 x^0$$

and hence $y\leqslant_1 x^0$. Conversely, if $y\leqslant_1 x^0$, then according to (P2) we get

$$L(x,y)\subseteq\Lambda(x,x^0)=0,$$

which implies $L(x, y) = 0$. This shows that $L(x, y) = 0$ is equivalent to $y \leqslant_1 x^0$. We conclude

$$\operatorname{Max}\{y \in P\colon\ L(x, y) = 0\} = \operatorname{Max}\{y \in P\colon\ y \leqslant_1 x^0\} = x^0.$$

The last equality can be seen as follows. Since both $\operatorname{Max}\{y \in P\colon\ y \leqslant_1 x^0\}$ and x^0 are antichains, it suffices to show $\operatorname{Max}\{y \in P\colon\ y \leqslant_1 x^0\} =_1 x^0$ according to (v) of Proposition 3.1. We have $\operatorname{Max}\{y \in P\colon\ y \leqslant_1 x^0\} \leqslant_1 x^0$ by definition, and $x^0 \subseteq \operatorname{Max}\{y \in P\colon\ y \leqslant_1 x^0\}$ implies $x^0 \leqslant_1 \operatorname{Max}\{y \in P\colon\ y \leqslant_1 x^0\}$. □

4. Unsharp implication and conjunction

Let us recall that Brouwerian semilattices are relatively pseudocomplemented meet-semilattices, see e.g. [15] and [16]. It is known that in logics defined on Brouwerian semilattices or on Heyting algebras the relative pseudocomplement is considered as the logical connective implication, i.e.,

$$x \to y = x * y = \operatorname{Max}\{z \in S\colon\ x \wedge z \leqslant y\}.$$

Of course, $\operatorname{Max}\{z \in S\colon\ x \wedge z \leqslant y\}$ is a singleton, thus $x * y$ is an element of S. In our case we will use formally the same definition, but the result $x \to y$ need not be an element of the poset $\mathbf{P}$ in question, it may be a subset of P in general. However, the elements of $x \to y$ are mutually incomparable. Hence, one cannot pick one of them to be the preferable element. Now we are going to define our main concept, i.e., the operator $\to$ which formalizes the logical connective implication. As mentioned above, for given entries x and y the result of $x \to y$ may be a subset of P. Due to this, if we combine this operator in various formulas, we must define its value also for entries which are subsets. We define:

Definition 4.1. Let $\mathbf{P} = (P, \leqslant, 0, 1)$ be a bounded poset satisfying the ACC, $a, b \in P$ and A, B nonempty subsets of P. Then

(I) $a \to b := \operatorname{Max}\{x \in P\colon\ L(a, x) \leqslant b\}$, $A \to B := \operatorname{Max}\{y \in P\colon\ L(x, y) \leqslant_1 B$ for all $x \in A\}$.

(C) $a \odot b := \operatorname{Max} L(a, b)$, $A \odot B := \operatorname{Max} \Lambda(A, B)$.

Of course, if $\mathbf{P}$ is a lattice, then $a \odot b = a \wedge b$. It is evident that $a \to 0 = a^0$ as usual in intuitionistic logic. In order to estimate how reasonable our definition of implication is, we can verify its relationship with conjunction. In the next proposition we show that these two unsharp logical connectives are related as follows:

(AD) $x \odot y \leqslant z$ if and only if $x \leqslant_1 y \to z$
or, even more general,

$$A \odot y \leqslant z \text{ if and only if } A \leqslant_1 y \to z$$

for every nonempty subset A of P.

This is a variant of adjointness of the operators $\odot$ and $\to$. Hence, the connectives introduced before seem to be sound. We list some of their properties. It is evident that $\odot$ is commutative and $x \odot 1 \approx 1 \odot x \approx x$.

Moreover, from $x \to y = x \to y$ we infer by (AD)

$$(x \to y) \odot x \leqslant_1 y,$$

which is a kind of the Modus Ponens derivation rule in intuitionistic logic. Namely, it says that from x and $x \to y$ we can derive y despite the fact that $x \to y$ may consist of a number of propositions.

For the operator $\to$ we prove the following result.

Proposition 4.2. *Let* $\mathbf{P} = (P, \leqslant, 0, 1)$ *be a bounded poset satisfying the* ACC, $a, b, c \in P$ *and* A, B *nonempty subsets of* P. *Then the following holds:*

(i) $a \to b$ *is an antichain,*

(ii) $b \leqslant_1 a \to b$,

(iii) $a \leqslant_1 (a \to b) \to b$,

(iv) $a \leqslant b$ *implies* $c \to a \leqslant_1 c \to b$ *and* $b \to c \leqslant_1 a \to c$,

(v) $\Lambda(a, a \to b) = L(a, b)$,

(vi) $\Lambda(a \to b, b) =_1 b$,

(vii) $1 \to A = \mathrm{Max} A$, especially $1 \to a = a$,

(viii) $A \to B = 1$ if and only if $A \leqslant_1 B$; especially $a \to b = 1$ if and only if $a \leqslant b$,

(ix) $A \odot b \leqslant c$ if and only if $A \leqslant_1 b \to c$,

(x) $a \odot (a \to b) = a \odot b$.

P r o o f. (i), (ii), (vii) and (ix) follow directly from the definition of $\to$.

(iii) follows from $L(x, a) = L(a, x) \leqslant b$ for all $x \in a \to b$.

(iv) holds since $a \leqslant b$ implies

$$\{x \in P\colon\ L(c, x) \leqslant a\} \subseteq \{x \in P\colon\ L(c, x) \leqslant b\},$$
$$\{x \in P\colon\ L(b, x) \leqslant c\} \subseteq \{x \in P\colon\ L(a, x) \leqslant c\}.$$

(v) If $c \in \Lambda(a, a \to b)$, then there exists some $d \in a \to b$ with $c \in L(a, d)$. Since $L(a, d) \leqslant b$, we have $c \in L(a, b)$. Conversely, assume $c \in L(a, b)$. Since $L(a, b) \leqslant b$, there exists some $d \in a \to b$ with $b \leqslant d$. Now $c \in L(a, d) \subseteq \Lambda(a, a \to b)$.

(vi) Since $L(a,b) \leqslant b$, there exists some $c \in a \to b$ with $b \leqslant c$. Now $b \in L(c,b) \subseteq \Lambda(a \to b, b)$ showing $b \leqslant_1 \Lambda(a \to b, b)$. Of course, $\Lambda(a \to b, b) \leqslant_1 b$.

(viii) The following are equivalent: $A \to B = 1$, $L(x,1) \leqslant_1 B$ for all $x \in A$, $A \leqslant_1 B$.

(x) According to (v) we have

$$a \odot (a \to b) = \operatorname{Max} \Lambda(a, a \to b) = \operatorname{Max} L(a,b) = a \odot b.$$

□

We can see that our operator $\to$ shares a lot of properties with the intuitionistic implication. In particular, our $\to$ is antitone in the first and monotone in the second entry.

Example 4.3. Consider the bounded poset $(\{0,a,b,c,d,e,1\}, \leqslant, 0, 1)$ visualized in Fig. 3:

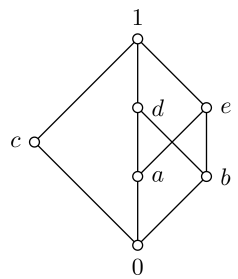

Figure 3. Bounded poset with unsharp implication and conjunction.

The operator tables of $\to$ and $\odot$ are as follows:

$\to$	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	bc	1	bc	bc	1	1	1
b	ac	ac	1	ac	1	1	1
c	de	de	de	1	de	de	1
d	c	ac	bc	c	1	ce	1
e	c	ac	bc	c	cd	1	1
1	0	a	b	c	d	e	1

$\odot$	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	a	0	0	a	a	a
b	0	0	b	0	b	b	b
c	0	0	0	c	0	0	c
d	0	a	b	0	d	ab	d
e	0	a	b	0	ab	e	e
1	0	a	b	c	d	e	1

Now we characterize the binary operator $\to$ in a similar way as it was done for the unary operator ^{0} in Theorem 3.4.

Theorem 4.4. *Let* $\mathbf{P} = (P, \leqslant, 0, 1)$ *be a bounded poset satisfying the* ACC *and* $\to$ *a binary operator on* P*. Then the following conditions* (1) *and* (2) *are equivalent:*

(1) $x \to y = \operatorname{Max}\{z \in P\colon L(x,z) \leqslant y\}$ *for all* $x, y \in P$,

(2) *the operator* $\to\colon 2^P \times 2^P \to 2^P$ (*restricted to* $P \times P$) *satisfies the following conditions for all* $x, y, z \in P$:

(R1) $x \to y$ *is an antichain*,

(R2) $L(x,z) \leqslant y$ *implies* $z \leqslant_1 x \to y$,

(R3) $\Lambda(x, x \to y) = L(x,y)$.

Proof. (1) $\Rightarrow$ (2): This follows from Proposition 4.2.

(2) $\Rightarrow$ (1): According to (R2) we have that $L(x,z) \leqslant y$ implies $z \leqslant_1 x \to y$. Conversely, if $z \leqslant_1 x \to y$, then by (R3) we obtain

$$L(x,z) \subseteq \Lambda(x, x \to y) = L(x,y) \leqslant y$$

and hence $L(x,z) \leqslant y$. This shows that $L(x,z) \leqslant y$ is equivalent to $z \leqslant_1 x \to y$. We conclude

$$\operatorname{Max}\{z \in S\colon L(x,z) \leqslant y\} = \operatorname{Max}\{z \in S\colon z \leqslant_1 x \to y\} = x \to y.$$

The last equality can be seen as follows. Both $\operatorname{Max}\{z \in S\colon z \leqslant_1 x \to y\} \leqslant_1 x \to y$ and $x \to y \leqslant_1 \operatorname{Max}\{z \in S\colon z \leqslant_1 x \to y\}$ follow immediately. This gives $\operatorname{Max}\{z \in S\colon z \leqslant_1 x \to y\} =_1 x \to y$. Since both sides are antichains, we find $\operatorname{Max}\{z \in S\colon z \leqslant_1 x \to y\} = x \to y$ according to (v) of Proposition 3.1. $\square$

Let us note that using both (R2) and (R3) we can prove the adjointness (AD) mentioned above.

Remark 4.5. It is of some interest that the unsharp operator $\to$ can be characterized by three exact and simple conditions in the language of posets equipped with the operator L of the lower cone.

5. Deductive systems

As mentioned in Section 4, we can derive the rule Modus Ponens. This rule is in fact closely related to the concept of a deductive system. Recall that in a Heyting algebra $\mathbf{A} = (A, \vee, \wedge, {}^*, 0, 1)$ a non-void subset D of A is called a deductive system if $x \in D$ and $x \to y \in D$ imply $y \in D$. The connection to Modus Ponens rule is evident. However, our unsharp connective implication is weaker than that in a Heyting algebra and hence we define our concept as follows. We show that it satisfies what is asked in a Heyting algebra and, moreover, this deductive system induces an equivalence relation on the poset in question, which is compatible with the logical connectives.

Definition 5.1. Let $\mathbf{P} = (P, \leqslant, 0, 1)$ be a bounded poset satisfying the ACC and let $\to$ be defined by (I) from Definition 4.1. A *deductive system* of $\mathbf{P}$ is a subset D of $2^P \setminus \{\emptyset\}$ satisfying the following conditions:

(i) $1 \in D$,

(ii) if $x, y, z, u \in P$, $x \to y \in D$ and $(x \to y) \to (z \to u) \in D$, then $z \to u \in D$,

(iii) if $x, y, z \in P$ and $x \to y, y \to x \in D$, then $(z \to x) \to (z \to y) \in D$ and $(x \to z) \to (y \to z) \in D$.

In the following we make use of the identities $x \odot 1 \approx 1 \odot x \approx x$ and $1 \to x \approx x$ (see Section 4). ($x \odot 1 \approx x$ means that $x \odot 1 = x$ for all x belonging to the corresponding base set.)

Remark 5.2. If $x \in D$, $y \in P$ and $x \to y \in D$, then according to (vii) of Proposition 4.2 we have $1 \to x = x \in D$ and $(1 \to x) \to (1 \to y) = x \to y \in D$ and hence because of (ii) of Definition 5.1 we get $y = 1 \to y \in D$. Therefore, $x \in D$ and $x \to y \in D$ imply $y \in D$ which justifies the name “deductive system” and illuminates the connection of such systems with the derivation rule Modus Ponens.

Example 5.3. Consider the bounded poset $\mathbf{P} = (P, \leqslant, 0, 1)$ depicted in Fig. 4:

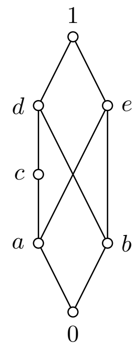

Figure 4. Bounded non-lattice poset.

The operator table for $\to$ is as follows:

$\to$	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	b	1	b	1	1	1	1
b	c	c	1	c	1	1	1
c	b	e	b	1	1	e	1
d	0	a	b	c	1	e	1
e	0	c	b	c	d	1	1
1	0	a	b	c	d	e	1

We want to show that $D := \{d, e, 1\}$ is a deductive system of $\mathbf{P}$. (The following considerations make heavy use of the operation table for $\to$.) In order to make the proof of this statement short and clear we define two subsets A and B of P^2 as follows:

$$A := \{(c,a),(c,e)\} \cup D^2 \cup \{(x,y) \in P^2 \colon\ x \leqslant y\},$$
$$B := \{a,c\}^2 \cup D^2 \cup \{(x,x) \colon\ x \in P\}.$$

Now let $x, y, z \in P$. Then we have

$$\begin{aligned}
&x \to y \in D \text{ if and only if } (x,y) \in A,\\
&x \to y, y \to x \in D \text{ if and only if } (x,y) \in B,\\
&1 \in D \text{ according to the definition of } D,\\
&\text{if } x \in D \text{ and } (x,y) \in A, \text{ then } y \in D,\\
&\text{if } (x,y) \in B, \text{ then } (z \to x, z \to y), (x \to z, y \to z) \in B
\end{aligned}$$

completing the proof that D is a deductive system of $\mathbf{P}$.

Since in posets we do not have everywhere defined lattice operations join and meet, we formulate a certain kind of compatibility in the following way.

Definition 5.4. Let $(P, \leqslant, 0, 1)$ be a bounded poset satisfying the ACC and Φ be an equivalence relation on $2^P \setminus \{\emptyset\}$. We say that Φ satisfies the *substitution property with respect to* $\odot$ if for all $x, y \in P$ we have

$$1 \;\Phi\; x \to y \text{ implies } x \odot 1 \;\Phi\; x \odot (x \to y).$$

We say that Φ satisfies the *substitution property with respect to* $\to$ if for all $x, y, z, u \in P$ we have

$$\begin{aligned}
x \;\Phi\; y &\text{ implies } x \to x \;\Phi\; x \to y,\\
x \;\Phi\; y &\text{ implies } (z \to x) \to (z \to x) \;\Phi\; (z \to x) \to (z \to y),\\
x \;\Phi\; y &\text{ implies } (x \to z) \to (x \to z) \;\Phi\; (x \to z) \to (y \to z),\\
1 \;\Phi\; x \to y &\text{ implies } 1 \to (z \to u) \;\Phi\; (x \to y) \to (z \to u).
\end{aligned}$$

If an equivalence relation on $2^P \setminus \{\emptyset\}$ satisfies the substitution property with respect to $\odot$ and $\to$, then we can easily describe the relationship to its kernel.

In the following, if Φ is an equivalence relation on a set A and $a \in A$, then $[a]\Phi$ denotes the equivalence class of a with respect to Φ.

Lemma 5.5. *Let $(P, \leqslant, 0, 1)$ be a bounded poset satisfying the* ACC*, Φ an equivalence relation on $2^P \setminus \{\emptyset\}$ satisfying the substitution property with respect to $\odot$ and $\to$, and $a, b \in P$. Then $(a, b) \in \Phi$ if and only if $a \to b, b \to a \in [1]\Phi$.*

P r o o f. If $(a,b)\in\Phi$, then

$$a\to b\in[a\to a]\Phi=[1]\Phi,\quad b\to a\in[b\to b]\Phi=[1]\Phi.$$

If, conversely, $a\to b, b\to a\in[1]\Phi$, then according to (x) of Proposition 4.2 we have

$$a=a\odot 1\;\Phi\;a\odot(a\to b)=a\odot b=b\odot a=b\odot(b\to a)\;\Phi\;b\odot 1=b.$$

$\square$

When studying congruences in varieties of algebras, an important congruence property is the so-called *weak regularity*. It means that if the algebra in question has a constant 1 and if for two of its congruences Φ and Ψ we have $[1]\Phi=[1]\Psi$ then $\Phi=\Psi$, see e.g. [5]. Surprisingly, we obtain a similar result for posets and equivalence relations having the substitution property with respect to $\odot$ and $\to$. In fact, this is a consequence of Lemma 5.5.

Corollary 5.6. *If $(P,\leqslant,0,1)$ is a bounded poset satisfying the* ACC, *Φ,Ψ are equivalence relations on $2^P\setminus\{\emptyset\}$ satisfying the substitution property with respect to $\odot$ and $\to$, and $[1]\Phi=[1]\Psi$, then $\Phi\cap P^2=\Psi\cap P^2$.*

For varieties of algebras, the mentioned weak regularity was characterized by Csákány, see e.g. [5], by means of certain binary terms satisfying a simple condition. This is not possible for posets, but we can show such a term, see the following proposition. Unfortunately, this term is not connected with the result presented in Corollary 5.6.

Proposition 5.7. *Let $(P,\leqslant,0,1)$ be a bounded poset satisfying the* ACC, *put $t(x,y):=(x\to y)\odot(y\to x)$ for all $x,y\in P$ and let $a,b\in P$. Then $t(a,b)=1$ if and only if $a=b$.*

P r o o f. According to (viii) of Proposition 4.2 the following are equivalent:

$$t(a,b)=1,$$
$$(a\to b)\odot(b\to a)=1,$$
$$\operatorname{Max}\Lambda(a\to b,b\to a)=1,$$
$$1\in\Lambda(a\to b,b\to a),$$
there exists some $x\in a\to b$ and some $y\in b\to a$ with $1\in L(x,y)$,

$$\text{there exists some } x \in a \to b \text{ and some } y \in b \to a \text{ with } x = y = 1,$$
$$1 \in (a \to b) \cap (b \to a),$$
$$a \to b = b \to a = 1,$$
$$a \leqslant b \leqslant a,$$
$$a = b.$$

□

We are going to show that the kernel of an equivalence relation satisfying the substitution property with respect to $\odot$ and $\to$ is just a deductive system.

Theorem 5.8. *Let $\mathbf{P} = (P, \leqslant, 0, 1)$ be a bounded poset satisfying the* ACC *and Φ an equivalence relation on $2^P \setminus \{\emptyset\}$ satisfying the substitution property with respect to $\odot$ and $\to$. Then $[1]\Phi$ is a deductive system of $\mathbf{P}$.*

Proof. We check (i)-(iii) of Definition 5.1. Let $a, b, c, d \in P$.

(i) This is clear.

(ii) If $a \to b, (a \to b) \to (c \to d) \in [1]\Phi$, then according to (vii) of Proposition 4.2 we have

$$c \to d = 1 \to (c \to d) \in [(a \to b) \to (c \to d)]\Phi = [1]\Phi.$$

(iii) If $a \to b, b \to a \in [1]\Phi$, then $(a, b) \in \Phi$ according to Lemma 5.5 and hence

$$(c \to a) \to (c \to b) \in [(c \to a) \to (c \to a)]\Phi = [1]\Phi,$$
$$(a \to c) \to (b \to c) \in [(a \to c) \to (a \to c)]\Phi = [1]\Phi.$$

□

The question if a given deductive system D on a bounded poset $(P, \leqslant, 0, 1)$ satisfying the ACC induces an equivalence relation on $2^P \setminus \{\emptyset\}$ with kernel $D \cap P$ having a property similar to the substitution property with respect to $\to$, is answered in the next result. At first, we define the relation induced by D.

Definition 5.9. For every bounded poset $(P, \leqslant, 0, 1)$ satisfying the ACC and every subset E of $2^P \setminus \{\emptyset\}$ define a binary relation $\Theta(E)$ on $2^P \setminus \{\emptyset\}$ as follows:

$$(A, B) \in \Theta(E) \text{ if and only if } A \to B, B \to A \in E$$

(A, B nonempty subsets of P). We call $\Theta(E)$ the *relation induced* by E.

In Theorem 5.8 we proved that for every equivalence relation Φ on $2^P \setminus \{\emptyset\}$ satisfying the substitution property with respect to $\odot$ and $\rightarrow$, its kernel $[1]\Phi$ is a deductive system of $\mathbf{P}$. The next theorem shows that there holds a converse version of this result if we consider the restriction of the relation induced by a deductive system of $\mathbf{P}$ to the base set P.

Theorem 5.10. *Let* $\mathbf{P} = (P, \leqslant, 0, 1)$ *be a bounded poset satisfying the* ACC *and* D *a deductive system of* $\mathbf{P}$. *Then the following hold:*

(i) $\Theta(D) \cap P^2$ *is an equivalence relation on* P,

(ii) $[1](\Theta(D) \cap P^2) = D \cap P$,

(iii) *if* $x, y, z \in P$ *and* $(x, y) \in \Theta(D)$, *then* $(z \rightarrow x, z \rightarrow y) \in \Theta(D)$ *and* $(x \rightarrow z, y \rightarrow z) \in \Theta(D)$.

P r o o f. Let $a, b, c \in P$.

(i) Evidently, $\Theta(D)$ is reflexive and symmetric. If $(a, b), (b, c) \in \Theta(D)$, then

$$b \rightarrow c, (b \rightarrow c) \rightarrow (a \rightarrow c), c \rightarrow b, (c \rightarrow b) \rightarrow (c \rightarrow a) \in D$$

and hence, $a \rightarrow c, c \rightarrow a \in D$ according to (ii) of Definition 5.1., i.e., $(a, c) \in \Theta(D)$. This shows that $\Theta(D)$ is transitive and therefore an equivalence relation on P.

(ii) The following are equivalent: $a \in [1](\Theta(D))$; $a \rightarrow 1, 1 \rightarrow a \in D$; $1, a \in D$; $a \in D$.

(iii) follows from Definition 5.1. $\square$

E x a m p l e 5.11. For the deductive system D from Example 5.3 we have

$$\Theta(D) \cap P^2 = D^2 \cup \{a, c\}^2 \cup \{b\}^2 \cup \{0\}^2.$$

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Authors' addresses: *Ivan Chajda*, Palacký University Olomouc, Faculty of Science, Department of Algebra and Geometry, 17. listopadu 12, 771 46 Olomouc, Czech Republic, e-mail: `ivan.chajda@upol.cz`; *Helmut Länger* (corresponding author), TU Wien, Fakultät für Mathematik und Geoinformation, Institut für Diskrete Mathematik und Geometrie, Wiedner Hauptstraße 8-10, 1040 Wien, Austria, and Palacký University Olomouc, Faculty of Science, Department of Algebra and Geometry, 17. listopadu 12, 771 46 Olomouc, Czech Republic, e-mail: `helmut.laenger@tuwien.ac.at`.