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BERTRAND PARTNER TRAJECTORIES RELATED TO PAFORS

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Abstract. In this study, we consider the concept of Bertrand partner trajectories related to Positional Adapted Frame on Regular Surfaces (shortly PAFORS) for the particles moving on the different regular surfaces in Euclidean 3-space. The relations between the PAFORS elements of the aforesaid trajectories are given. Also, the relations between Darboux basis vectors of them are found. Furthermore, some characterizations are given for some special cases of these trajectories with the aid of their PAFORS elements.

Keywords: kinematics of a particle; Bertrand curve; PAFORS

MSC 2020: 53A04, 57R25, 70B05

1. Introduction

In classical differential geometry, the theory of surfaces plays quite an important role. It has several applications in different types of disciplines such as differential geometry, computer graphics, architecture, and engineering. The concept of moving frames, which is a part of these working areas of differential geometry, has attracted several researchers from the past to the present.

The Serret-Frenet frame, which is studied independently of each other by researchers Serret [22] and Frenet [10], is the most popular moving frame. With a similar logic, Darboux frame was constructed on regular surfaces by Darboux [4]. In the existing literature, many different types of other moving frames were studied such as Bishop frame (type-1, type-2, type-3) [2], [24], [28], N-Bishop frame [14], q-frame [7], Flc-frame [5] and N-C-W frame [21]. Recently, a new type of moving frame called Positional Adapted Frame (shortly PAF) has been investigated by Özen and Tosun [17] in Euclidean 3-space $\mathbb{E}^3$. They have presented this frame for the trajectories having nonvanishing angular momentum. Also, inspired by the concept of Darboux frame, Özen and Tosun obtained the other new type of moving frame on

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regular surfaces in $\mathbb{E}^3$ for the trajectories with nonvanishing angular momentum. They called this frame Positional Adapted Frame on Regular Surfaces (PAFORS). Also, the characterizations on geodesic asymptotic and slant helical trajectories with respect to PAFORS were given in [18]. Then, Ertuğ Gürbüz introduced the frames PAF and PAFORS in the Minkowski 3-space [9]. In [19] and [25], Smarandache curves with respect to PAF were studied.

On the other hand, constructing new types of special curves has been one of the most interesting working areas of the theory of curves for a long time. Various studies have been done in the existing literature to create new types of curves and curve couples (pairs, mates). One of the most popular of them is the Bertrand curve couple. Bertrand curve couple is characterized as follows: the principal normal line of one of these partner curves coincides with the principal normal line of the other partner curve at the corresponding points of these curves. This definition was presented by French mathematician Joseph Louis François Bertrand in 1850 [1]. This study includes the characterization of these curves according to their curvatures and torsions. Several researchers studied the Bertrand curve couples according to various moving frames. Yerlikaya et al. [27] examined the Bertrand curves with respect to the type-2-Bishop frame. Also, Kazaz et al. [13] determined this special curve couple related to Darboux frame. Then, Dede et al. investigated the Bertrand curve couples with respect to the q-frame [6]. Additionally, several mathematicians studied the notion of the Bertrand curve couple with different perspectives. The studies can be recommended for readers [3], [11], [12], [20], [26]. At the beginning of this study, we started with the following question and the results aroused our curiosity: "What kind of results and geometric interpretations do we get if we combine Bertrand curve couples, which is a basic geometric concept, and PAFORS, which is a new type of special attracted moving frame on a regular surface?". In order to answer this question, we want to discuss this topic.

This study is organized as follows. In Section 2, we remind some required information, notions, and notations in order to understand the ensuing section. In Section 3, Bertrand partner trajectories related to PAFORS in Euclidean 3-space are introduced. We obtain and scrutinize the relations between the PAFORS elements of the aforesaid partners. Then, we give the relations between the Darboux basis vectors of Bertrand partner trajectories related to PAFORS. Afterward, to be an asymptotic curve and geodesic curve of one of these partners, we give the necessary conditions with respect to the PAFORS curvatures of the other partner.

2. Preliminaries

In this section, we remind some basic notions and notations that are required and used from the beginning to the end of this study.

In Euclidean 3-space $\mathbb{E}^3$, let $\boldsymbol{\rho} = (\rho_1, \rho_2, \rho_3)$, $\boldsymbol{\varrho} = (\varrho_1, \varrho_2, \varrho_3) \in \mathbb{R}^3$ be considered. The standard dot product of these vectors and the norm of $\boldsymbol{\rho}$ are presented as $\langle \boldsymbol{\rho}, \boldsymbol{\varrho} \rangle = \rho_1\varrho_1 + \rho_2\varrho_2 + \rho_3\varrho_3$ and $\|\boldsymbol{\rho}\| = \sqrt{\langle \boldsymbol{\rho}, \boldsymbol{\rho} \rangle}$, respectively. A differentiable curve $\alpha = \alpha(s)\colon I \subset \mathbb{R} \to \mathbb{E}^3$ is called a unit speed curve if $\|\mathrm{d}\alpha/\mathrm{d}s\| = 1$ is satisfied for all $s \in I$. Then, s is called the arc-length parameter of the curve α. If the derivative of a differentiable curve never vanishes along this curve, it is called a regular curve. Any regular curve always has a parameterization such that it is a unit speed curve [23]. Throughout this study, we will care to show the differentiation according to the arc-length parameter s with the symbol prime "$\prime$".

Suppose that a point particle P of constant mass moves on the regular surface M in the Euclidean 3-space and along the trajectory $\alpha = \alpha(s)$ which is a unit speed curve. Thus, we can write $\alpha\colon I \subset \mathbb{R} \to M \subset \mathbb{E}^3$. For α, the unit tangent vector is calculated as $\mathbf{T}(s) = \alpha'(s)$. Also, the system of the Darboux base vectors of α is given as $\{\mathbf{T}(s), \mathbf{Y}(s), \mathbf{U}(s)\}$ along the curve α, where $\mathbf{T}$ is the unit tangent vector as mentioned earlier. On the other hand, $\mathbf{U}$ is the unit normal vector of the surface M, and $\mathbf{Y}$ is the unit vector determined by $\mathbf{Y} = \mathbf{U} \times \mathbf{T}$. It should be indicated that the curves considered in this paper have nonzero second-order derivatives (that is, $\alpha''(s)$ is always nonzero). The derivative formulas of the Darboux frame are given as

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{Y}'(s) \\ \mathbf{U}'(s) \end{pmatrix} = \begin{pmatrix} 0 & k_g(s) & k_n(s) \\ -k_g(s) & 0 & \tau_g(s) \\ -k_n(s) & -\tau_g(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{Y}(s) \\ \mathbf{U}(s) \end{pmatrix},$$

where k_g is a geodesic curvature, k_n is a normal curvature, and τ_g is a geodesic torsion of the curve α [8], [15], [23].

Suppose that the angular momentum vector of the aforesaid particle P about the origin is equal to zero nowhere. Then, PAFORS $\{\mathbf{T}(s), \mathbf{G}(s), \mathbf{H}(s)\}$ is well defined during the motion along $\alpha = \alpha(s)$. The basis vectors of PAFORS are calculated as in the following:

$$\begin{aligned} \mathbf{T}(s) &= \mathbf{T}(s), \\ \mathbf{G}(s) &= \frac{\langle \alpha(s), \mathbf{U}(s) \rangle}{\sqrt{\langle \alpha(s), \mathbf{Y}(s) \rangle^2 + \langle \alpha(s), \mathbf{U}(s) \rangle^2}} \mathbf{Y}(s) \\ &\quad + \frac{\langle \alpha(s), \mathbf{Y}(s) \rangle}{\sqrt{\langle \alpha(s), \mathbf{Y}(s) \rangle^2 + \langle \alpha(s), \mathbf{U}(s) \rangle^2}} \mathbf{U}(s), \end{aligned}$$

$$\mathbf{H}(s)=\frac{\langle -\alpha(s),\mathbf{Y}(s)\rangle}{\sqrt{\langle \alpha(s),\mathbf{Y}(s)\rangle^2+\langle \alpha(s),\mathbf{U}(s)\rangle^2}}\mathbf{Y}(s)$$
$$+\frac{\langle \alpha(s),\mathbf{U}(s)\rangle}{\sqrt{\langle \alpha(s),\mathbf{Y}(s)\rangle^2+\langle \alpha(s),\mathbf{U}(s)\rangle^2}}\mathbf{U}(s).$$

The relation between the Darboux frame and PAFORS can be given as

$$\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{G}(s)\\ \mathbf{H}(s)\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\phi(s) & -\sin\phi(s)\\ 0 & \sin\phi(s) & \cos\phi(s)\end{pmatrix}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{Y}(s)\\ \mathbf{U}(s)\end{pmatrix}, \tag{2.1}$$

where $\phi(s)$ is the angle between the vector $\mathbf{U}(s)$ and the vector $\mathbf{H}(s)$ which is positively oriented from the vector $\mathbf{U}(s)$ to the vector $\mathbf{H}(s)$. Additionally, the derivative formulas of PAFORS can be written as [16]

$$\begin{pmatrix}\mathbf{T}'(s)\\ \mathbf{G}'(s)\\ \mathbf{H}'(s)\end{pmatrix}=\begin{pmatrix}0 & k_1(s) & k_2(s)\\ -k_1(s) & 0 & k_3(s)\\ -k_2(s) & -k_3(s) & 0\end{pmatrix}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{G}(s)\\ \mathbf{H}(s)\end{pmatrix}, \tag{2.2}$$

where

$$\begin{cases} k_1(s)=k_g(s)\cos\phi(s)-k_n(s)\sin\phi(s),\\ k_2(s)=k_g(s)\sin\phi(s)+k_n(s)\cos\phi(s),\\ k_3(s)=\tau_g(s)-\phi'(s).\end{cases}$$

Also, the rotation angle $\phi(s)$ is expressed with the help of the following equation [16]:

$$\phi(s)=\begin{cases}\arctan\Big(-\dfrac{\langle \alpha(s),\mathbf{Y}(s)\rangle}{\langle \alpha(s),\mathbf{U}(s)\rangle}\Big) & \text{if } \langle \alpha(s),\mathbf{U}(s)\rangle>0,\\ \arctan\Big(-\dfrac{\langle \alpha(s),\mathbf{Y}(s)\rangle}{\langle \alpha(s),\mathbf{U}(s)\rangle}\Big)+\pi & \text{if } \langle \alpha(s),\mathbf{U}(s)\rangle<0,\\ -\dfrac{\pi}{2} & \text{if } \langle \alpha(s),\mathbf{U}(s)\rangle=0,\ \langle \alpha(s),\mathbf{Y}(s)\rangle>0,\\ \dfrac{\pi}{2} & \text{if } \langle \alpha(s),\mathbf{U}(s)\rangle=0,\ \langle \alpha(s),\mathbf{Y}(s)\rangle<0.\end{cases}$$

The elements of the set $\{\mathbf{T}(s),\mathbf{G}(s),\mathbf{H}(s),k_1(s),k_2(s),k_3(s)\}$ are called PAFORS apparatuses of $\alpha=\alpha(s)$ [16].

Let us remind the conditions of being an asymptotic curve and geodesic curve in Euclidean 3-space [15]:

- ▷ $\alpha=\alpha(s)$ is an asymptotic curve if and only if $k_n=0$,
- ▷ $\alpha=\alpha(s)$ is a geodesic curve if and only if $k_g=0$.

Theorem 2.1 ([18]). *Suppose that $\alpha = \alpha(s)$ is an asymptotic curve on the regular surface M with the condition $k_g \neq 0$. Then $\alpha = \alpha(s)$ is a trajectory whose position vector lies in the corresponding plane* $\mathrm{Sp}\{\mathbf{T}(s), \mathbf{Y}(s)\}$ *if and only if $k_1 = 0$.*

Theorem 2.2 ([18]). *Assume that $\alpha = \alpha(s)$ is a geodesic curve on the regular surface M with the condition $k_n \neq 0$. Then, $\alpha = \alpha(s)$ is a trajectory whose position vector lies in the corresponding plane* $\mathrm{Sp}\{\mathbf{T}(s), \mathbf{U}(s)\}$ *if and only if $k_1 = 0$.*

For more detailed information with respect to this new type of special and attractive frame PAFORS, we can refer to the studies [9], [16], [18].

3. Bertrand partner trajectories related to PAFORS lying on different regular surfaces

Bertrand partner trajectories related to PAFORS are defined and some characterizations are given for them in this section.

Definition 3.1. Let R and $\overline{R}$ be the moving point particles on regular surfaces M and $\overline{M}$ in 3-dimensional Euclidean space. Show the unit speed parameterization of the trajectories of R and $\overline{R}$ with $\alpha = \alpha(s)$ and $\overline{\alpha} = \overline{\alpha}(\overline{s})$, respectively. Let $\{\mathbf{T}, \mathbf{G}, \mathbf{H}, k_1, k_2, k_3\}$ and $\{\overline{\mathbf{T}}, \overline{\mathbf{G}}, \overline{\mathbf{H}}, \overline{k}_1, \overline{k}_2, \overline{k}_3\}$ represent the PAFORS apparatus of the trajectories α and $\overline{\alpha}$, respectively. If the PAFORS base vector $\mathbf{G}$ coincides with the PAFORS base vector $\overline{\mathbf{G}}$ at the corresponding points of the trajectories α and $\overline{\alpha}$, $\overline{\alpha}$ is said to be a Bertrand partner trajectory of α related to PAFORS. Furthermore, the pair $\{\alpha, \overline{\alpha}\}$ is called a Bertrand pair related to PAFORS.

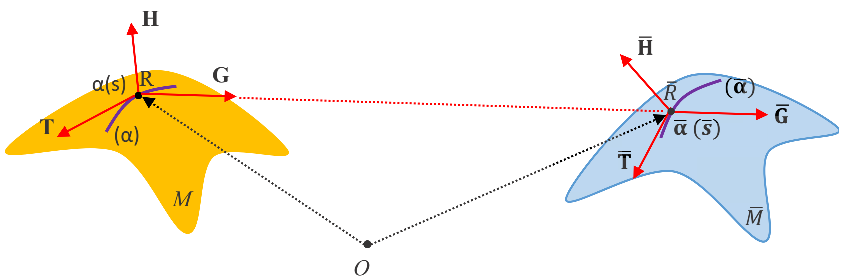

Figure 1. Bertrand partner trajectories related to PAFORS.

Thanks to the definition of Bertrand pair related to PAFORS, we can write the equation

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{G} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} \cos\gamma & 0 & -\sin\gamma \\ 0 & 1 & 0 \\ \sin\gamma & 0 & \cos\gamma \end{pmatrix} \begin{pmatrix} \overline{\mathbf{T}} \\ \overline{\mathbf{G}} \\ \overline{\mathbf{H}} \end{pmatrix}, \tag{3.1}$$

where γ is the angle between the tangent vectors $\mathbf{T}$ and $\overline{\mathbf{T}}$.

Theorem 3.1. *Suppose that $\{\alpha=\alpha(s),\overline{\alpha}=\overline{\alpha}(\overline{s})\}$ is any Bertrand pair related to PAFORS. In this case, the distance between the corresponding points of $\overline{\alpha}$ and α is constant.*

Proof. With the aid of the definition of Bertrand trajectories related to PAFORS, the equation

$$\alpha(s)=\overline{\alpha}(\overline{s})+\lambda(\overline{s})\overline{\mathbf{G}}(\overline{s}) \tag{3.2}$$

can be written, where λ is a real valued smooth function of $\overline{s}$ (see Figure 1). If we differentiate equation (3.2) with respect to $\overline{s}$ and use equation (2.2), we obtain:

$$\mathbf{T}\frac{\mathrm{d}s}{\mathrm{d}\overline{s}}=(1-\lambda\overline{k}_1)\overline{\mathbf{T}}+\lambda'\overline{\mathbf{G}}+\lambda\overline{k}_3\overline{\mathbf{H}}. \tag{3.3}$$

Because the scalar products of the vectors $\overline{\mathbf{T}}$, $\mathbf{T}$ and $\overline{\mathbf{H}}$ with the vector $\overline{\mathbf{G}}$ equal to zero, we find $\lambda'=0$. Thus, λ is a nonzero constant and equation (3.3) can be rewritten as:

$$\mathbf{T}\frac{\mathrm{d}s}{\mathrm{d}\overline{s}}=(1-\lambda\overline{k}_1)\overline{\mathbf{T}}+\lambda\overline{k}_3\overline{\mathbf{H}}. \tag{3.4}$$

Lastly, the distance between the corresponding points of $\overline{\alpha}$ and α can be given as

$$d(\alpha(s),\overline{\alpha}(\overline{s}))=\|\alpha(s)-\overline{\alpha}(\overline{s})\|=\|\lambda\overline{\mathbf{G}}\|=|\lambda|=\text{constant}$$

since $\lambda(\neq 0)$ is constant. □

Theorem 3.2. *Assume that $\{\alpha=\alpha(s),\overline{\alpha}=\overline{\alpha}(\overline{s})\}$ is a Bertrand pair related to PAFORS. Then the equation below holds:*

$$\begin{pmatrix}\mathbf{T}\\ \mathbf{G}\\ \mathbf{H}\end{pmatrix}=\begin{pmatrix}(1-\lambda\overline{k}_1)\dfrac{\mathrm{d}\overline{s}}{\mathrm{d}s} & 0 & \lambda\overline{k}_3\dfrac{\mathrm{d}\overline{s}}{\mathrm{d}s}\\ 0 & 1 & 0\\ -\lambda\overline{k}_3\dfrac{\mathrm{d}\overline{s}}{\mathrm{d}s} & 0 & (1-\lambda\overline{k}_1)\dfrac{\mathrm{d}\overline{s}}{\mathrm{d}s}\end{pmatrix}\begin{pmatrix}\overline{\mathbf{T}}\\ \overline{\mathbf{G}}\\ \overline{\mathbf{H}}\end{pmatrix}. \tag{3.5}$$

Proof. Let $\{\alpha,\overline{\alpha}\}$ be a Bertrand pair related to PAFORS. If we use the equations (3.1) and (3.4), we find:

$$\cos\gamma\frac{\mathrm{d}s}{\mathrm{d}\overline{s}}\overline{\mathbf{T}}-\sin\gamma\frac{\mathrm{d}s}{\mathrm{d}\overline{s}}\overline{\mathbf{H}}=(1-\lambda\overline{k}_1)\overline{\mathbf{T}}+\lambda\bar{k}_3\overline{\mathbf{H}}.$$

The last equation yields:

$$\begin{cases}\cos\gamma=(1-\lambda\overline{k}_1)\dfrac{\mathrm{d}\overline{s}}{\mathrm{d}s},\\[2ex] \sin\gamma=-\lambda\overline{k}_3\dfrac{\mathrm{d}\overline{s}}{\mathrm{d}s}.\end{cases} \tag{3.6}$$

If equation (3.6) is substituted into equation (3.1), the desired result is obtained. □

Corollary 3.1. *The tangent of the angle between the unit tangent vectors of the Bertrand partner trajectories (related to PAFORS)* $\alpha = \alpha(s)$ *and* $\overline{\alpha} = \overline{\alpha}(\overline{s})$ *is expressed as*

$$\tan\gamma = \frac{-\lambda\overline{k}_3}{1-\lambda\overline{k}_1}. \tag{3.7}$$

Theorem 3.3. *Let* $\{\alpha = \alpha(s), \overline{\alpha} = \overline{\alpha}(\overline{s})\}$ *be a Bertrand pair related to PAFORS and their Darboux frame be denoted by* $\{\mathbf{T}, \mathbf{Y}, \mathbf{U}\}$ *and* $\{\overline{\mathbf{T}}, \overline{\mathbf{Y}}, \overline{\mathbf{U}}\}$*, respectively. In this case, the relations between the Darboux base vectors of this pair are given by:*

$$\begin{aligned}
\overline{\mathbf{T}} &= (1-\lambda\overline{k}_1)\frac{\mathrm{d}\overline{s}}{\mathrm{d}s}\mathbf{T} - \lambda\overline{k}_3\sin\Omega\frac{\mathrm{d}\overline{s}}{\mathrm{d}s}\mathbf{Y} - \lambda\overline{k}_3\cos\Omega\frac{\mathrm{d}\overline{s}}{\mathrm{d}s}\mathbf{U},\\
\overline{\mathbf{Y}} &= \lambda\overline{k}_3\sin\overline{\Omega}\frac{\mathrm{d}\overline{s}}{\mathrm{d}s}\mathbf{T} + \left(\cos\overline{\Omega}\cos\Omega + (1-\lambda\overline{k}_1)\sin\overline{\Omega}\sin\Omega\frac{\mathrm{d}\overline{s}}{\mathrm{d}s}\right)\mathbf{Y}\\
&\quad + \left(-\cos\overline{\Omega}\sin\Omega + (1-\lambda\overline{k}_1)\sin\overline{\Omega}\cos\Omega\frac{\mathrm{d}\overline{s}}{\mathrm{d}s}\right)\mathbf{U},\\
\overline{\mathbf{U}} &= \lambda\overline{k}_3\cos\overline{\Omega}\frac{\mathrm{d}\overline{s}}{\mathrm{d}s}\mathbf{T} + \left(-\sin\overline{\Omega}\cos\Omega + (1-\lambda\overline{k}_1)\cos\overline{\Omega}\sin\Omega\frac{\mathrm{d}\overline{s}}{\mathrm{d}s}\right)\mathbf{Y}\\
&\quad + \left(\sin\overline{\Omega}\sin\Omega + (1-\lambda\overline{k}_1)\cos\overline{\Omega}\cos\Omega\frac{\mathrm{d}\overline{s}}{\mathrm{d}s}\right)\mathbf{U}.
\end{aligned}$$

Here Ω *is the angle between the vectors* $\mathbf{U}$ *and* $\mathbf{H}$ *and also,* $\overline{\Omega}$ *is the angle between the vectors* $\overline{\mathbf{U}}$ *and* $\overline{\mathbf{H}}$.

P r o o f. Taking into account equation (2.1), the equations

$$\begin{pmatrix}\mathbf{T}\\ \mathbf{G}\\ \mathbf{H}\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\Omega & -\sin\Omega\\ 0 & \sin\Omega & \cos\Omega\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{Y}\\ \mathbf{U}\end{pmatrix} \tag{3.8}$$

and

$$\begin{pmatrix}\overline{\mathbf{T}}\\ \overline{\mathbf{Y}}\\ \overline{\mathbf{U}}\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\overline{\Omega} & \sin\overline{\Omega}\\ 0 & -\sin\overline{\Omega} & \cos\overline{\Omega}\end{pmatrix}\begin{pmatrix}\overline{\mathbf{T}}\\ \overline{\mathbf{G}}\\ \overline{\mathbf{H}}\end{pmatrix} \tag{3.9}$$

can be written. On the other hand, it is not difficult to see

$$\begin{pmatrix}\overline{\mathbf{T}}\\ \overline{\mathbf{G}}\\ \overline{\mathbf{H}}\end{pmatrix} = \begin{pmatrix}(1-\lambda\overline{k}_1)\dfrac{\mathrm{d}\overline{s}}{\mathrm{d}s} & 0 & -\lambda\overline{k}_3\dfrac{\mathrm{d}\overline{s}}{\mathrm{d}s}\\ 0 & 1 & 0\\ \lambda\overline{k}_3\dfrac{\mathrm{d}\overline{s}}{\mathrm{d}s} & 0 & (1-\lambda\overline{k}_1)\dfrac{\mathrm{d}\overline{s}}{\mathrm{d}s}\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{G}\\ \mathbf{H}\end{pmatrix} \tag{3.10}$$

from (3.5). If equation (3.10) is used in equation (3.9),

$$\begin{pmatrix}\overline{\mathbf{T}}\\ \overline{\mathbf{Y}}\\ \overline{\mathbf{U}}\end{pmatrix} = \begin{pmatrix}(1-\lambda\overline{k}_1)\dfrac{\mathrm{d}\overline{s}}{\mathrm{d}s} & 0 & -\lambda\overline{k}_3\dfrac{\mathrm{d}\overline{s}}{\mathrm{d}s}\\ \lambda\overline{k}_3\sin\overline{\Omega}\dfrac{\mathrm{d}\overline{s}}{\mathrm{d}s} & \cos\overline{\Omega} & (1-\lambda\overline{k}_1)\sin\overline{\Omega}\dfrac{\mathrm{d}\overline{s}}{\mathrm{d}s}\\ \lambda\overline{k}_3\cos\overline{\Omega}\dfrac{\mathrm{d}\overline{s}}{\mathrm{d}s} & -\sin\overline{\Omega} & (1-\lambda\overline{k}_1)\cos\overline{\Omega}\dfrac{\mathrm{d}\overline{s}}{\mathrm{d}s}\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{G}\\ \mathbf{H}\end{pmatrix} \tag{3.11}$$

is obtained. Substituting equation (3.8) into equation (3.11) finishes the proof. □

Theorem 3.4. *Let $\{\alpha=\alpha(s),\overline{\alpha}=\overline{\alpha}(\overline{s})\}$ be a Bertrand pair related to PAFORS. Then the following relations hold:*

(1) $k_1=(\overline{k}_1-\lambda\overline{k}_1^2-\lambda\overline{k}_3^2)/(1-2\lambda\overline{k}_1+\lambda^2(\overline{k}_1^2+\overline{k}_3^2))$,
(2) $\overline{k}_1=(k_1-\zeta k_1^2-\zeta k_3^2)/(1-2\zeta k_1+\zeta^2(k_1^2+k_3^2))$,
where ζ is a constant satisfying $|\zeta|=|\lambda|$.

Proof. (1) Let $\{\alpha,\overline{\alpha}\}$ be a Bertrand pair related to PAFORS. By means of the equality $\cos^2\gamma+\sin^2\gamma=1$ and equation (3.6), we find

$$\left(\frac{\mathrm{d}s}{\mathrm{d}\overline{s}}\right)^2=1-2\lambda\overline{k}_1+\lambda^2(\overline{k}_1^2+\overline{k}_3^2). \tag{3.12}$$

If equation (3.4) is differentiated with respect to $\overline{s}$ and equation (2.2) is utilized,

$$\begin{aligned}&\frac{\mathrm{d}^2s}{\mathrm{d}\overline{s}^2}\mathbf{T}+k_1\left(\frac{\mathrm{d}s}{\mathrm{d}\overline{s}}\right)^2\mathbf{G}+k_2\left(\frac{\mathrm{d}s}{\mathrm{d}\overline{s}}\right)^2\mathbf{H}\\ &=(-\lambda\overline{k}_1'-\lambda\overline{k}_2\overline{k}_3)\overline{\mathbf{T}}+(\overline{k}_1(1-\lambda\overline{k}_1)-\lambda\overline{k}_3^2)\overline{\mathbf{G}}+(\overline{k}_2(1-\lambda\overline{k}_1)+\lambda\overline{k}_3')\overline{\mathbf{H}}\end{aligned} \tag{3.13}$$

is obtained. Thus, we get

$$k_1\left(\frac{\mathrm{d}s}{\mathrm{d}\overline{s}}\right)^2=(1-\lambda\overline{k}_1)\overline{k}_1-\lambda\overline{k}_3^2 \tag{3.14}$$

using the inner product in equation (3.13). Substituting equation (3.12) into equation (3.14) finishes the proof.

(2) Thanks to the definition of the Bertrand pair related to PAFORS, we can write

$$\overline{\alpha}(\overline{s})=\alpha(s)+\zeta\mathbf{G}(s),$$

where ζ is a constant which satisfies the equality $|\zeta|=|\lambda|$ (see Figure 1). Let us differentiate the last equation with respect to s twice. In this case, we get:

$$\overline{\mathbf{T}}\frac{\mathrm{d}\overline{s}}{\mathrm{d}s}=(1-\zeta k_1)\mathbf{T}+\zeta k_3\mathbf{H} \tag{3.15}$$

and

$$\frac{\mathrm{d}^2\overline{s}}{\mathrm{d}s^2}\overline{\mathbf{T}} + \overline{k}_1\left(\frac{\mathrm{d}\overline{s}}{\mathrm{d}s}\right)^2\overline{\mathbf{G}} + \overline{k}_2\left(\frac{\mathrm{d}\overline{s}}{\mathrm{d}s}\right)^2\overline{\mathbf{H}} \tag{3.16}$$
$$= (-\zeta k_1' - \zeta k_2 k_3)\mathbf{T} + (k_1(1-\zeta k_1) - \zeta k_3^2)\mathbf{G} + (k_2(1-\zeta k_1) + \zeta k_3')\mathbf{H}.$$

On the other hand, $\overline{\mathbf{T}} = \cos\gamma\mathbf{T} + \sin\gamma\mathbf{H}$ can be written with the aid of equation (3.1). In that case, we get

$$\cos\gamma\frac{\mathrm{d}\overline{s}}{\mathrm{d}s}\mathbf{T} + \sin\gamma\frac{\mathrm{d}\overline{s}}{\mathrm{d}s}\mathbf{H} = (1-\zeta k_1)\mathbf{T} + \zeta k_3\mathbf{H}$$

by utilizing equation (3.15). From here we find

$$\cos\gamma\frac{\mathrm{d}\overline{s}}{\mathrm{d}s} = 1-\zeta k_1 \quad \text{and} \quad \sin\gamma\frac{\mathrm{d}\overline{s}}{\mathrm{d}s} = \zeta k_3.$$

These equations yield the following:

$$\left(\frac{\mathrm{d}\overline{s}}{\mathrm{d}s}\right)^2 = 1 - 2\zeta k_1 + \zeta^2(k_1^2 + k_3^2). \tag{3.17}$$

Also, the inner product with $\mathbf{G}$ in equation (3.16) gives us

$$\overline{k}_1\left(\frac{\mathrm{d}\overline{s}}{\mathrm{d}s}\right)^2 = k_1 - \zeta k_1^2 - \zeta k_3^2. \tag{3.18}$$

Thus, we get the desired result taking into consideration equation (3.17). □

By means of Theorem 2.1, Theorem 2.2 and Theorem 3.4, we can state the next corollaries.

Corollary 3.2. *Let $\{\alpha = \alpha(s), \overline{\alpha} = \overline{\alpha}(\overline{s})\}$ be a Bertrand pair related to PAFORS. In that case, the following expressions hold:*

(1) *Assume that the geodesic curvature of α never equals to zero. Then $\alpha = \alpha(s)$ is an asymptotic trajectory whose position vector always lies in the corresponding plane* $\mathrm{Sp}\{\mathbf{T}(\mathbf{s}), \mathbf{Y}(\mathbf{s})\}$ *if and only if*

$$\frac{\overline{k}_1 - \lambda\overline{k}_1^2 - \lambda\overline{k}_3^2}{1 - 2\lambda\overline{k}_1 + \lambda^2(\overline{k}_1^2 + \overline{k}_3^2)} = 0.$$

(2) *Assume that the geodesic curvature of $\overline{\alpha}$ never equals to zero. Then $\overline{\alpha} = \overline{\alpha}(\overline{s})$ is an asymptotic trajectory whose position vector always lies in the corresponding plane* $\mathrm{Sp}\{\overline{\mathbf{T}}(\overline{\mathbf{s}}), \overline{\mathbf{Y}}(\overline{\mathbf{s}})\}$ *if and only if*

$$\frac{k_1 - \zeta k_1^2 - \zeta k_3^2}{1 - 2\zeta k_1 + \zeta^2(k_1^2 + k_3^2)} = 0.$$

Corollary 3.3. *Let $\{\alpha = \alpha(s), \overline{\alpha} = \overline{\alpha}(\overline{s})\}$ be a Bertrand pair related to PAFORS. In that case, the following expressions hold:*

(1) *Suppose that the normal curvature of α never equals to zero. Then $\alpha = \alpha(s)$ is a geodesic trajectory whose position vector always lies in the corresponding plane $\mathrm{Sp}\{\mathbf{T}(\mathbf{s}), \mathbf{U}(\mathbf{s})\}$ if and only if*

$$\frac{\overline{k}_1 - \lambda\overline{k}_1^2 - \lambda\overline{k}_3^2}{1 - 2\lambda k_1 + \lambda^2(\overline{k}_1^2 + \overline{k}_3^2)} = 0.$$

(2) *Suppose that the normal curvature of $\overline{\alpha}$ never equals to zero. Then $\overline{\alpha} = \overline{\alpha}(\overline{s})$ is a geodesic trajectory whose position vector always lies in the corresponding plane $\mathrm{Sp}\{\overline{\mathbf{T}}(\overline{\mathbf{s}}), \overline{\mathbf{U}}(\overline{\mathbf{s}})\}$ if and only if*

$$\frac{k_1 - \zeta k_1^2 - \zeta k_3^2}{1 - 2\zeta k_1 + \zeta^2(k_1^2 + k_3^2)} = 0.$$

4. Conclusions

In this paper, we introduced the Bertrand partner trajectories related to PAFORS for the particles moving on the different regular surfaces in Euclidean 3-space. Also, we obtain the relations between the PAFORS elements of the aforesaid trajectories. Then the relations between Darboux basis vectors of them were investigated. Additionally, some characterizations were presented for some special cases of these trajectories by their PAFORS elements.

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