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CONFORMAL QUASI-HEMI-SLANT $\xi^\perp$ -RIEMANNIAN SUBMERSIONS FROM SASAKIAN MANIFOLDS

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Abstract. We introduce some geometric properties of a horizontally conformal quasi-hemi-slant Riemannian submersion from a Sasakian manifold, normal to the characteristic vector field, supported by an example. Under some conditions, we obtain geometric configurations of fibres and the base manifold of such submersions. We also give a characterization theorem for the proper horizontal conformal quasi-hemi-slant Riemannian submersions with totally umbilical fibres.

Keywords: conformal submersion; slant submersion; quasi-hemi-slant submersion; Sasakian manifold

MSC 2020: 53C15, 53C25

1. INTRODUCTION

In differential geometry, the concepts of immersion and submersion are special tools and play important roles in Riemannian geometry and other related topics. In [7], for instance, Fischer introduced Riemannian maps between Riemannian manifolds as a unification and generalization of the notions of isometric immersions and Riemannian submersions. The latter theory was initiated by Gray [8] and O’Neil [12]. Riemannian submersion is in fact the dual notion of isometric and this was defined by O’Neill and he obtained fundamental equations such as Gauss, Codazzi, Ricci equations, etc. Riemannian submersions have been studied by several authors (see [1]–[21], for more details and references therein) from almost Hermitian, almost contact manifolds, etc. Watson in [21] considered Riemannian submersions between almost Hermitian manifolds, named almost Hermitian submersion, in which the underlying submersion is an almost complex map.

In this paper, we introduce and study a new class of conformal submersions including the ones of invariant, anti-invariant and slant submersions. We particularly pay attention to those from almost contact manifolds in which the characteristic vector field is normal. This submersion generalizes the quasi-hemi-slant Riemannian submersion and was first introduced in [10] from almost Hermitian manifolds. In general, conformal submersions are studied, for instance, in [1], [9], [11], [14] and [18] and references therein. In [11], Mustafa obtained a characterization of totally geodesic horizontally conformal maps via Bochner techniques for harmonic morphisms. This result led to the existence of non-constant harmonic morphism from a compact Riemannian manifold of non-negative Ricci curvature to a compact Riemannian.

The paper is organized as follows. Section 2 is devoted to some basic definitions and properties to be used in the underlying submersions. In Section 3, we introduce the new class of conformal Riemannian submersions. We also investigate the geometry of leaves of the vertical and horizontal distributions of a conformal quasi-hemi-slant Riemannian submersion, including the decomposition theorems of the distributions. We finally give, in Section 5, a characterization theorem for such submersions being proper with totally umbilical fibres. We also prove that, under a certain condition, there exist no horizontal conformal quasi-hemi-slant $\xi^\perp$ -Riemannian submersions with totally geodesic fibres.

2. PRELIMINARIES

Let M be a $(2n + 1)$ -dimensional manifold endowed with an almost contact structure (ϕ, ξ, η) , i.e., ϕ is a tensor field of type $(1, 1)$, ξ is a vector field, and η is a 1-form satisfying

$$(2.1) \quad \phi^2 = -\mathbb{I} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0 \quad \text{and} \quad \phi\xi = 0.$$

Then (ϕ, ξ, η, g) is called an almost contact metric structure on M if (ϕ, ξ, η) is an almost contact structure on M and g is a Riemannian metric on M such that for any vector fields X and Y on M ,

$$(2.2) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

which implies that $\eta(X) = g(\xi, X)$. Moreover, an almost contact metric manifold M is said to be a Sasakian manifold if and only if [4], Theorem 6.3

$$(2.3) \quad (\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X$$

for any vector fields X and Y on M , which implies $\nabla_X \xi = -\phi X$. Here ∇ is the Levi-Civita connection on M with respect to g .

Throughout this note, manifolds are assumed to be of class $\mathcal{C}^\infty$ and connected, and all tensor fields are of class $\mathcal{C}^\infty$. We will denote the $\mathcal{F}(M)$ module of smooth sections of a vector bundle Ξ with $\Gamma(\Xi)$.

Let (M, g_M) and (N, g_N) be two smooth Riemannian manifolds, and $F: M \rightarrow N$ be a surjective smooth map between the two manifolds. Then F is a smooth submersion if it has maximal rank at any point of M .

Let $\mathcal{C}_F := \{x \in M: \text{rank } F_{*x} < \dim N\}$ be its critical point set. The points of the set $M \setminus \mathcal{C}_F$ are called regular points. For each $x \in M \setminus \mathcal{C}_F$, the *vertical space* $\mathcal{V}_x$ at x is defined as the kernel of the differential of F at x , i.e., $\mathcal{V}_x = \ker F_{*x}$. This leads to an integrable distribution $\mathcal{V}$, which is called *vertical distribution* and corresponds to the foliation of M determined by the fibres of F . The *horizontal distribution* $\mathcal{H} = (\ker F_*)^\perp$ is given by the orthogonal complement of $\mathcal{V}$ so that the tangent bundles splits as $TM = \mathcal{V} \oplus \mathcal{H}$. The smooth submersion F is called a Riemannian submersion if, at each point x of M , F_{*x} preserves the length of the horizontal vectors. A horizontal vector field X on M is said to be basic if X is F -related to a vector field X' on N . It is clear that every vector field X' on N has a unique horizontal lift X to M and X is basic.

It is well known that for a Riemannian submersion $F: (M, g_M) \rightarrow (N, g_N)$, there are two (1, 2)-O'Neill tensors fields $\mathcal{T}$ and $\mathcal{A}$ on (M, g_M) defined by the formulas

$$(2.4) \quad \mathcal{T}_E F = \mathcal{H} \nabla_{\mathcal{V}E} \mathcal{V}F + \mathcal{V} \nabla_{\mathcal{V}E} \mathcal{H}F,$$

$$(2.5) \quad \mathcal{A}_E F = \mathcal{V} \nabla_{\mathcal{H}E} \mathcal{H}F + \mathcal{H} \nabla_{\mathcal{H}E} \mathcal{V}F$$

for any vector fields E, F on M , where ∇ is the Levi-Civita connection of g_M . It is easy to see that $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators on the tangent bundle of M reversing the vertical and the horizontal distributions. We summarize the properties of the tensor fields $\mathcal{T}$ and $\mathcal{A}$. From (2.4) and (2.5) we have

$$(2.6) \quad \nabla_X Y = \mathcal{T}_X Y + \mathcal{V} \nabla_X Y,$$

$$(2.7) \quad \nabla_X V = \mathcal{H} \nabla_X V + \mathcal{T}_X V,$$

$$(2.8) \quad \nabla_V X = \mathcal{A}_V X + \mathcal{V} \nabla_V X,$$

$$(2.9) \quad \nabla_V W = \mathcal{H} \nabla_V W + \mathcal{A}_V W$$

for any $X, Y \in \Gamma(\ker F_*)$ and $V, W \in \Gamma(\ker F_*)^\perp$, where $\mathcal{H} \nabla_X V = \mathcal{A}_V X$ if V is basic. The covariant derivatives of φ and ω in (3.7) are defined as

$$(2.10) \quad (\nabla_X \varphi)Y = \mathcal{V} \nabla_X \varphi Y - \varphi \mathcal{V} \nabla_X Y,$$

$$(2.11) \quad (\nabla_X \omega)Y = \mathcal{H} \nabla_X \omega Y - \omega \mathcal{H} \nabla_X Y,$$

$$(2.12) \quad (\nabla_V \mathcal{B})W = \mathcal{V} \nabla_V \mathcal{B}W - \mathcal{B} \mathcal{H} \nabla_V W,$$

$$(2.13) \quad (\nabla_V \mathcal{C})W = \mathcal{H} \nabla_V \mathcal{B}W - \mathcal{C} \mathcal{H} \nabla_V W$$

for any $X, Y \in \Gamma(\ker F_*)$ and $V, W \in \Gamma(\ker F_*)^\perp$. The tensor $\mathcal{T}$ is symmetric on the vertical distribution and $\mathcal{T}_X Y$ coincides with the second fundamental form of the immersion of the fibre submanifolds. In particular, $\mathcal{T} = 0$ if and only if any fibre of F is a totally geodesic submanifold of M .

Let $F: M \rightarrow N$ be a smooth map between the two manifolds. Then the second fundamental form of F is given by

$$(2.14) \quad \nabla F_*(X, Y) = \nabla_X^F F_*(Y) - F_*(\nabla_X Y)$$

for any vector fields X and $Y \in \Gamma(TM)$, where ∇^F is the pullback connection along F and ∇ denotes conveniently the Levi-Civita connections of the metrics g_M and g_N . It is known that the second fundamental form is symmetric (see [14] for more details and reference therein), and F is a totally geodesic map if $\nabla F_*(X, Y) = 0$ for all $X, Y \in \Gamma(TM)$.

Definition 2.1 ([9]). Let F be a smooth submersion from a Riemannian manifold (M, g_M) to a Riemannian manifold (N, g_N) . Then F is called a *horizontally conformal submersion* if there is a positive function $\lambda: M \setminus \mathcal{C}_F \rightarrow \mathbb{R}^+$ such that

$$(2.15) \quad \lambda^2 g_M(X, Y) = g_N(F_* X, F_* Y)$$

for every $X, Y \in \Gamma(\mathcal{H})$. The function λ is then extended to the whole of M by putting $\lambda|_{\mathcal{C}_F} \equiv 0$. The extended function $\lambda: M \rightarrow \mathbb{R}^+ \cup 0$ is called the *dilation* of F .

It is easy to see that every Riemannian submersion is a particular horizontally conformal submersion with $\lambda = 1$. We recall the following.

Lemma 2.1 ([1]). Let $F: (M, g_M) \rightarrow (N, g_N)$ be a horizontally conformal submersion. Then for any horizontal vector fields U, V and vertical vector fields X, Y , we have

- (i) $\nabla F_*(U, V) = U(\ln \lambda)F_* V + V(\ln \lambda)F_* U - g_M(U, V)F_*(\text{grad}_V \ln \lambda)$,
- (ii) $\nabla F_*(X, Y) = -F_*(\mathcal{T}_X Y)$,
- (iii) $\nabla F_*(U, X) = -F_*(\nabla_U X) = -F_*(\mathcal{A}_U X)$.

If $F: (M, g_M) \rightarrow (N, g_N)$ is a horizontally conformal submersion, then the O'Neill tensor $\mathcal{A}$ is given by

$$(2.16) \quad \mathcal{A}_V W = -\mathcal{A}_W V = \frac{1}{2} \left\{ \mathcal{V}[V, W] - \lambda^2 g(V, W) \text{grad}_V \left(\frac{1}{\lambda^2} \right) \right\}$$

for any $V, W \in \Gamma((\ker F_*)^\perp)$, where $\text{grad}_V(f)$ is the vertical component of the gradient on M , contains a skew-symmetric part that measures the obstruction to integrability of the horizontal distribution $\mathcal{H}$, and also means that $\mathcal{A}$ is alternating on the horizontal distribution (see [6] for more details).

3. QUASI-HEMI-SLANT $\xi^\perp$ -RIEMANNIAN SUBMERSIONS

In this section, we introduce and study a new Riemannian submersion which is a generalization of the existing one, namely, the hemi-slant, semi-slant, semi-invariant, etc. $\xi^\perp$ -Riemannian submersions from a Sasakian manifold onto a Riemannian manifold.

Let F be a Riemannian submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) , i.e.,

$$F: (M, \phi, \xi, \eta, g_M) \rightarrow (N, g_N).$$

Then the Kernel of the tangent map F_* is defined as

$$\ker F_* = \{X \in \Gamma(TM): F_*X = 0\}.$$

This means that $\ker F_* \subseteq TM$ and we therefore decompose the tangent space of M as

$$TM = \ker F_* \oplus (\ker F_*)^\perp.$$

The distributions $\ker F_*$ and $(\ker F_*)^\perp$ are referred to as the vertical and horizontal distributions, respectively.

Let us consider $\mathbb{R}^{2n+1}$ with its usual contact structure given by

$$(3.1) \quad \begin{aligned} \eta &= \frac{1}{2} \left\{ dz - \sum_{i=1}^n y^i dx^i \right\}, \quad \xi = 2 \frac{\partial}{\partial z}, \\ g &= \eta \otimes \eta + \frac{1}{4} \sum_{i=1}^n \{ dx^i \otimes dx^i + dy^i \otimes dy^i \}, \\ \phi \left(\sum_{i=1}^n \left(X_i \frac{\partial}{\partial x^i} + Y_i \frac{\partial}{\partial y^i} \right) + Z \frac{\partial}{\partial z} \right) &= \sum_{i=1}^n \left(Y_i \frac{\partial}{\partial x^i} - X_i \frac{\partial}{\partial y^i} \right) + \sum_{i=1}^n Y_i y^i \frac{\partial}{\partial z}, \end{aligned}$$

where $(x_1, \dots, x_n, y_1, \dots, y_n, z)$ denotes the Cartesian coordinates on $\mathbb{R}^{2n+1}$. Then $(\mathbb{R}^{2n+1}, \phi, \eta, \xi, g)$ is a Sasakian manifold (see [4] for more details).

Let $F: \mathbb{R}^{13} \rightarrow \mathbb{R}^6$ be a Riemannian map defined by

$$F(x_1, \dots, x_6, y_1, \dots, y_6, z) = e^2 \left(\frac{x_1 - x_2}{\sqrt{2}}, x_3, x_5, y_2, y_4, z \right).$$

It is easy to see that its Jacobian matrix is of rank 6 and therefore F is a submersion. The vector fields

$$(3.2) \quad \begin{aligned} X_1 &= \frac{1}{\sqrt{2}} \left\{ \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right\}, \quad X_2 = \frac{1}{\sqrt{2}} \left\{ \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right\}, \\ X_3 &= \frac{\partial}{\partial x_3}, \quad X_4 = \frac{\partial}{\partial x_4}, \quad X_5 = \frac{\partial}{\partial x_5}, \quad X_6 = \frac{\partial}{\partial x_6}, \\ Y_1 &= \frac{\partial}{\partial y_1}, \quad Y_2 = \frac{\partial}{\partial y_2}, \quad Y_3 = \frac{\partial}{\partial x_3}, \quad Y_4 = \frac{\partial}{\partial y_4}, \\ Y_5 &= \frac{\partial}{\partial y_5}, \quad Y_6 = \frac{\partial}{\partial y_6}, \quad \xi = 2 \frac{\partial}{\partial z}, \end{aligned}$$

are linearly independent. It is not difficult to show that $F_*X_1 = 0$, $F_*X_4 = 0$, $F_*X_6 = 0$, $F_*Y_1 = 0$, $F_*Y_3 = 0$, $F_*Y_5 = 0$, $F_*Y_6 = 0$, which means that

$$\ker F_* = \text{span}\{X_1, X_4, X_6, Y_1, Y_3, Y_5, Y_6\}.$$

Also, since the vector fields X_2, X_3, X_5, Y_2, Y_4 and ξ are orthogonal and we have

$$(\ker F_*)^\perp = \text{span}\{X_2, X_3, X_5, Y_2, Y_4, \xi\}.$$

Let $\mathcal{D} = \text{span}\{X_6, Y_6\}$, $\mathcal{D}^\perp = \text{span}\{X_4, Y_3, Y_5\}$ and $\mathcal{D}^\theta = \text{span}\{X_1, Y_1\}$. Then using the $(1, 1)$ -tensor in (3.1), one has $\phi\mathcal{D} = \mathcal{D}$, $\phi\mathcal{D}^\perp \subseteq (\ker F_*)^\perp$, the angle between $\phi\mathcal{D}^\theta$ and $\mathcal{D}^\theta$ is constant, given by $\theta = \pi/4$. Therefore, F is a conformal quasi-hemi-slant Riemannian submersion with the quasi-hemi-slant angle $\theta = \pi/4$ and dilation $\lambda = e^2$ such that its fibre is decomposed as $\ker F_* = \mathcal{D} \oplus \mathcal{D}^\theta \oplus \mathcal{D}^\perp$.

Therefore, we have the following result.

Theorem 3.1. *Let $(M, \phi, \xi, \eta, g_M)$ be a $(2n+1)$ -dimensional Sasakian manifold. Then there is a conformal $\xi^\perp$ -Riemannian submersion $F: (M, \phi, \eta, \xi, g_M) \rightarrow (N, g_N)$ such that its vertical distribution $\ker F_*$ admits three orthogonal distributions $\mathcal{D}$, $\mathcal{D}^\theta$, $\mathcal{D}^\perp$, i.e.,*

$$(3.3) \quad \ker F_* = \mathcal{D} \oplus \mathcal{D}^\theta \oplus \mathcal{D}^\perp,$$

where the distribution $\mathcal{D}$ is invariant, i.e., $\phi\mathcal{D} = \mathcal{D}$, the distribution $\mathcal{D}^\perp$ is anti-invariant, i.e., $\phi\mathcal{D}^\perp \subseteq (\ker F_*)^\perp$, $\mathcal{D}^\theta$ is the slant distribution, and the angle between $\phi\mathcal{D}^\theta$ and $\mathcal{D}^\theta$ is a constant θ .

We observe that if we denote the dimensions of $\mathcal{D}$, $\mathcal{D}^\theta$ and $\mathcal{D}^\perp$ by m_1, m_2 and m_3 , respectively, then we get the following cases:

- (1) If $m_1 = 0$, then F is a conformal hemi-slant $\xi^\perp$ -Riemannian submersion [19].
- (2) If $m_2 = 0$, then F is a conformal semi-invariant $\xi^\perp$ -Riemannian submersion [3].
- (3) If $m_3 = 0$, then F is a conformal semi-slant $\xi^\perp$ -Riemannian submersion [2].

The above mentioned Riemannian submersion in Theorem (3.1) shall be called a *Conformal quasi-hemi-slant $\xi^\perp$ -Riemannian submersion* and the angle θ is called the *quasi-hemi-slant angle* of the submersion. This submersion generalises hemi-slant, semi-invariant and semi-slant $\xi^\perp$ -Riemannian submersions.

If $\mathcal{D} \neq \{0\}$, $\mathcal{D}^\perp \neq \{0\}$ and the quasi-hemi-slant angle $\theta \neq 0, \pi/2$, then the quasi-hemi-slant $\xi^\perp$ -Riemannian submersion $F: (M, \phi, \eta, \xi, g_M) \rightarrow (N, g_N)$ is *proper*.

Let F be a horizontally conformal quasi-hemi-slant $\xi^\perp$ -Riemannian submersion from a Sasakian manifold $(M, \phi, \eta, \xi, g_M)$ onto a Riemannian manifold (N, g_N) . Then we have

$$(3.4) \quad TM = \ker F_* \oplus (\ker F_*)^\perp.$$

Let us define the projections p and q on the tangent vectors TM of M by $p: TM \rightarrow \ker F_*$ and $q: TM \rightarrow (\ker F_*)^\perp$, respectively. Now, for any $X \in \Gamma(TM)$,

$$(3.5) \quad X = pX + qX,$$

where $pX \in \Gamma(\ker F_*)$ and $qX \in \Gamma((\ker F_*)^\perp)$.

Now, for any $X \in \Gamma(\ker F_*)$, the vector field X can be written as

$$(3.6) \quad X = \mathcal{P}X + \mathcal{Q}X + \mathcal{R}X,$$

where $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ are projections of $\ker F_*$ onto $\mathcal{D}, \mathcal{D}^\theta$ and $\mathcal{D}^\perp$, respectively. Put

$$(3.7) \quad \phi X = \varphi X + \omega X,$$

where $\varphi X \in \Gamma(\ker F_*)$ and $\omega X \in \Gamma((\ker F_*)^\perp)$. From (3.6) and (3.7) we have

$$\phi X = \varphi \mathcal{P}X + \omega \mathcal{P}X + \varphi \mathcal{Q}X + \omega \mathcal{Q}X + \varphi \mathcal{R}X + \omega \mathcal{R}X.$$

Since $\phi \mathcal{D} = \mathcal{D}$ and $\phi \mathcal{D}^\perp \subseteq (\ker F_*)^\perp$, we have $\omega \mathcal{P}X = 0$ and $\varphi \mathcal{R}X = 0$, and so

$$(3.8) \quad \phi X = \varphi \mathcal{P}X + \varphi \mathcal{Q}X + \omega \mathcal{Q}X + \omega \mathcal{R}X.$$

This means that

$$(3.9) \quad \phi \ker F_* = \mathcal{D} \oplus \varphi \mathcal{D}^\theta \oplus \omega \mathcal{D}^\theta \oplus \phi \mathcal{D}^\perp,$$

where $\oplus$ is the orthogonal sum of bundles. Since $\omega \mathcal{D}^\theta \subseteq (\ker F_*)^\perp$ and $\phi \mathcal{D}^\perp \subseteq (\ker F_*)^\perp$, one obtains

$$(3.10) \quad (\ker F_*)^\perp = \omega \mathcal{D}^\theta \oplus \phi \mathcal{D}^\perp \oplus \nu,$$

where ν is the orthogonal complement of $\omega\mathcal{D}^\theta \oplus \phi\mathcal{D}^\perp$ in $(\ker F_*)^\perp$ and it is invariant with respect to ϕ and contains ξ . The latter means

$$(3.11) \quad \nu = \phi\nu \oplus \mathbb{R}\xi,$$

where $\mathbb{R}\xi = \text{span}\{\xi\}$.

Also, for any $V \in (\ker F_*)^\perp$ we have

$$(3.12) \quad \phi V = \mathcal{B}V + \mathcal{C}V,$$

where $\mathcal{B}V \in \Gamma(\mathcal{D}^\theta \oplus \mathcal{D}^\perp)$ and $\mathcal{C}V \in \Gamma(\nu)$. From (3.7) and (3.12), we have the following lemma.

Lemma 3.1. *Let F be a horizontally conformal quasi-hemi-slant $\xi^\perp$ -Riemannian submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) . Then we have*

$$(3.13) \quad \varphi^2 + \mathcal{B}\omega = -\mathbb{I}, \quad \omega\varphi + \mathcal{C}\omega = 0,$$

$$(3.14) \quad \varphi\mathcal{B} + \mathcal{B}\mathcal{C} = 0, \quad \mathcal{C}^2 + \omega\mathcal{B} = -\mathbb{I} + \eta \otimes \xi.$$

By definition of underlying $\xi^\perp$ -Riemannian submersion and (3.10), we have the following.

Lemma 3.2. *Let F be a horizontally conformal quasi-hemi-slant $\xi^\perp$ -Riemannian submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) . Then we have*

$$\varphi\mathcal{D} = \mathcal{D}, \quad \varphi\mathcal{D}^\perp = \{0\}, \quad \mathcal{B}\omega\mathcal{D}^\theta = \mathcal{D}^\theta, \quad \mathcal{B}(\omega\mathcal{D}^\perp) = \mathcal{D}^\perp, \quad \varphi\mathcal{D}^\theta = \mathcal{D}^\theta.$$

Now, as the $\xi^\perp$ -Riemannian submersion from the Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) , one has, for any $X, Y \in \Gamma(\ker F_*)$,

$$(3.15) \quad \begin{aligned} g(X, Y)\xi &= \mathcal{T}_X\varphi Y + \mathcal{V}\nabla_X\varphi Y + \mathcal{H}\nabla_X\omega Y + \mathcal{T}_X\omega Y - \mathcal{B}\mathcal{T}_X Y \\ &\quad - \mathcal{C}\mathcal{T}_X Y - \varphi\mathcal{V}\nabla_X Y - \omega\mathcal{V}\nabla_X Y. \end{aligned}$$

By comparing tangential and normal components of (3.15), one has

$$\begin{aligned} \mathcal{V}\nabla_X\varphi Y + \mathcal{T}_X\omega Y &= \mathcal{B}\mathcal{T}_X Y + \varphi\mathcal{V}\nabla_X Y, \\ \mathcal{T}_X\varphi Y + \mathcal{H}\nabla_X\omega Y &= \mathcal{C}\mathcal{T}_X Y + \omega\mathcal{V}\nabla_X Y + g(X, Y)\xi. \end{aligned}$$

For any $X \in \Gamma(\ker F_*)$ and $V \in \Gamma(\ker F_*)^\perp$,

$$\begin{aligned} -\eta(V)X &= (\nabla_X \phi)V = \mathcal{T}_X \mathcal{B}V + \mathcal{V} \nabla_X \mathcal{B}V + \mathcal{H} \nabla_X \mathcal{C}V + \mathcal{T}_X \mathcal{C}V - \varphi \mathcal{T}_X V \\ &\quad - \omega \mathcal{T}_X V - \mathcal{B} \mathcal{H} \nabla_X V - \mathcal{C} \mathcal{H} \nabla_X V, \end{aligned}$$

which leads to

$$\begin{aligned} (3.16) \quad \mathcal{V} \nabla_X \mathcal{B}V + \mathcal{T}_X \mathcal{C}V &= \mathcal{B} \mathcal{H} \nabla_X V + \varphi \mathcal{T}_X V - \eta(V)X, \\ \mathcal{T}_X \mathcal{B}V + \mathcal{H} \nabla_X \mathcal{C}V &= \omega \mathcal{T}_X V + \mathcal{C} \mathcal{H} \nabla_X V. \end{aligned}$$

Also, we have

$$\begin{aligned} 0 &= (\nabla_V \phi)V = \mathcal{A}_V \varphi X + \mathcal{V} \nabla_V \varphi X + \mathcal{H} \nabla_V \omega X + \mathcal{A}_V \omega X \\ &\quad - \mathcal{B} \mathcal{A}_V X - \mathcal{C} \mathcal{A}_V X - \varphi \mathcal{V} \nabla_V X - \omega \mathcal{V} \nabla_V X, \end{aligned}$$

which implies that

$$\begin{aligned} (3.17) \quad \mathcal{V} \nabla_V \varphi X + \mathcal{A}_V \omega X &= \mathcal{B} \mathcal{A}_V X + \varphi \mathcal{V} \nabla_V X, \\ \mathcal{A}_V \varphi X + \mathcal{H} \nabla_V \omega X &= \mathcal{C} \mathcal{A}_V X + \omega \mathcal{V} \nabla_V X. \end{aligned}$$

For any $V, W \in \Gamma(\ker F_*)^\perp$,

$$\begin{aligned} g(V, W)\xi - \eta(W)V &= \mathcal{A}_V \mathcal{B}W + \mathcal{V} \nabla_V \mathcal{B}W + \mathcal{A}_V \mathcal{C}W + \mathcal{H} \nabla_V \mathcal{C}W \\ &\quad - \mathcal{B} \mathcal{H} \nabla_V W - \mathcal{C} \mathcal{H} \nabla_V W - \varphi \mathcal{A}_V W - \omega \mathcal{A}_V W, \end{aligned}$$

which, by comparing tangential and normal components, gives

$$\begin{aligned} (3.18) \quad \mathcal{V} \nabla_V \mathcal{B}W + \mathcal{A}_V \mathcal{C}W &= \mathcal{B} \mathcal{H} \nabla_V W + \varphi \mathcal{A}_V W, \\ \mathcal{A}_V \mathcal{B}W + \mathcal{H} \nabla_V \mathcal{C}W &= \mathcal{C} \mathcal{H} \nabla_V W + \omega \mathcal{A}_V W + g(V, W)\xi - \eta(W)V. \end{aligned}$$

Therefore, from (3.15)–(3.18), we have the following.

Lemma 3.3. *Let F be a horizontally conformal quasi-hemi-slant $\xi^\perp$ -Riemannian submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) . Then we have*

$$(3.19) \quad (\nabla_X \varphi)Y = \mathcal{B} \mathcal{T}_X Y - \mathcal{T}_X \omega Y,$$

$$(3.20) \quad (\nabla_X \omega)Y = \mathcal{C} \mathcal{T}_X Y - \mathcal{T}_X \varphi Y + g_M(X, Y)\xi,$$

$$(3.21) \quad (\nabla_V \mathcal{B})W = \varphi \mathcal{A}_V W - \mathcal{A}_V \mathcal{C}W,$$

$$(3.22) \quad (\nabla_V \mathcal{C})W = \omega \mathcal{A}_V W - \mathcal{A}_V \mathcal{B}W + g_M(V, W)\xi - \eta(W)V$$

for any $X, Y \in \Gamma(\ker F_*)$ and $V, W \in \Gamma(\ker F_*)^\perp$.

From relations (3.19), (3.20), (3.21) and (3.22), and using (2.16), we have

$$(3.23) \quad \nabla_Y \phi X - \nabla_X \phi Y + \phi[X, Y] = \mathcal{T}_X \phi Y - \mathcal{T}_Y \phi X,$$

$$(3.24) \quad \nabla_W \phi V - \nabla_V \phi W + \phi[V, W] = \phi(\mathcal{A}_W V - \mathcal{A}_V W) + \mathcal{A}_V \phi W \\ - \mathcal{A}_W \phi V + \eta(W)V - \eta(V)W$$

for any $X, Y \in \Gamma(\ker F_*)$ and $V, W \in \Gamma(\ker F_*)^\perp$.

Lemma 3.4. *Let F be a horizontally conformal quasi-hemi-slant $\xi^\perp$ -Riemannian submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) . Then for any $X, Y \in \Gamma(\mathcal{D}^\perp)$ we have*

$$(3.25) \quad \phi[X, Y] = \mathcal{T}_X \phi Y - \mathcal{T}_Y \phi X.$$

Proof. The proof follows directly from (3.23). □

Now, if the tensors φ and ω are parallel with respect to the connection ∇ on M , respectively, then for any $X, Y \in \Gamma(\ker F_*)$,

$$(3.26) \quad \mathcal{B}\mathcal{T}_X Y = \mathcal{T}_X \omega Y \quad \text{and} \quad \mathcal{C}\mathcal{T}_X Y = \mathcal{T}_X \varphi Y + g_M(X, Y)\xi,$$

respectively. We also state the following lemma.

Lemma 3.5. *Let F be a horizontally conformal quasi-hemi-slant $\xi^\perp$ -Riemannian submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) . Then we have*

$$(3.27) \quad \varphi^2 X = -(\cos^2 \theta)X,$$

$$(3.28) \quad g_M(\varphi X, \varphi Y) = \cos^2 \theta g_M(X, Y),$$

$$(3.29) \quad g_N(\omega X, \omega Y) = \sin^2 \theta g_M(X, Y)$$

for any $X, Y \in \Gamma(\mathcal{D}^\theta)$.

Proof. The proof follows directly from the one given in [5]. □

From the first relation in (3.13) and Lemma 3.5, we deduce the following lemma.

Lemma 3.6. *Let F be a horizontally conformal quasi-hemi-slant $\xi^\perp$ -Riemannian submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) . Then we have*

$$(3.30) \quad \mathcal{B}\omega X = -(\sin^2 \theta)X$$

for any $X \in \Gamma(\mathcal{D}^\theta)$.

Lemma 3.7. *Let F be a horizontally conformal quasi-hemi-slant $\xi^\perp$ -Riemannian submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) . If ω is parallel with respect to ∇ on $\mathcal{D}^\theta$, then we have*

$$\mathcal{T}_{\varphi X} \varphi X = -\cos^2 \theta \mathcal{T}_X X$$

for any $X \in \Gamma(\mathcal{D}^\theta)$.

Proof. If ω is parallel with respect to ∇ on $\mathcal{D}^\theta$, then by (3.20) and for any $X, Y \in \Gamma(\mathcal{D}^\theta)$, $\mathcal{C}\mathcal{T}_X Y = \mathcal{T}_X \varphi Y - g_M(X, Y)\xi$. Interchanging X and Y in this relation and using the symmetry of $\mathcal{T}$, one has $\mathcal{T}_{\varphi X} \varphi X = -\cos^2 \theta \mathcal{T}_X X$. $\square$

4. INTEGRABILITY, GEODESIBILITY OF DISTRIBUTIONS AND THE DECOMPOSITION THEOREMS

In this section, we investigate the integrability and geodesibility conditions of distributions in the decomposition of the tangent bundle.

We start by examining the integrability conditions for the slant, anti-invariant and invariant distributions as well as the ones of the distributions $\ker F_*$ and $(\ker F_*)^\perp$.

Theorem 4.1. *Let F be a horizontally conformal quasi-hemi-slant $\xi^\perp$ -Riemannian submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) with a quasi-hemi-slant angle θ . Then the slant distribution $\mathcal{D}^\theta$ is integrable if and only if*

$$\begin{aligned} (4.1) \quad & g_M(\mathcal{T}_X \omega \varphi Y - \mathcal{T}_Y \omega \varphi X + \varphi \mathcal{T}_X \omega Y - \varphi \mathcal{T}_Y \omega X, Z) \\ &= \frac{1}{\lambda^2} g_N(\nabla F_*(Y, \omega X) - \nabla F_*(X, \omega Y), F_*(\phi Z)) \\ &\quad + \frac{1}{\lambda^2} g_N(\nabla_X^F F_*(\omega Y) - \nabla_Y^F F_*(\omega X), F_*(\phi Z)) \end{aligned}$$

for any $X, Y \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D} \oplus \mathcal{D}^\perp)$.

Proof. Note that $\mathcal{D}^\theta$ is integrable if and only if for any $X, Y \in \Gamma(\mathcal{D}^\theta)$, $g_M([X, Y], Z) = 0$ and $g_M([X, Y], W) = 0$ for any $Z \in \Gamma(\mathcal{D} \oplus \mathcal{D}^\perp)$ and $W \in \Gamma(\ker F_*)^\perp$. The distribution $\ker F_*$ being always integrable, $g_M([X, Y], W) = 0$. So, $\mathcal{D}^\theta$ is integrable if and only if $g_M([X, Y], Z) = 0$. Now, using the identities in Lemma 3.2, Lemma 3.5 and the fact that $\mathcal{D}$ is an invariant distribution, we get

$$\begin{aligned} (4.2) \quad & g_M([X, Y], Z) = g_M(\phi \nabla_X Y, \phi Z) - g_M(\phi \nabla_Y X, \phi Z) \\ &= \cos^2 \theta g_M([X, Y], Z) + g_M(\mathcal{T}_Y \omega \varphi X - \mathcal{T}_X \omega \varphi Y, Z) \\ &\quad + g_M(\mathcal{T}_X \omega Y - \mathcal{T}_Y \omega X, \phi Z) \\ &\quad + g_M(\mathcal{H} \nabla_X \omega Y - \mathcal{H} \nabla_Y \omega X, \phi Z). \end{aligned}$$

Since F is horizontally conformal submersion, using (2.15), one obtains

$$(4.3) \quad \sin^2 \theta g_M([X, Y], Z) = g_M(\mathcal{T}_Y \omega \varphi X - \mathcal{T}_X \omega \varphi Y + \varphi \mathcal{T}_Y \omega X - \varphi \mathcal{T}_X \omega Y, Z) \\ + \frac{1}{\lambda^2} g_N(F_*(\nabla_X \omega Y) - F_*(\nabla_Y \omega X), F_*(\phi Z)),$$

which completes the proof, using (2.15). $\square$

Theorem 4.2. *Let F be a horizontally conformal quasi-hemi-slant $\xi^\perp$ -Riemannian submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) . Then the anti-invariant distribution $\mathcal{D}^\perp$ is integrable if and only if*

$$(4.4) \quad \mathcal{T}_X \phi Y = \mathcal{T}_Y \phi X$$

for any $X, Y \in \Gamma(\mathcal{D}^\perp)$.

Proof. The proof follows from Lemma 3.4. $\square$

Theorem 4.3. *Let F be a horizontally conformal quasi-hemi-slant $\xi^\perp$ -Riemannian submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) . Then the invariant distribution $\mathcal{D}$ is integrable if and only if*

$$g_M(\mathcal{T}_X \phi Y - \mathcal{T}_Y \phi X, \omega Z) = g_M(\mathcal{V} \nabla_Y \phi X - \mathcal{V} \nabla_X \phi Y, \varphi Z)$$

for any $X, Y \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(\mathcal{D} \oplus \mathcal{D}^\perp)$.

Proof. The invariant distribution $\mathcal{D}$ is integrable if and only if for any $X, Y \in \Gamma(\mathcal{D})$, $g_M([X, Y], Z) = 0$ and $g_M([X, Y], W) = 0$ for any $Z \in \Gamma(\mathcal{D} \oplus \mathcal{D}^\perp)$ and $W \in \Gamma(\ker F_*)^\perp$. The distribution $\ker F_*$ being always integrable, $g_M([X, Y], W) = 0$. So, $\mathcal{D}$ is integrable if and only if $g_M([X, Y], Z) = 0$. Therefor we have $g_M([X, Y], Z) = g_M(\mathcal{T}_X \phi Y - \mathcal{T}_Y \phi X, \omega Z) + g_M(\mathcal{V} \nabla_X \phi Y - \mathcal{V} \nabla_Y \phi X, \varphi Z)$, and this completes the proof. $\square$

Let V and W be two vector fields on $(\ker F_*)^\perp$. The horizontal distribution $(\ker F_*)^\perp$ is integrable if and only if $g_M([V, W], Z) = 0$ and $g_M([V, W], W) = 0$ for any $Z \in \Gamma(\ker F_*)$. Using the fact that F is horizontally conformal submersion, we have

$$g_M([V, W], Z) = g_M(\mathcal{V} \nabla_V BW - \mathcal{V} \nabla_W BV + \mathcal{A}_V CW - \mathcal{A}_W CV, \varphi Z) \\ + \frac{1}{\lambda^2} g_N(\nabla_V^F F_*(CW) - \nabla_W^F F_*(CV), F_*(\omega Z)) \\ + \frac{1}{\lambda^2} g_N(\nabla F_*(W, CV) - \nabla F_*(V, CW), F_*(\omega Z)) \\ + g_M(\mathcal{A}_V BW - \mathcal{A}_W BV, \omega Z) + \eta(W)g(V, \omega Z) - \eta(V)g(W, \omega Z).$$

Using Lemma 2.1, we obtain

$$\begin{aligned}
 (4.5) \quad \nabla F_*(W, CV) - \nabla F_*(V, CW) \\
 = W(\ln \lambda)F_*CV + CV(\ln \lambda)F_*W - V(\ln \lambda)F_*CW \\
 - CW(\ln \lambda)F_*V + 2g_M(V, CW)F_*(\text{grad} \ln \lambda),
 \end{aligned}$$

which leads to

$$\begin{aligned}
 (4.6) \quad \frac{1}{\lambda^2}g_N(\nabla F_*(W, CV) - \nabla F_*(V, CW), F_*(\omega Z)) \\
 = g_M(W(\ln \lambda)CV + CV(\ln \lambda)W \\
 - V(\ln \lambda)CW - CW(\ln \lambda)V + 2g_M(V, CW)\text{grad} \ln \lambda, \omega Z).
 \end{aligned}$$

From (4.5) we get

$$\begin{aligned}
 (4.7) \quad g_M([V, W], Z) = g_M(\mathcal{V}\nabla_V BW - \mathcal{V}\nabla_W BV + \mathcal{A}_V CW - \mathcal{A}_W CV, \varphi Z) \\
 + \frac{1}{\lambda^2}g_N(\nabla_V^F F_*(CW) - \nabla_W^F F_*(CV), F_*(\omega Z)) \\
 + g_M(W(\ln \lambda)CV + CV(\ln \lambda)W - V(\ln \lambda)CW - CW(\ln \lambda)V \\
 + 2g_M(V, CW)\text{grad} \ln \lambda, \omega Z) + g_M(\mathcal{A}_V BW - \mathcal{A}_W BV, \omega Z) \\
 + \eta(W)g(V, \omega Z) - \eta(V)g(W, \omega Z).
 \end{aligned}$$

Therefore, we have the following.

Theorem 4.4. *Let F be a horizontally conformal quasi-hemi-slant $\xi^\perp$ -Riemannian submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) . Then distribution $(\ker F_*)^\perp$ is integrable if and only if*

$$\begin{aligned}
 \frac{1}{\lambda^2}g_N(\nabla_W^F F_*(CV) - \nabla_V^F F_*(CW), F_*(\omega Z)) \\
 = g_M(\mathcal{V}\nabla_V BW - \mathcal{V}\nabla_W BV + \mathcal{A}_V CW \\
 - \mathcal{A}_W CV, \varphi Z) + g_M(\mathcal{A}_V BW - \mathcal{A}_W BV + W(\ln \lambda)CV \\
 + CV(\ln \lambda)W - V(\ln \lambda)CW - CW(\ln \lambda)V \\
 + 2g_M(V, CW)\text{grad} \ln \lambda, \omega Z) + \eta(W)g(V, \omega Z) - \eta(V)g(W, \omega Z)
 \end{aligned}$$

for any $V, W \in \Gamma((\ker F_*)^\perp)$ and $Z \in \Gamma(\ker F_*)$.

Next, we investigate the geometry of leaves of the distributions by starting with the one of the invariant, slant distribution and anti-invariant $\mathcal{D}$, $\mathcal{D}^\theta$ and $\mathcal{D}^\perp$, respectively.

For any $X, Y \in \Gamma(\mathcal{D})$, $Z = \mathcal{Q}Z + \mathcal{R}Z \in \Gamma(\mathcal{D}^\theta \oplus \mathcal{D}^\perp)$ and $V \in \Gamma((\ker F_*)^\perp)$,

$$\begin{aligned} g_M(\nabla_X Y, Z) &= g_M(\phi \nabla_X Y, \phi \mathcal{Q}Z) + g_M(\phi \nabla_X Y, \phi \mathcal{R}Z) \\ &= \cos^2 \theta g_M(\nabla_X Y, \mathcal{Q}Z) - g_M(\mathcal{T}_X Y, \omega \varphi \mathcal{Q}Z) + g_M(\mathcal{T}_X \phi Y, \omega Z). \end{aligned}$$

That is,

$$(4.8) \quad \sin^2 \theta g_M(\nabla_X Y, \mathcal{Q}Z) + g_M(\nabla_X Y, \mathcal{R}Z) = -g_M(\mathcal{T}_X Y, \omega \varphi \mathcal{Q}Z) + g_M(\mathcal{T}_X \phi Y, \omega Z).$$

Also,

$$(4.9) \quad g_M(\nabla_X Y, V) = g_M(\mathcal{T}_X Y, V).$$

Therefore, we have the following.

Theorem 4.5. *Let F be a horizontally conformal quasi-hemi-slant $\xi^\perp$ -Riemannian submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) . Then the invariant distribution $\mathcal{D}$ defines a totally geodesic foliation on M if and only if*

$$(4.10) \quad \mathcal{T}_X Y = 0 \quad \forall X, Y \in \Gamma(\mathcal{D}).$$

Proof. The proof follows from a straightforward calculation using (4.8) and (4.9). $\square$

For any $X, Y \in \Gamma(\ker F_*)$ and using the second relation in (3.26), when the tensors φ and ω given in (3.7) are parallel, we have

$$(4.11) \quad \mathcal{C}\mathcal{T}_X Y = \mathcal{T}_X \varphi Y + g_M(X, Y)\xi.$$

Now, if $\mathcal{D}$ defines a totally geodesic foliation on M , then for any vector fields X and Y in $\mathcal{D}$, together with the hypotheses of Theorem 4.5, $\mathcal{T}_X Y = 0$. This also implies that $\mathcal{T}_X \phi Y = 0$ as the distribution $\mathcal{D}$ is invariant with respect to ϕ . In this context, and using (4.7), we have $g_M(X, Y)\xi = 0$, a contradiction. Therefore, we have the following.

Theorem 4.6. *There is no horizontally conformal quasi-hemi-slant $\xi^\perp$ -Riemannian submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) such that the invariant distribution $\mathcal{D}$ given in (3.3) is totally geodesic and tensor ω is parallel.*

For any $X, Y \in \Gamma(\mathcal{D}^\theta)$, $Z = \mathcal{P}Z + \mathcal{R}Z \in \Gamma(\mathcal{D} \oplus \mathcal{D}^\perp)$ and $V \in \Gamma((\ker F_*)^\perp)$,

$$g_M(\nabla_X Y, Z) = \cos^2 \theta g_M(\nabla_X Y, Z) - g_M(\mathcal{T}_X \omega \varphi Y, Z) \\ + g_M(\mathcal{H} \nabla_X \omega Y, \phi Z) + g_M(\mathcal{T}_X \omega Y, \phi Z).$$

That is,

$$\sin^2 \theta g_M(\nabla_X Y, Z) = -g_M(\mathcal{T}_X \omega \varphi Y, Z) + g_M(\mathcal{H} \nabla_X \omega Y, \phi Z) + g_M(\mathcal{T}_X \omega Y, \phi Z).$$

Since F is a horizontally conformal submersion, using Lemma 2.1 we have

$$(4.12) \quad \sin^2 \theta g_M(\nabla_X Y, Z) = g_M(\mathcal{T}_X \omega Y, \phi \mathcal{P}Z) - g_M(\mathcal{T}_X \omega \varphi Y, Z) \\ + \frac{1}{\lambda^2} g_N(\nabla_X^F F_*(\omega Y), F_*(\phi \mathcal{R}Z)) \\ - \frac{1}{\lambda^2} g_N(\nabla F_*(X, \omega Y), F_*(\phi \mathcal{R}Z)) \\ = g_M(\mathcal{T}_X \omega Y, \phi \mathcal{P}Z) - g_M(\mathcal{T}_X \omega \varphi Y, Z) \\ + \frac{1}{\lambda^2} g_N(\nabla_X^F F_*(\omega Y), F_*(\phi \mathcal{R}Z)) + g_M(\mathcal{A}_{\omega Y} X, \phi \mathcal{R}Z).$$

Also,

$$(4.13) \quad g_M(\nabla_X Y, V) = \cos^2 \theta g_M(\nabla_X Y, V) - g_M(\mathcal{H} \nabla_X \omega \varphi Y, V) \\ + g_M(\mathcal{H} \nabla_X \omega Y, \mathcal{C}V) + g_M(\mathcal{T}_X \omega Y, \mathcal{B}V).$$

That is,

$$(4.14) \quad \sin^2 \theta g_M(\nabla_X Y, V) = g_M(\mathcal{T}_X \omega Y + \mathcal{A}_{\omega Y} X, \mathcal{B}V + \mathcal{C}V) - g_M(\mathcal{A}_{\omega \varphi Y} X, V) \\ + \frac{1}{\lambda^2} g_N(\nabla_X^F F_*(\omega Y), F_*(\mathcal{C}V)) \\ - \frac{1}{\lambda^2} g_N(\nabla_X^F F_*(\omega \varphi Y), F_*(V)).$$

Therefore, we have the following result.

Theorem 4.7. *Let F be a horizontally conformal quasi-hemi-slant $\xi^\perp$ -Riemannian submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) . Then the slant distribution $\mathcal{D}^\theta$ defines a totally geodesic foliation on M if and only if*

$$(4.15) \quad \frac{1}{\lambda^2} g_N(\nabla_X^F F_*(\omega Y), F_*(\phi \mathcal{R}Z)) = -g_M(\mathcal{T}_X \omega Y + \mathcal{A}_{\omega Y} X, \phi Z) + g_M(\mathcal{T}_X \omega \varphi Y, Z),$$

and

$$(4.16) \quad \frac{1}{\lambda^2} g_N(\nabla_X^F F_*(\omega\varphi Y), F_*(V)) - \frac{1}{\lambda^2} g_N(\nabla_X^F F_*(\omega Y), F_*(CV)) \\ = g_M(\mathcal{T}_X \omega Y + \mathcal{A}_{\omega Y} X, \phi V) - g_M(\mathcal{A}_{\omega\varphi Y} X, V)$$

for any $X, Y \in \Gamma(\mathcal{D}^\theta)$, $Z \in \Gamma(\mathcal{D} \oplus \mathcal{D}^\perp)$ and $V \in \Gamma((\ker F_*)^\perp)$.

Theorem 4.8. *Let F be a horizontally conformal quasi-hemi-slant $\xi^\perp$ -Riemannian submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) . Then the anti-invariant distribution $\mathcal{D}^\perp$ defines a totally geodesic foliation on M if and only if*

$$(4.17) \quad \frac{1}{\lambda^2} g_N(\nabla_X^F F_*(\phi Y), F_*(\omega Z)) = g_M(\mathcal{T}_X Y, \omega\varphi Z) - g_M(\mathcal{A}_{\phi Y} X, \omega Z),$$

$$(4.18) \quad \frac{1}{\lambda^2} g_N(\nabla_X^F F_*(\phi Y), F_*(CV)) = g_M(\mathcal{T}_X \phi Y + \mathcal{A}_{\phi Y} X, \phi V)$$

for any $X, Y \in \Gamma(\mathcal{D}^\perp)$, $Z \in \Gamma(\mathcal{D} \oplus \mathcal{D}^\theta)$ and $V \in \Gamma((\ker F_*)^\perp)$.

Proof. For any $X, Y \in \Gamma(\mathcal{D}^\perp)$, $Z = \mathcal{P}Z + \mathcal{Q}Z \in \Gamma(\mathcal{D} \oplus \mathcal{D}^\theta)$ and $V \in \Gamma((\ker F_*)^\perp)$,

$$(4.19) \quad g_M(\nabla_X Y, Z) = \cos^2 \theta g_M(\nabla_X Y, Z) - g_M(\nabla_X Y, \omega\varphi Z) + g_M(\nabla_X \phi Y, \omega Z) \\ = \cos^2 \theta g_M(\nabla_X Y, Z) - g_M(\mathcal{T}_X Y, \omega\varphi Z) + g_M(\mathcal{H}\nabla_X \phi Y, \omega Z).$$

That is,

$$\sin^2 \theta g_M(\nabla_X Y, Z) = -g_M(\mathcal{T}_X Y, \omega\varphi Z) + g_M(\mathcal{H}\nabla_X \phi Y, \omega Z).$$

From F being a horizontally conformal submersion and Lemma 2.1, we have

$$(4.20) \quad \sin^2 \theta g_M(\nabla_X Y, Z) = -g_M(\mathcal{T}_X Y, \omega\varphi Z) \\ + \frac{1}{\lambda^2} g_N(\nabla_X^F F_*(\phi Y), F_*(\omega Z)) + g_M(\mathcal{A}_{\phi Y} X, \omega Z).$$

Since F is a horizontally conformal submersion, from Lemma 2.1 we have

$$(4.21) \quad g_M(\nabla_X Y, V) = g_M(\nabla_X \phi Y, \phi V) = g_M(\mathcal{T}_X \phi Y, \mathcal{B}V) + g_M(\mathcal{H}\nabla_X \phi Y, \mathcal{C}V) \\ = g_M(\mathcal{T}_X \phi Y + \mathcal{A}_{\phi Y} X, \mathcal{B}V + \mathcal{C}V) \\ + \frac{1}{\lambda^2} g_N(\nabla_X^F F_*(\phi Y), F_*(\mathcal{C}V)),$$

which completes the proof. □

From Theorem 4.5, Theorem 4.7 and Theorem 4.8, we have the following decomposition result.

Theorem 4.9. *Let F be a horizontally conformal quasi-hemi-slant $\xi^\perp$ -Riemannian submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) . Then the fibres of F are local product Riemannian manifold of the form*

$$M_{\mathcal{D}} \times M_{\mathcal{D}^\theta} \times M_{\mathcal{D}^\perp},$$

where $M_{\mathcal{D}}$, $M_{\mathcal{D}^\theta}$ and $M_{\mathcal{D}^\perp}$ are leaves of $\mathcal{D}$, $\mathcal{D}^\theta$ and $\mathcal{D}^\perp$, respectively, if and only if conditions (4.10), (4.15), (4.16), (4.17) and (4.18) are satisfied.

For any $V, W \in \Gamma((\ker F_*)^\perp)$ and $Z \in \Gamma(\ker F_*)$ and using (3.6) one has

$$\begin{aligned}
 (4.22) \quad g_M(\nabla_V W, Z) &= g_M(\nabla_V \phi W, \phi \mathcal{P}Z) + g_M(\nabla_V \phi W, \phi \mathcal{Q}Z) \\
 &\quad + g_M(\nabla_V \phi W, \phi \mathcal{R}Z) - g_M((\nabla_V \phi)W, \phi Z) \\
 &= \eta(W)g_M(V, \omega Z) + g_M(\nabla_V \phi W, \phi \mathcal{P}Z) + g_M(\phi \nabla_V W, \varphi \mathcal{Q}Z) \\
 &\quad + g_M(\nabla_V \phi W, \omega \mathcal{Q}Z) + g_M(\nabla_V \phi W, \phi \mathcal{R}Z) \\
 &= \eta(W)g_M(V, \omega Z) + g_M(\nabla_V \phi W, \phi \mathcal{P}Z) \\
 &\quad + \cos^2 \theta g_M(\mathcal{A}_V W, \mathcal{Q}Z) \\
 &\quad - g_M(\mathcal{H} \nabla_V W, \omega \varphi \mathcal{Q}Z) + g_M(\nabla_V \phi W, \omega \mathcal{Q}Z + \phi \mathcal{R}Z).
 \end{aligned}$$

Since F is a horizontally conformal submersion, $\omega Z = \omega \mathcal{Q}Z + \phi \mathcal{R}Z$ and the fact that $g_M(\nabla_V \phi W, \phi \mathcal{P}Z) = g_M(\mathcal{A}_V W, Z)$, we have

$$\begin{aligned}
 (4.23) \quad g_M(\nabla_V W, Z) &= \eta(W)g_M(V, \omega Z) + g_M(\mathcal{A}_V W, Z) \\
 &\quad + \cos^2 \theta g_M(\mathcal{A}_V W, \mathcal{Q}Z) + g_M(\mathcal{H} \nabla_V CW, \omega Z) \\
 &\quad - g_M(\mathcal{H} \nabla_V W, \omega \varphi \mathcal{Q}Z) + g_M(\mathcal{A}_V BW, \omega Z) \\
 &= \eta(W)g_M(V, \omega Z) + g_M(\mathcal{A}_V W, Z) + \cos^2 \theta g_M(\mathcal{A}_V W, \mathcal{Q}Z) \\
 &\quad + \frac{1}{\lambda^2} g_N(\nabla_V^F F_*(CW), F_*(\omega Z)) \\
 &\quad - \frac{1}{\lambda^2} g_N(\nabla_V^F F_*(W), F_*(\omega \varphi \mathcal{Q}Z)) \\
 &\quad - \frac{1}{\lambda^2} g_N(\nabla F_*(V, CW), F_*(\omega Z)) \\
 &\quad + \frac{1}{\lambda^2} g_N(\nabla F_*(V, W), F_*(\omega \varphi \mathcal{Q}Z)) + g_M(\mathcal{A}_V BW, \omega Z).
 \end{aligned}$$

Now, using Lemma 2.1, one gets

$$\begin{aligned}
(4.24) \quad & g_M(\nabla_V W, Z) \\
&= \eta(W)g_M(V, \omega Z) + g_M(\mathcal{A}_V W, Z) + \cos^2 \theta g_M(\mathcal{A}_V W, \mathcal{Q}Z) \\
&\quad + \frac{1}{\lambda^2} g_N(\nabla_V^F F_*(\mathcal{C}W), F_*(\omega Z)) - \frac{1}{\lambda^2} g_N(\nabla_V^F F_*(W), F_*(\omega \varphi \mathcal{Q}Z)) \\
&\quad - g(V, \text{grad } \ln \lambda) g_M(\mathcal{C}W, \omega Z) - g(\mathcal{C}W, \text{grad } \ln \lambda) g_M(V, \omega Z) \\
&\quad + g_M(V, \mathcal{C}W) g_M(\text{grad } \ln \lambda, \omega Z) + g(V, \text{grad } \ln \lambda) g_M(W, \omega \varphi \mathcal{Q}Z) \\
&\quad + g(W, \text{grad } \ln \lambda) g_M(V, \omega \varphi \mathcal{Q}Z) - g_M(V, W) g_M(\text{grad } \ln \lambda, \omega \varphi \mathcal{Q}Z) \\
&\quad + g_M(\mathcal{A}_V \mathcal{B}W, \omega Z).
\end{aligned}$$

Therefore, we have the following.

Theorem 4.10. *Let F be a horizontally conformal quasi-hemi-slant $\xi^\perp$ -Riemannian submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) . Then the distribution $(\ker F_*)^\perp$ defines a totally geodesic foliation on M if and only if*

$$\begin{aligned}
(4.25) \quad & \frac{1}{\lambda^2} g_N(\nabla_V^F F_*(W), F_*(\omega \varphi \mathcal{Q}Z)) - \frac{1}{\lambda^2} g_N(\nabla_V^F F_*(\mathcal{C}W), F_*(\omega Z)) \\
&= \eta(W)g_M(V, \omega Z) + g_M(\mathcal{A}_V W, Z) + \cos^2 \theta g_M(\mathcal{A}_V W, \mathcal{Q}Z) \\
&\quad - g(V, \text{grad } \ln \lambda) g_M(\mathcal{C}W, \omega Z) - g(\mathcal{C}W, \text{grad } \ln \lambda) g_M(V, \omega Z) \\
&\quad + g_M(V, \mathcal{C}W) g_M(\text{grad } \ln \lambda, \omega Z) + g(V, \text{grad } \ln \lambda) g_M(W, \omega \varphi \mathcal{Q}Z) \\
&\quad + g(W, \text{grad } \ln \lambda) g_M(V, \omega \varphi \mathcal{Q}Z) - g_M(V, W) g_M(\text{grad } \ln \lambda, \omega \varphi \mathcal{Q}Z) \\
&\quad + g_M(\mathcal{A}_V \mathcal{B}W, \omega Z)
\end{aligned}$$

for any $V, W \in \Gamma((\ker F_*)^\perp)$ and $Z \in \Gamma(\ker F_*)$.

Theorem 4.11. *Let F be a horizontally conformal quasi-hemi-slant $\xi^\perp$ -Riemannian submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) . Then the distribution $\ker F_*$ defines a totally geodesic foliation on M if and only if*

$$\begin{aligned}
(4.26) \quad & \frac{1}{\lambda^2} g_N(\nabla_X^F F_*(\omega \varphi \mathcal{Q}Y), F_*(V)) - \frac{1}{\lambda^2} g_N(\nabla_X^F F_*(\omega Y), F_*(\mathcal{C}V)) \\
&= g_M(\mathcal{T}_X \mathcal{P}Y, V) - g_M(\varphi X, \mathcal{P}Y) \eta(V) + \cos^2 \theta g_M(\mathcal{T}_X \mathcal{Q}Y, V) \\
&\quad + g_M(\mathcal{T}_X \omega Y, \mathcal{B}V) + g_M(\mathcal{A}_{\omega \varphi \mathcal{Q}Y} X, V) + g_M(\mathcal{A}_{\omega Y} X, \mathcal{C}V)
\end{aligned}$$

for any $X, Y \in \Gamma(\ker F_*)$ and $V \in \Gamma((\ker F_*)^\perp)$.

Proof. For any $X, Y \in \Gamma(\ker F_*)$ and $V \in \Gamma((\ker F_*)^\perp)$,

$$\begin{aligned}
g_M(\nabla_X Y, V) - \eta(\nabla_X Y)\eta(V) &= g_M(\nabla_X \phi Y, \phi V) = g_M(\nabla_X \phi \mathcal{P}Y, \phi V) \\
&+ \cos^2 \theta g_M(\nabla_X \mathcal{Q}Y, V) - g_M(\nabla_X \omega \varphi \mathcal{Q}Y, V) \\
&+ g_M(\nabla_X \omega \mathcal{Q}Y, \phi V) + g_M(\nabla_X \phi \mathcal{R}Y, \phi V) \\
&= g_M(\mathcal{T}_X \mathcal{P}Y, V) - g_M(\varphi X, \mathcal{P}Y)\eta(V) + \cos^2 \theta g_M(\nabla_X \mathcal{Q}Y, V) \\
&- g_M(\mathcal{H} \nabla_X \omega \varphi \mathcal{Q}Y, V) + g_M(\nabla_X (\omega \mathcal{Q}Y + \nabla_X \phi \mathcal{R}Y), \mathcal{B}V) \\
&+ g_M(\nabla_X (\omega \mathcal{Q}Y + \phi \mathcal{R}Y), \mathcal{C}V).
\end{aligned}$$

Since $\omega Y = \omega \mathcal{Q}Y + \phi \mathcal{R}Y$ and using Lemma 2.1, we have

$$\begin{aligned}
g_M(\nabla_X Y - \eta(\nabla_X \xi), V) &= g_M(\mathcal{T}_X \mathcal{P}Y, V) - g_M(\varphi X, \mathcal{P}Y)\eta(V) + \cos^2 \theta g_M(\mathcal{T}_X \mathcal{Q}Y, V) \\
&+ g_M(\mathcal{T}_X \omega Y, \mathcal{B}V) - \frac{1}{\lambda^2} g_N(\nabla_X^F F_*(\omega \varphi \mathcal{Q}Y), F_*(V)) \\
&+ \frac{1}{\lambda^2} g_N(\nabla_X^F F_*(\omega Y), F_*(\mathcal{C}V)) \\
&+ g_M(\mathcal{A}_{\omega \varphi \mathcal{Q}Y} X, V) + g_M(\mathcal{A}_{\omega Y} X, \mathcal{C}V),
\end{aligned}$$

which completes the proof. $\square$

From Theorem 4.10 and Theorem 4.11, we have the following decomposition result.

Theorem 4.12. *Let F be a horizontally conformal quasi-hemi-slant $\xi^\perp$ -Riemannian submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) . Then the total space M is a local product Riemannian manifold of the form*

$$M_{(\ker F_*)^\perp} \times M_{\ker F_*},$$

where $M_{(\ker F_*)^\perp}$ and $M_{\ker F_*}$ are leaves of the distributions $(\ker F_*)^\perp$ and $\ker F_*$, respectively, if and only if conditions (4.25) and (4.26) are satisfied.

5. CONFORMAL QUASI-HEMI-SLANT $\xi^\perp$ -RIEMANNIAN SUBMERSIONS WITH TOTALLY UMBILICAL OR GEODESIC FIBRES

Now, we study the geometry of the conformal quasi-hemi-slant $\xi^\perp$ -Riemannian submersions with totally umbilical and geodesic fibres.

Note that a fibre of a conformal Riemannian submersion F is totally umbilical if

$$(5.1) \quad \mathcal{T}_X Y = g(X, Y)H$$

for any $X, Y \in \Gamma(\ker F_*)$, where H is the mean curvature vector field of the fibre in M . When $H = 0$, the fibre is called minimal.

Let F be a proper horizontally conformal quasi-hemi-slant $\xi^\perp$ -Riemannian submersion from a Sasakian manifold $(M, \phi, \eta, \xi, g_M)$ onto a Riemannian manifold (N, g_N) . Then $\mathcal{D} \neq \{0\}$, $\mathcal{D}^\perp \neq \{0\}$. This means that the dimensions of $\mathcal{D}$ and $\mathcal{D}^\perp$ is either 2 and 1, respectively, that is, $\dim \mathcal{D} = 2$ and $\dim \mathcal{D}^\perp = 1$ or $\dim \mathcal{D} > 2$ and $\dim \mathcal{D}^\perp > 1$.

(i) Case 1: $\dim \mathcal{D} = 2$ and $\dim \mathcal{D}^\perp = 1$, it is obvious.

(ii) Case 2: $\dim \mathcal{D} > 2$ and $\dim \mathcal{D}^\perp > 1$. Choose $X, Y \in \Gamma(\mathcal{D}^\perp)$ such that $g_N(X, Y) = 0$. Using (3.7), (3.12) and (2.7), one has

$$\begin{aligned} \mathcal{H}\nabla_X \phi Y + \mathcal{T}_X \phi Y &= \nabla_X \phi Y = g_M(X, Y)\xi + \phi \nabla_X Y \\ &= g_M(X, Y)\xi + \varphi \mathcal{V} \nabla_X Y + \omega \mathcal{V} \nabla_X Y + \mathcal{B}\mathcal{T}_X Y + \mathcal{C}\mathcal{T}_X Y. \end{aligned}$$

Taking the g_M with X , we obtain

$$(5.2) \quad g_M(\mathcal{T}_X \phi Y, X) = g_M(\mathcal{B}\mathcal{T}_X Y, X) = g_M(\mathcal{T}_X Y, \phi X).$$

If the fibres are totally umbilical, then from (5.1) and (5.2) we have

$$\begin{aligned} (5.3) \quad 0 &= g_M(X, \phi Y)g_M(H, X) = g_M(\mathcal{T}_X \phi Y, X) \\ &= g_M(\mathcal{T}_X Y, \phi X) = g_M(X, Y)g_M(H, \phi X). \end{aligned}$$

That is, $g_M(H, \phi X) = 0$. Thus, we deduce that

$$(5.4) \quad H \perp \phi \mathcal{D}^\perp.$$

This means that the mean curvature vector H is perpendicular to $\mathcal{D}^\perp$. For $Z \in \Gamma(\mathcal{D}^\theta)$, one gets

$$(5.5) \quad g_M(X, Y)g_M(H, \omega Z) = -g_M(Y, \mathcal{T}_X \omega Z) = g_M(Y, (\nabla_X \varphi)Z).$$

Also, for any $V \in \Gamma(\nu)$ such that $V \perp \xi$, and using $\varphi \mathcal{D}^\perp = \{0\}$ and (3.20), we have

$$(5.6) \quad g_M(X, Y)g_M(H, \phi V) = -g_M(\mathcal{C}\mathcal{T}_X Y, V) = -g_M((\nabla_X \omega)Y, V),$$

and for the characteristic vector field $\xi \in \nu$ we have

$$(5.7) \quad g_M(X, Y)g_M(H, \xi) = g_M(\mathcal{T}_X Y, \xi) \neq 0,$$

as $\mathcal{T}_X Y \in (\ker F_*)^\perp$.

For any $X_1, X_2 \in \Gamma(\mathcal{D})$ and $X \in \Gamma(\mathcal{D}^\perp)$ and using (3.19), we obtain

$$(5.8) \quad g_M(X_1, X_2)g_M(H, \phi X) = -g_M(X_2, \mathcal{T}_{X_1}\omega X) = g_M(X_2, (\nabla_{X_1}\phi)X).$$

Now, for any $Z \in \Gamma(\mathcal{D}^\theta)$,

$$(5.9) \quad g_M(X_1, X_2)g_M(H, \omega Z) = -g_M(X_2, \mathcal{T}_{X_1}\omega Z) = g_M(X_2, (\nabla_{X_1}\phi)Z).$$

For any $V \in \Gamma(\nu)$ such that $V \perp \xi$,

$$(5.10) \quad \begin{aligned} g_M(X_1, X_2)g_M(H, \phi V) &= -g_M(\mathcal{C}\mathcal{T}_{X_1}X_2, V) \\ &= -g_M((\nabla_{X_1}\omega)X_2, V) - g_M(X_1, \phi X_2)g_M(H, V), \end{aligned}$$

and also, for the characteristic vector field $\xi \in \nu$ we have

$$(5.11) \quad g_M(X_1, X_2)g_M(H, \xi) = g_M(\mathcal{T}_{X_1}X_2, \xi) \neq 0.$$

Moreover, if the tensor ϕ is parallel, then

$$(5.12) \quad H \perp \omega\mathcal{D}^\perp.$$

From (5.4) and (5.12), and using (3.10) we deduce that the mean curvature vector $H \in \nu$. If ω is parallel, using (5.6), then $H \perp \phi\nu$, and since $g_M(H, \xi) \neq 0$, by virtue of (5.7), $H \in \mathbb{R}\xi$, where $\mathbb{R}\xi = \text{span}\{\xi\}$ defined in (3.11). Therefore, we have the following.

Theorem 5.1. *Let F be a proper horizontally conformal quasi-hemi-slant $\xi^\perp$ -Riemannian submersion with totally umbilical fibres from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) . Then either*

- (i) *the invariant and anti-invariant distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are of dimension 2 and 1, respectively, or*
- (ii) *the mean curvature vector field H of any fibre $F^{-1}(y)$, $y \in N$ is perpendicular $\phi\mathcal{D}^\perp$.*

Moreover, if ϕ is parallel, then $H \in \nu$. Furthermore, if ω is parallel, then $H \in \mathbb{R}\xi$.

The O'Neill tensor fields $\mathcal{A}$ and $\mathcal{T}$, defined in (2.4) and (2.5), and their covariant derivatives play a fundamental role in expressing the Riemannian curvature R of (M, g_M) .

Let $R^\mathcal{V}$, R and R^N be the Riemannian curvature tensors of the fibre $\ker F_* = \mathcal{V}$, M and N , respectively. The curvature tensors R and $R^\mathcal{V}$ are related as (see [6] for more details)

$$(5.13) \quad g_M(R(X, Y)Z, W) = g_M(R^\mathcal{V}(X, Y)Z, W) + g_M(\mathcal{T}_X Z, \mathcal{T}_Y W) - g_M(\mathcal{T}_Y Z, \mathcal{T}_X W)$$

for any $X, Y, Z, W \in \Gamma(\ker F_*)$.

A plane section in TM is called a ϕ -section if there exists a vector $X \in TM$ orthogonal to ξ such that $\{X, \phi X\}$ span the section. The sectional curvature $K(X, \phi)$, denoted $H(X)$, is called ϕ -sectional curvature (see [4] for more details and references therein). Then we have

$$(5.14) \quad g_M(R(X, \phi X)\phi X, X) = g_M(R^\mathcal{V}(X, \phi X)\phi X, X) + g_M(\mathcal{T}_X \phi X, \mathcal{T}_{\phi X} X) - g_M(\mathcal{T}_{\phi X} \phi X, \mathcal{T}_X X),$$

which implies that

$$(5.15) \quad H(X) = H^\mathcal{V}(X) + \|\mathcal{T}_X \phi X\|_M^2 - g_M(\mathcal{T}_{\phi X} \phi X, \mathcal{T}_X X),$$

where $H(X) = g_M(R(X, \phi X)\phi X, X)$ and $H^\mathcal{V}(X) = g_M(R^\mathcal{V}(X, \phi X)\phi X, X)$ are the ϕ -sectional curvatures of M and $\ker F_*$, respectively, and $\|\cdot\|_M^2 := g_M(\cdot, \cdot)$. Therefore, we have the following result.

Theorem 5.2. *Let F be a horizontally conformal quasi-hemi-slant $\xi^\perp$ -Riemannian submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) with integrable anti-invariant distribution $\mathcal{D}^\perp$. Then the holomorphic sectional curvatures H and $H^\mathcal{V}$ of M and $\ker F_*$, respectively, satisfy the inequality*

$$(5.16) \quad H(X) \geq H^\mathcal{V}(X) \quad \forall X \in \Gamma(\mathcal{D}^\perp), \quad \|X\|_M = 1,$$

and the equality holds if and only if the fibre $\ker F_*$ is $\mathcal{D}^\perp$ -totally geodesic and mixed $(\mathcal{D}^\perp, \phi\mathcal{D}^\perp)$ -totally geodesic.

Proof. Note that if $\mathcal{D}^\perp$ is integrable, then, by Theorem 4.2, $\mathcal{T}_{\phi X} \phi X = -\mathcal{T}_X X$ and the first assertion follows. The equality holds if and only if $\mathcal{T}_X X = 0$ and $\mathcal{T}_X \phi X = 0$, $\forall X \in \Gamma(\mathcal{D}^\perp)$ and $\|X\|_M = 1$. By linearity of $\mathcal{T}$ it is easy to see that $\mathcal{T}_X Y = 0$, for any $X, Y \in \Gamma(\mathcal{D}^\perp)$ and $\mathcal{T}_X V = 0$, $V \in \Gamma(\phi\mathcal{D}^\perp)$. $\square$

Mustafa in [11], page 49, achieved a wonderful characterization of totally geodesic horizontally conformal maps between Riemannian manifolds. Its conformal quasi-hemi-slant version states as follows.

Theorem 5.3. *Let F be a horizontally conformal quasi-hemi-slant $\xi^\perp$ -Riemannian submersion from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) . Then F is totally geodesic if and only if F has constant dilation, totally geodesic fibres and integrable horizontal distribution.*

Proof. The proof is similar to the one given in [11], Theorem 2.3. □

Now, using relation (3.26) and Theorem 5.3, we have the following existence theorem.

Theorem 5.4. *There exist no horizontally conformal quasi-hemi-slant $\xi^\perp$ -Riemannian submersions with totally geodesic fibres from a Sasakian manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) such that the tensor ω is parallel.*

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