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INERTIAL ENERGETIC SOLUTIONS FOR SMALL-STRAIN SINGLE-SLIP ELASTO–PLASTICITY

Akira Ishikawa

We propose an energetic formulation for a small-strain single-slip elasto–plastic body with inertia and Kelvin–Voigt viscosity. The evolution couples a weak momentum balance for the displacement with a rate-independent semistability condition for the slip variable, and it is driven by a total energy that includes elastic and hardening contributions together with a strain-gradient regularization of the slip. Under variable material coefficients and H^1-regular loadings, we establish the existence of inertial energetic solutions.

Keywords: rate-independent systems, inertia, small-strain elastoplasticity, single-slip

Classification: 74C05, 74D05, 74H20

1. INTRODUCTION

Rate-independent systems (RIS) provide a variational description of activated processes when the intrinsic time scale is negligible relative to the external loading. A widely used approach is the energetic formulation, where the evolution is encoded through a stored energy, a dissipation distance, a stability condition, and an energy balance; see [8]. When inertial effects are not negligible, the evolution couples a second-order equation for the displacement with a rate-independent subsystem for an internal variable; cf. [8, Sec. 5.1].

Kink-band formation is a characteristic deformation mechanism in layered crystalline solids and has been discussed, for instance, in connection with strengthening of Mg-based long-period stacking ordered (LPSO) phases; see, e. g., [5, 10]. More broadly, energetic approaches to rate-independent crystal plasticity provide a natural variational framework for analyzing the nucleation and propagation of localized deformation bands. Compression experiments on layered structures reported in [3] show that localized kink-bands nucleate and propagate in sequence, with load drops correlated with each event (see, for instance, Figure 5 and the discussion on p. 12 in [3]). This sequential character suggests that inertia may influence the timing and selection of deformation patterns. Motivated by this observation, we aim at an energetic framework that incorporates inertia. As a first step, the present paper develops a dynamic small-strain single-slip formulation in a setting with variable coefficients, providing a baseline analytical framework for

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subsequent extensions.

Our solution concept follows the dynamic energetic framework developed in [8]. In particular, [8, Def. 5.1.1] combines a weak formulation of the momentum balance with a semistability condition for the rate-independent subsystem and an energy inequality accounting for kinetic energy and viscous dissipation. An abstract existence result is stated in [8, Thm. 5.1.2]. For concrete PDE models, however, applying such an abstract result requires verifying compactness and closedness properties needed to pass to the limit. This issue is especially delicate when the dissipation distance acts pointwise in the internal variable; compare [8, Rem. 4.2.6]. In the present single-slip setting, where γ denotes the scalar slip variable (the magnitude of slip), we therefore incorporate an explicit strain-gradient regularization $\kappa\|\nabla\gamma\|^{\alpha}_{L^{\alpha}(\Omega)}$ and choose α so that the required compactness is guaranteed. Without an additional regularization mechanism of this type, the needed compactness of the slip variable γ is not apparent, and the limit passage in the time-discrete construction seems to require further justification. The present paper makes this requirement explicit by including a strain-gradient term and provides a complete existence proof for the model under explicit and verifiable assumptions on the data and coefficients.

In our model, we take the slip γ as internal variable and prescribe the plastic strain $e^{\mathrm{p}} = \gamma S$ for a fixed symmetric slip-system projector S. The stored energy consists of a position-dependent quadratic elastic part and a hardening potential $w(\gamma)$, supplemented by the strain-gradient regularization of γ. Viscosity enters through a quadratic Kelvin–Voigt term in the strain rate, while inertia is retained through the mass density $\rho > 0$ in the momentum balance. The evolution is driven by the total energy $E(t, u, \gamma)$ and the corresponding rate-independent dissipation distance D. Under mixed boundary conditions (prescribed displacement and traction) and H^1-regular body forces and boundary tractions, we prove that the initial-value problem admits at least one inertial energetic solution (Theorem 4.1).

The paper is organized as follows. Section 2 introduces the geometric and constitutive setting of the model, together with the associated energy and dissipation functionals. In Section 3, we formulate the notion of energetic solutions with inertia and viscosity and state the corresponding weak momentum balance, semistability condition, and energy inequality. Section 4 is devoted to the proof of existence of such inertial energetic solutions under the general assumptions of Section 2. Finally, Section 5 addresses uniqueness: assuming that the stored energy density, excluding the external loading terms, is quadratic both in the displacement u and in the slip variable γ, we prove uniqueness of the solution constructed in the proof of Theorem 4.1.

2. PROBLEM SETTING

Geometric and kinematic setting. Let $\Omega \subset \mathbb{R}^d$ ($d \in \{2, 3\}$) be a bounded Lipschitz domain whose boundary is decomposed into a Dirichlet part Γ_0 and a Neumann part Γ_1, with $\Gamma_0 \cap \Gamma_1 = \emptyset$ and

$$\mathcal{H}^{d-1}(\Gamma_0) > 0, \qquad \partial\Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1}.$$

A fixed final time $T > 0$ is prescribed.

We work in the infinitesimal-strain regime. The deformation is $y : [0,T] \times \Omega \to \mathbb{R}^d$, the displacement is $u := y - \mathrm{id}$, and the symmetric strain is

$$e(u) := \frac{1}{2}\big(\nabla u + \nabla u^{\top}\big) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}.$$

Single-slip plasticity. The plastic distortion is assumed to take the additive single-slip form

$$e^{\mathrm{p}} := \gamma\,\frac{1}{2}\big(s \otimes m + m \otimes s\big) =: \gamma\, S, \qquad s, m \in \mathbb{R}^d,\ s \perp m,\ |s| = |m| = 1,$$

with scalar slip field $\gamma : [0,T] \times \Omega \to \mathbb{R}$, and $S := \frac{1}{2}(s\otimes m + m\otimes s)$ denotes the (constant) slip-system projector. The elastic strain therefore reads

$$e^{\mathrm{e}} := e(u) - e^{\mathrm{p}} = e(u) - \gamma S.$$

Stored energy. We define the elastic energy density by

$$W(x, e^{\mathrm{e}}) := \frac{1}{2}\,\mathbb{C}(x)\big(e(u) - \gamma S\big) : \big(e(u) - \gamma S\big),$$

where $\mathbb{C}(x) : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is a fourth-order elasticity tensor defined for a.e. $x \in \Omega$. We assume the positive definiteness and boundedness

$$\exists\ 0 < \lambda_0 \le \Lambda_0 < \infty \ \text{ s.t. for a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}} : \quad \lambda_0|\xi|^2 \le \mathbb{C}(x)\xi : \xi \le \Lambda_0|\xi|^2,$$

and the symmetry on symmetric tensors

$$\mathbb{C}(x)\xi : \eta = \mathbb{C}(x)\eta : \xi \quad \text{for all } \xi, \eta \in \mathbb{R}^{d\times d}_{\mathrm{sym}},\ \text{a.e. } x \in \Omega.$$

Material coefficients and external data. The following list collects all material parameters and prescribed loadings that will enter the energetic formulation.

- **Mass density** $\rho \in L^\infty(\Omega)$ with $0 < \rho_- \le \rho(x)$ for some constant ρ_-.
- **Viscous dissipation potential** We define the viscous dissipation potential as

  $$V(\dot u) := \frac{1}{2}\int_\Omega \mathbb{D}(x)\, e(\dot u) : e(\dot u)\,\mathrm{d}x,$$

  where $\mathbb{D}(x) : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is a fourth-order viscosity tensor. We assume the positive definiteness and boundedness

  $$\exists\ 0 < \delta_0 \le \Delta_0 < \infty \ \text{ s.t. for a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}} : \\ \delta_0|\xi|^2 \le \mathbb{D}(x)\xi : \xi \le \Delta_0|\xi|^2,$$

  and the symmetry on symmetric tensors

  $$\mathbb{D}(x)\xi : \eta = \mathbb{D}(x)\eta : \xi, \quad \text{for all } \xi, \eta \in \mathbb{R}^{d\times d}_{\mathrm{sym}},\ \text{a.e. } x \in \Omega.$$

- **Slip resistance** $\theta \in C(\overline{\Omega})$ with $0 < \theta_0 \le \theta(x)$ for some constant θ_0.
- **Hardening potential** $w : \mathbb{R} \to \mathbb{R}_{\ge 0}$ is strictly convex and satisfies the q-growth condition

$$C_1|\gamma|^q - c_0 \;\le\; w(\gamma) \;\le\; C_2\big(1 + |\gamma|^q\big), \qquad C_1, C_2 > 0,\; c_0 \ge 0,\; q \ge 2.$$

- **Strain-gradient regularization** We add the term $\kappa|\nabla\gamma|^\alpha$ to the free energy of the model, with exponent $\alpha > 0$ and coefficient $\kappa > 0$.

  The compactness requirement is given by $\alpha \ge q$. This condition is related to the Gagliardo–Nirenberg inequality: define $\eta \in \mathbb{R}$ by

$$\frac{1}{\alpha} = \frac{1-\eta}{q} + \eta\Big(\frac{1}{\alpha} - \frac{1}{d}\Big) \quad\Longleftrightarrow\quad \eta = \frac{\frac{1}{q} - \frac{1}{\alpha}}{\frac{1}{d} + \frac{1}{q} - \frac{1}{\alpha}} = \frac{d\Big(\frac{1}{q} - \frac{1}{\alpha}\Big)}{1 + d\Big(\frac{1}{q} - \frac{1}{\alpha}\Big)}.$$

  Then, by the assumption $\alpha \ge q$, we have $\eta \in [0, 1)$. By the Gagliardo–Nirenberg theorem, for a bounded domain $\Omega \subset \mathbb{R}^d$ it holds that

$$\|\gamma\|_{L^\alpha(\Omega)} \le C\Big(\|\nabla\gamma\|^{\eta}_{L^\alpha(\Omega)}\, \|\gamma\|^{1-\eta}_{L^q(\Omega)} + \|\gamma\|_{L^q(\Omega)}\Big). \tag{1}$$

  Moreover, when $\alpha \ge q$, by the Rellich–Kondrachov theorem, $W^{1,\alpha}(\Omega)$ is compactly embedded into $L^q(\Omega)$.
- **Boundary displacement and external loads** We work with homogeneous Dirichlet data on the designated part Γ_0 of the boundary. We define the closed subspace of $H^1(\Omega; \mathbb{R}^d)$ as

$$H^1_{\Gamma_0}(\Omega; \mathbb{R}^d) := \big\{\, v \in H^1(\Omega; \mathbb{R}^d) \;\big|\; \operatorname{Tr} v = 0 \text{ on } \Gamma_0 \,\big\},$$

  where Tr denotes the trace operator. At each time $t \in [0, T]$ the state variables (u, γ) are restricted to the set

$$\mathcal{Q} := H^1_{\Gamma_0}(\Omega; \mathbb{R}^d) \times W^{1,\alpha}(\Omega).$$

  The external loads consist of a body force $f \in H^1\big(0, T; L^2(\Omega; \mathbb{R}^d)\big)$ and a boundary traction $g \in H^1\big(0, T; L^2(\Gamma_1; \mathbb{R}^d)\big)$.

  Let

$$\sigma := \mathbb{C}\big(e(u) - \gamma S\big) + \mathbb{D}\, e(\dot{u})$$

  denote the Cauchy stress and n the unit outward normal to $\partial\Omega$. The body force acts in the bulk through the balance

$$\rho\, \ddot{u} - \operatorname{div} \sigma = f(t) \quad \text{in } \Omega \text{ for a.e. } t \in (0, T),$$

  while the traction is imposed on Γ_1 by the Neumann condition

$$\sigma\, n = g(t) \quad \text{on } \Gamma_1 \text{ for a.e. } t \in (0, T).$$

Energy functional and rate-independent dissipation. We now specify the kinetic energy K, the total free energy E, the rate-independent slip dissipation R, and the corresponding dissipation distance D that drive the evolution.

$$K(\dot u) := \int_\Omega \frac{\rho(x)}{2}\,|\dot u|^2\,\mathrm{d}x,$$

$$E(t,u,\gamma) := \int_\Omega \Big[W\big(e^{\mathrm{e}}\big) + w(\gamma) + \kappa|\nabla\gamma|^\alpha\Big]\mathrm{d}x + L(t,u),$$

$$R(\dot\gamma) := \int_\Omega \theta(x)\,|\dot\gamma(x)|\,\mathrm{d}x, \qquad D(\gamma_1,\gamma_2) := \int_\Omega \theta(x)\,|\gamma_1-\gamma_2|\,\mathrm{d}x,$$

where

$$L(t,u) := -\int_\Omega f(t)\cdot u\,\mathrm{d}x - \int_{\Gamma_1} g(t)\cdot u\,\mathrm{d}S.$$

3. ENERGETIC FORMULATION WITH INERTIA

Definition 3.1. (Local solution with inertia and viscosity) A pair (u,γ) satisfying

$$u \in H^1\big(0,T;H^1_{\Gamma_0}(\Omega;\mathbb{R}^d)\big), \quad \dot u \in C_w([0,T];L^2(\Omega;\mathbb{R}^d)),$$
$$\gamma \in BV\big([0,T];L^1(\Omega)\big) \cap L^\alpha\big(0,T;W^{1,\alpha}(\Omega)\big)$$

is called a local solution with inertia and viscosity if the following hold:

1. **Weak momentum balance**

$$\int_0^T \left[\int_\Omega \Big(\mathbb{C}\big(e(u(t)) - \gamma(t)S\big) + \mathbb{D}\,e(\dot u(t))\Big) : e(\phi(t))\,\mathrm{d}x - \int_\Omega \rho\,\dot u(t)\cdot\dot\phi(t)\,\mathrm{d}x\right]\mathrm{d}t$$
$$+ \int_\Omega \rho\,\dot u(T)\cdot\phi(T)\,\mathrm{d}x$$
$$= \int_\Omega \rho\,\dot u(0)\cdot\phi(0)\,\mathrm{d}x - \int_0^T L(t,\phi(t))\,\mathrm{d}t,$$

for all $\phi \in H^1\big(0,T;H^1_{\Gamma_0}(\Omega;\mathbb{R}^d)\big)$. Here $\dot u(0)$ and $\dot u(T)$ denote the endpoint values of the weakly continuous $L^2(\Omega;\mathbb{R}^d)$-representative of $\dot u$.

2. **Local semi-stability**

For a.e. $t \in [0,T]$,

$$E(t,u(t),\gamma(t)) \;\le\; E(t,u(t),\widetilde\gamma) + D(\gamma(t),\widetilde\gamma) \quad \forall\,\widetilde\gamma \in W^{1,\alpha}(\Omega).$$

3. **Energy inequality**

For a.e. $t_1, t_2\,(0 \le t_1 \le t_2 \le T)$,

$$K(\dot u(t_2)) + E(t_2,u(t_2),\gamma(t_2)) + \mathrm{Var}_D(\gamma;[t_1,t_2]) + \int_{t_1}^{t_2}\int_\Omega \mathbb{D}\,e(\dot u(s)) : e(\dot u(s))\,\mathrm{d}x\mathrm{d}s$$
$$\le K(\dot u(t_1)) + E(t_1,u(t_1),\gamma(t_1)) + \int_{t_1}^{t_2} \partial_s L(s,u(s))\,\mathrm{d}s,$$

where, for $\gamma \in BV([0,T];L^1(\Omega))$, the D-variation of γ on $[t_1,t_2]$ is defined by

$$\mathrm{Var}_D(\gamma;[t_1,t_2]) := \sup\left\{ \sum_{j=1}^{N} D\big(\gamma(s_{j-1}),\gamma(s_j)\big) \;\middle|\; N\in\mathbb{N},\ t_1 = s_0 < s_1 < \cdots < s_N = t_2 \right\}.$$

4. EXISTENCE THEOREM

Theorem 4.1. (Existence of inertial energetic solutions) Assume the data and constitutive functions stated in Section 2, and in particular assume that $\alpha \geq q$. For every initial triple $(u^0,v^0,\gamma^0) \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^d)\times L^2(\Omega;\mathbb{R}^d)\times W^{1,\alpha}(\Omega)$, there exists at least one energetic solution with inertia in the sense of Definition 3.1, satisfying $u(0) = u^0$, $\dot u(0) = v^0$, and $\gamma(0) = \gamma^0$.

Although it is necessary to include the kinetic energy term and to establish the weak momentum balance, the overall structure of the proof is the same as that of the existence proof for the energetic solutions of classical RIS. Namely, we first formulate the minimization problem at each discrete time step and prove the existence of its solution (Proposition 4.2). Next, we derive uniform estimates that are independent of the time discretization parameter (Proposition 4.4, Corollary 4.5). Finally, by exploiting appropriate compactness results deduced from these uniform estimates, we obtain various convergences and pass to the limit in the discrete weak momentum balance, the discrete stability condition, and the discrete energy inequality obtained from the time-discrete minimization problem.

Discrete minimisation at a fixed time-step

Fix a step–size $\tau = T/N > 0$ ($N\in\mathbb{N}$), and a previous state $(u^k,\gamma^k)\in\mathcal{Q}$ with $t_k = k\tau$. Denote $t_{k+1} = t_k+\tau$. For $k=0$ the second order difference term $\frac{1}{\tau^2}(u-2u^k+u^{k-1})$ involves a fictitious value u^{-1}; we set $u^{-1} := u^0 - \tau v^0$, so that the discrete velocity $(u^0-u^{-1})/\tau$ matches the prescribed initial velocity v^0. We introduce the functional

$$\mathcal{J}^k_\tau(u,\gamma) := \frac{1}{2\tau^2}\int_\Omega \rho\,\big|\,u-2u^k+u^{k-1}\big|^2\,\mathrm{d}x \;+\; \tau\,V\Big(\tfrac{u-u^k}{\tau}\Big) \;+\; E\big(t_{k+1},u,\gamma\big). \tag{2}$$

Proposition 4.2. (Well-posedness of one time-step) Under the hypotheses of Theorem 4.1 and for every $\tau>0$, $N\in\mathbb{N}$, and $k=0,\dots,N-1$, consider the following two incremental minimization problems:

$$\left.\begin{array}{rl} \text{minimize} & \mathcal{J}^k_\tau(u,\gamma^k) \\ \text{subject to} & u\in H^1_{\Gamma_0}(\Omega;\mathbb{R}^d), \end{array}\right\} \tag{3}$$

and, for any u^{k+1} chosen as a minimiser of the first problem,

$$\left.\begin{array}{rl} \text{minimize} & E(t_{k+1},u^{k+1},\gamma)+D(\gamma^k,\gamma) \\ \text{subject to} & \gamma\in W^{1,\alpha}(\Omega). \end{array}\right\} \tag{4}$$

Then both problems admit a unique solution. In particular, there exists a unique $u^{k+1} \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^d)$ solving (3), and, for this u^{k+1}, there exists a unique $\gamma^{k+1} \in W^{1,\alpha}(\Omega)$ solving (4).

In what follows, whenever no confusion can arise, we omit Ω and $\mathbb{R}^d$ from the notation for L^p and Sobolev spaces.

P r o o f. **Step 1: Coercivity.** Both minimization problems consist of nonnegative terms and include the functional E. Therefore, it is sufficient to establish the coercivity of $E(t,u,\gamma)$.

For the external loading terms, since $\mathcal{H}^{d-1}(\Gamma_0) > 0$ and $u \in H^1_{\Gamma_0}$, we can use the Poincaré inequality and the trace theorem: there exists $\widetilde{C}_{\mathrm{tr}} = \widetilde{C}_{\mathrm{tr}}(\Omega,\Gamma_0) > 0$ such that for all u satisfying $u \in H^1_{\Gamma_0}$,

$$\|u\|_{L^2(\Gamma_1)} \;\le\; \|u\|_{L^2(\partial\Omega)} \;\le\; \widetilde{C}_{\mathrm{tr}}\,\|\nabla u\|_{L^2}. \tag{5}$$

Hence, by Young's inequality, for any $\varepsilon_1 > 0$,

$$\left|\int_\Omega f\cdot u\,\mathrm{d}x + \int_{\Gamma_1} g\cdot u\,\mathrm{d}S\right| \;\le\; \varepsilon_1\|\nabla u\|^2_{L^2} + C_{\varepsilon_1}\big(\|f\|^2_{L^2} + \|g\|^2_{L^2(\Gamma_1)}\big). \tag{6}$$

For the positive contributions of elasticity, hardening, and strain-gradient regularization, the coercivity of $\mathbb{C}$ with constant λ_0 implies

$$\int_\Omega \Big[\frac{1}{2}\mathbb{C}\big(e^{\mathrm{e}}(u)\big) : (e^{\mathrm{e}}(u)) + w(\gamma) + \kappa|\nabla\gamma|^\alpha\Big]\,\mathrm{d}x$$
$$\ge\; \frac{\lambda_0}{2}\|e^{\mathrm{e}}(u)\|^2_{L^2} + C_1\|\gamma\|^q_{L^q} + \kappa\|\nabla\gamma\|^\alpha_{L^\alpha} - c_0|\Omega|. \tag{7}$$

By Young's inequality, we obtain

$$\|e(u)\|^2_{L^2} \;\le\; 2\|e^{\mathrm{e}}(u)\|^2_{L^2} + 2|S|^2\|\gamma\|^2_{L^2} \;=\; 2\|e^{\mathrm{e}}(u)\|^2_{L^2} + \|\gamma\|^2_{L^2}. \tag{8}$$

Since the domain Ω is bounded and $q \ge 2$, we have the embedding $L^q \hookrightarrow L^2$. First, consider the case where $q > 2$. In this case, for any $\varepsilon_2 > 0$, by choosing C_{ε_2} sufficiently large, we obtain

$$\|\gamma\|^2_{L^2} \;\le\; C_\Omega\,\|\gamma\|^2_{L^q} \;\le\; C_{\varepsilon_2} + \varepsilon_2\|\gamma\|^q_{L^q}. \tag{9}$$

Combining (8) and (9) yields

$$\|e(u)\|^2_{L^2} \;\le\; 2\|e^{\mathrm{e}}(u)\|^2_{L^2} + C_{\varepsilon_2} + \varepsilon_2\|\gamma\|^q_{L^q}. \tag{10}$$

Combining (6), (7), Korn's inequality for $u \in H^1_{\Gamma_0}$, and (10), we find that, by taking $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ sufficiently small, the negative terms can be absorbed. Therefore, there exist constants $c, C_0 > 0$ such that

$$E(t,u,\gamma) \;\ge\; c\Big(\|\nabla u\|^2_{L^2} + \|\gamma\|^q_{L^q} + \|\nabla\gamma\|^\alpha_{L^\alpha}\Big) - C_0. \tag{11}$$

By the Gagliardo–Nirenberg inequality, the $W^{1,\alpha}$-norm of γ can also be bounded from above in terms of E.
In the case $q = 2$, we take λ_1 such that $0 < \lambda_1 < \min\{\lambda_0, 4C_1\}$. The estimate (7) can be relaxed as follows:

$$\int_\Omega \Big[\frac{1}{2}\mathbb{C}\big(e^{\mathrm{e}}(u)\big) : (e^{\mathrm{e}}(u)) + w(\gamma) + \kappa|\nabla\gamma|^\alpha\Big]\,\mathrm{d}x \\ \geq \frac{\lambda_1}{2}\|e^{\mathrm{e}}(u)\|^2_{L^2} + C_1\|\gamma\|^q_{L^q} + \kappa\|\nabla\gamma\|^\alpha_{L^\alpha} - c_0|\Omega|.$$

With this relaxation, from (8), we can again absorb the negative term with respect to γ, and thus the required coercivity is obtained.

Step 2: Compactness. Let $\{u_n\}_{n\in\mathbb{N}}$ be a minimizing sequence for (3). By the coercivity established in Step 1, we may extract a subsequence (not relabelled) such that

$$u_n \rightharpoonup u^* \quad \text{in } H^1.$$

The boundary space $H^1_{\Gamma_0}$ is weakly closed; thus $u^* \in H^1_{\Gamma_0}$.

Similarly, let $\{\gamma_m\}$ be a minimizing sequence for (4). Then, after extracting a subsequence,

$$\gamma_m \rightharpoonup \gamma^* \quad \text{in } W^{1,\alpha}.$$

Because the embedding $W^{1,\alpha} \hookrightarrow L^q$ is compact, $\gamma_m \to \gamma^*$ strongly in L^q.

Step 3: Lower semicontinuity. The inertial term $u \mapsto \int_\Omega \rho\,|u - 2u^k + u^{k-1}|^2\,\mathrm{d}x$ is the square of a weighted L^2-norm; since $0 < \rho_- \leq \rho \leq \rho^+$ a.e., it is convex and weakly l.s.c. in L^2. The viscous and elastic contributions, given by integrals of positive-definite quadratic forms in $e\Big(\frac{u-u^k}{\tau}\Big)$ and $e(u) - \gamma S$, are weakly l.s.c. under $u_n \rightharpoonup u$ in H^1. Moreover, $\|\nabla\gamma\|^\alpha_{L^\alpha}$ is weakly l.s.c. in $W^{1,\alpha}$, while $D(\gamma,\gamma^k) = \int_\Omega \theta|\gamma - \gamma^k|\,\mathrm{d}x$ and $\int_\Omega w(\gamma)\,\mathrm{d}x$ are continuous along the established strong L^q-convergence of γ_m; the loading terms are linear and hence weakly continuous. Therefore, the objective functionals are sequentially weakly lower semicontinuous on the admissible sets.

u^* and γ^*, by lower semicontinuity, actually attain the lower limit of the minimizing sequence and therefore realize the minimum. Moreover, all the functionals appearing above, except for the dissipation distance D and the linear loading terms, are not only convex but strictly convex. Since D and the linear loading terms are convex and added to strictly convex functionals, both mappings

$$u \mapsto \mathcal{J}^k_\tau(u,\gamma^k) \quad \text{and} \quad \gamma \mapsto E(t_{k+1}, u^{k+1}, \gamma) + D(\gamma^k, \gamma)$$

are strictly convex on their respective domains. Therefore, each of the minimization problems (3) and (4) admits a unique minimizer.

Consequently, $u^{k+1} := u^*$ and $\gamma^{k+1} := \gamma^*$ are unique minimizers of (3) and (4), respectively. $\square$

Time interpolation and uniform τ−estimates

Let $\{(u^k,\gamma^k)\}_{k=0}^N$ be the sequence of discrete solutions obtained by successive minimisation (Proposition 4.2) with time grid $t_k = k\tau$ $(k = 0,\dots,N)$ and set $t_{-1} = -\tau$. Recalling $u^{-1} := u^0 - \tau v^0$, we also set $\gamma^{-1} := \gamma^0$.

Define the piecewise affine/constant interpolants on the interval $[-\tau, T]$ as follows: For $k = 0, \ldots, N$ and $t \in [t_{k-1}, t_k)$, set

$$\begin{aligned}
u_\tau(t) &:= u^{k-1} + \frac{t - t_{k-1}}{\tau}\big(u^k - u^{k-1}\big), \\
\overline{u}_\tau(t) &:= u^{k-1}, \\
\gamma_\tau(t) &:= \gamma^{k-1} + \frac{t - t_{k-1}}{\tau}\big(\gamma^k - \gamma^{k-1}\big), \\
\overline{\gamma}_\tau(t) &:= \gamma^{k-1}.
\end{aligned}$$

At the endpoint $t = T$, we set

$$u_\tau(T) := u^N, \qquad \overline{u}_\tau(T) := u^N, \qquad \gamma_\tau(T) := \gamma^N, \qquad \overline{\gamma}_\tau(T) := \gamma^N.$$

The difference quotients

$$v^k := \frac{u^k - u^{k-1}}{\tau}, \qquad \beta^k := \frac{\gamma^k - \gamma^{k-1}}{\tau}, \qquad k = 1, \ldots, N,$$

represent the velocity vectors on each time interval. They satisfy

$$\dot{u}_\tau(t) = v^k, \qquad \dot{\gamma}_\tau(t) = \beta^k, \qquad t \in [t_{k-1}, t_k), \quad k = 1, \ldots, N,$$

while at the endpoint $t = T$ we have

$$\dot{u}_\tau(T) = v^N, \qquad \dot{\gamma}_\tau(T) = \beta^N.$$

Note that for the piecewise affine interpolation γ_τ, one has

$$\mathrm{Var}_D(\gamma_\tau; [t_m, t_n]) = \sum_{k=m}^{n-1} D(\gamma^k, \gamma^{k+1}) = \int_{t_m}^{t_n} R(\dot{\gamma}_\tau(s))\, \mathrm{d}s \qquad (0 \le m < n \le N).$$

In addition, we define the piecewise constant functions K_τ and E_τ on $[0, T]$ by

$$\begin{aligned}
K_\tau(t) &:= K^k := K(v^k) = \frac{1}{2}\int_\Omega \rho\, |v^k|^2\, \mathrm{d}x, \quad && t \in [t_k, t_{k+1}), \quad k = 0, \ldots, N-1, \\
E_\tau(t) &:= E^k := E(t_k, u^k, \gamma^k), && t \in [t_k, t_{k+1}), \quad k = 0, \ldots, N-1.
\end{aligned}$$

At the endpoint $t = T$, we set

$$K_\tau(T) := K^N := \frac{1}{2}\int_\Omega \rho\, |v^N|^2\, \mathrm{d}x, \qquad E_\tau(T) := E^N := E(T, u^N, \gamma^N).$$

The initial values are

$$K_\tau(0) := K^0 = K(v^0), \qquad E_\tau(0) := E^0 = E(0, u^0, \gamma^0),$$

where the triplet (u^0, v^0, γ^0) is prescribed in Theorem 4.1.

Lemma 4.3. (Discrete energy inequality) For every $t \in \{0, \tau, \cdots N\tau\}$,

$$K_\tau(t) + E_\tau(t) + \int_0^t \Big[2V\big(\dot{u}_\tau(s)\big) + R\big(\dot{\gamma}_\tau(s)\big)\Big]\mathrm{d}s \;\le\; K^0 + E^0 + \int_0^t \partial_s L\big(s, \overline{u}_\tau(s)\big)\mathrm{d}s \quad (12)$$

Proof. **Step 1: Minimality.** Following the approach in [8] (see pp. 468–469), we consider the Euler–Lagrange equation associated with (3) and test it by $v^{k+1} := (u^{k+1} - u^k)/\tau$. Rearranging terms and using the standard estimate $a\cdot(a-b) \ge \big(|a|^2 - |b|^2\big)/2$ for the quadratic inertial and elastic contributions yields a discrete energy inequality. Moreover, testing (4) with the competitor γ^k (so that $D(\gamma^k, \gamma^{k+1}) = \tau R(\beta^{k+1})$) and combining the two estimates gives

$$K^{k+1} + 2\tau V(v^{k+1}) + \tau R(\beta^{k+1}) + E^{k+1} \le K^k + E(t_{k+1}, u^k, \gamma^k). \quad (13)$$

Step 2: Isolating the explicit time derivative. Because $E(t, u, \gamma)$ depends on t only through L, we have

$$E(t_k, u^k, \gamma^k) = E(t_{k+1}, u^k, \gamma^k) - \int_{t_k}^{t_{k+1}} \partial_s L\big(s, u^k\big)\mathrm{d}s. \quad (14)$$

Subtracting (14) from (13) and rearranging yields

$$\big[K^{k+1} + E^{k+1}\big] - \big[K^k + E^k\big] + 2\tau V(v^{k+1}) + \tau R(\beta^{k+1}) \le \int_{t_k}^{t_{k+1}} \partial_s L\big(s, u^k\big)\mathrm{d}s. \quad (15)$$

Step 3: Telescopic sum. For n with $n \in \{1, \cdots N\}$ sum (15) over $k = 0, \ldots, n-1$:

$$K^n + E^n + 2\sum_{k=0}^{n-1} \tau V(v^{k+1}) + \sum_{k=0}^{n-1} \tau R(\beta^{k+1}) \le K^0 + E^0 + \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \partial_s L\big(s, u^k\big)\mathrm{d}s. \quad (16)$$

Step 4: Identification with the interpolants. Since $\dot{u}_\tau$ and $\dot{\gamma}_\tau$ are interval-wise constants v^{k+1} and β^{k+1}, the Riemann sums equal the integrals for $t = n\tau$:

$$\sum_{k=0}^{n-1} \tau V(v^{k+1}) = \int_0^t V(\dot{u}_\tau)\,\mathrm{d}s, \quad \sum_{k=0}^{n-1} \tau R(\beta^{k+1}) = \int_0^t R(\dot{\gamma}_\tau)\,\mathrm{d}s.$$

Moreover $K^n = K_\tau(n\tau)$, $E^n = E_\tau(n\tau)$, and $\overline{u}_\tau(s) = u^k$ for $s \in [t_k, t_{k+1})$. Thus (16) becomes (12).
It is straightforward to extend this result to arbitrary $t, t' \in \{0, \tau, \ldots, N\tau\}$ with $t \le t'$ as follows:

$$K_\tau(t') + E_\tau(t') + \int_t^{t'} \Big[2V\big(\dot{u}_\tau(s)\big) + R\big(\dot{\gamma}_\tau(s)\big)\Big]\,\mathrm{d}s \;\le\; K_\tau(t) + E_\tau(t) + \int_t^{t'} \partial_s L\big(s, \overline{u}_\tau(s)\big)\,\mathrm{d}s. \quad (17)$$

□

Proposition 4.4. (Uniform τ–bounds) There exists a constant $C > 0$, independent of τ, such that

$$\|\dot{u}_\tau\|_{L^\infty(0,T;L^2)} \leq C, \qquad \|\nabla u_\tau\|_{L^\infty(0,T;L^2)} \leq C, \tag{18}$$

$$\|\gamma_\tau\|_{L^\infty(0,T;L^q)} \leq C, \qquad \|\nabla \gamma_\tau\|_{L^\infty(0,T;L^\alpha)} \leq C, \tag{19}$$

$$\|e(\dot{u}_\tau)\|_{L^2(0,T;L^2)} \leq C, \qquad \int_0^T R(\dot{\gamma}_\tau)\,\mathrm{d}s \leq C. \tag{20}$$

Proof. **Step 1: Forcing term estimate.** For $t = n\tau$ with $0 \leq n \leq N$, we have $\overline{u}_\tau(t) \in H^1_{\Gamma_0}$. Hence, the Poincaré inequality yields

$$\|\overline{u}_\tau(t)\|_{H^1} \leq C_P \|\nabla \overline{u}_\tau(t)\|_{L^2}$$

and the trace estimate yields

$$\|\overline{u}_\tau(t)\|_{L^2(\Gamma_1)} \leq C_{\mathrm{tr}} \|\overline{u}_\tau(t)\|_{H^1}.$$

For the loading potential we compute

$$\partial_t L\big(t, \overline{u}_\tau(t)\big) = -\int_\Omega \dot{f}(t) \cdot \overline{u}_\tau(t)\,\mathrm{d}x - \int_{\Gamma_1} \dot{g}(t) \cdot \overline{u}_\tau(t)\,\mathrm{d}S,$$

so that by Cauchy–Schwarz, Poincaré, and the trace inequality,

$$\big|\partial_t L\big(t, \overline{u}_\tau(t)\big)\big| \leq C\big(\|\dot{f}(t)\|_{L^2} + \|\dot{g}(t)\|_{L^2(\Gamma_1)}\big)\|\nabla \overline{u}_\tau(t)\|_{L^2}.$$

By Hölder and Young inequalities, for any $\delta > 0$ we obtain

$$\begin{aligned}\int_0^{n\tau} \big|\partial_t L\big(t, \overline{u}_\tau(t)\big)\big|\,\mathrm{d}t &\leq C\Big(\|\dot{f}\|_{L^2(0,T;L^2)} + \|\dot{g}\|_{L^2(0,T;L^2(\Gamma_1))}\Big)\|\nabla \overline{u}_\tau\|_{L^2(0,T;L^2)} \\ &\leq \delta\,\|\nabla \overline{u}_\tau\|^2_{L^2(0,T;L^2)} + C_L,\end{aligned}$$

where $C_L > 0$ is a constant depending on δ, $\|\dot{f}\|_{L^2(0,T;L^2)}$, $\|\dot{g}\|_{L^2(0,T;L^2(\Gamma_1))}$.

Step 2: Lower estimate of the internal energy. Applying (11) with $(u, \gamma) = (u^n, \gamma^n)$, we find

$$E^n \geq c\left(\|\nabla u^n\|^2_{L^2} + \|\gamma^n\|^q_{L^q} + \|\nabla \gamma^n\|^\alpha_{L^\alpha}\right) - C_0. \tag{21}$$

Similarly, $K^n \geq \frac{\rho_-}{2}\|v^n\|^2_{L^2}$.

Step 3: Comparison and absorption. Let $\|\nabla u^m\|_{L^2} = \max_n \{\|\nabla u^n\|_{L^2}\}$, then,

$$\|\nabla \overline{u}_\tau\|^2_{L^2(0,T;L^2)} \leq T\,\|\nabla u^m\|^2_{L^2}.$$

Discarding the non-negative viscous and dissipation terms in the discrete energy inequality, combining with (21), and gathering constants gives for any n $(1 \leq n \leq N)$

$$\frac{\rho_-}{2}\|v^n\|^2_{L^2} + c\big(\|\nabla u^n\|^2_{L^2} + \|\gamma^n\|^q_{L^q} + \|\nabla \gamma^n\|^\alpha_{L^\alpha}\big)$$

$$\begin{aligned}&\le\ K^0+E^0+C_0+\delta\,\|\nabla\overline{u}_\tau\|^2_{L^2(0,T;L^2)}+C_L\\&\le\ K^0+E^0+C_0+\delta T\,\|\nabla u^m\|^2_{L^2}+C_L.\end{aligned}$$

By taking $n=m$ on the left-hand side, choosing $\delta=c/(2T)$ and defining $C_{\text{in}}:=K^0+E^0+C_0+C_L$, we conclude

$$\frac{c}{2}\|\nabla u^m\|^2_{L^2}\le\ C_{\text{in}}$$

where C_{in} and c are independent of τ. So, $\|\nabla u^n\|_{L^2}\le\|\nabla u^m\|_{L^2}\le\sqrt{2C_{\text{in}}/c}$, and

$$\frac{\rho_-}{2}\|v^n\|^2_{L^2}+c\,(\|\gamma^n\|^q_{L^q}+\|\nabla\gamma^n\|^\alpha_{L^\alpha})\ \le\ C.$$

Here, C does not depend on τ or n, either. From these, (18) and (19) follow. Returning to the discrete energy inequality, by uniformly estimating the right-hand side, we obtain (20). □

Corollary 4.5. There exists a constant $C>0$, independent of τ, such that

$$\|\ddot{u}_\tau\|_{\mathscr{M}([0,T];\,H^{-1}_{\Gamma_0})}\le C.$$

Here, $\mathscr{M}\left([0,T];H^{-1}_{\Gamma_0}\right)$ denotes the Banach space of $H^{-1}_{\Gamma_0}$-valued Radon measures on $[0,T]$.

P r o o f. This can be proved analogously to (5.1.36) in [8]; see p. 471. □

With the above preparations, we now prove the existence of (local) energetic solutions in accordance with Definition 3.1.

P r o o f. (of Theorem 4.1) Since the boundary Γ_0 has positive measure, by Korn's inequality with (18) and (20), we have

$$\exists\,C_K>0:\quad\|\dot{u}_\tau(t)\|_{H^1}\le C_K\Big(\|e(\dot{u}_\tau(t))\|_{L^2}+\|\dot{u}_\tau(t)\|_{L^2}\Big)\le C,$$

and the Poincaré inequality together with (18) yields

$$\|u_\tau\|_{H^1(0,T;H^1_{\Gamma_0})}\le C,\tag{22}$$

so that $\{u_\tau\}$ is bounded in $H^1(0,T;H^1_{\Gamma_0})$. Hence, there exist u and a subsequence (not relabeled) such that

$$u_\tau\rightharpoonup u\quad\text{in }H^1(0,T;H^1_{\Gamma_0}).$$

On the other hand, by (1) with (19),

$$\|\gamma_\tau\|_{L^\infty(0,T;W^{1,\alpha})}<C.$$

So, by the Banach–Alaoglu theorem, there exist γ and a subsequence such that

$$\gamma_\tau\overset{*}{\rightharpoonup}\gamma\quad\text{in }L^\infty\big(0,T;W^{1,\alpha}\big).$$

In the following, we may extract further subsequences without further mention.

Step 1: Compactness

Let us show that Proposition 4.4 and Aubin–Lions compactness theory give

$$
\begin{aligned}
&u_\tau \to u && \text{in } C\big([0,T];L^2\big), && \text{(23a)}\\
&u_\tau(t) \to u(t) && \text{for a.e. } t\in[0,T] \text{ in } H^1, && \text{(23b)}\\
&\dot u_\tau \rightharpoonup \dot u && \text{in } L^2\big(0,T;H^1\big), && \text{(23c)}\\
&\dot u_\tau \to \dot u && \text{in } L^2\big(0,T;L^2\big), && \text{(23d)}\\
&\gamma_\tau \to \gamma && \text{in } L^q\big(0,T;L^q\big), && \text{(23e)}\\
&\dot\gamma_\tau \overset{*}{\rightharpoonup} \dot\gamma && \text{in } \mathscr{M}\big((0,T)\times\Omega\big). && \text{(23f)}
\end{aligned}
$$

Strong compactness of u_τ in $C([0,T];L^2)$ (23a):
Since u_τ is bounded in $L^\infty(0,T;H^1)$ and $\dot u_\tau$ is bounded in $L^\infty(0,T;L^2)\subset L^2(0,T;H^{-1})$, we can apply the Simon's version of the Aubin–Lions theorem (see Theorem 7.1 below) with

$$H^1 \hookrightarrow\hookrightarrow L^2 \hookrightarrow H^{-1} \tag{24}$$

to obtain (23a).

Strong compactness of $\dot u_\tau$ in $L^2(0,T;L^2)$ (23d):
From (22), $\dot u_\tau$ is bounded in $L^2(0,T;H^1)$. Since we already know that $\ddot u_\tau \in \mathscr{M}(0,T;H^{-1}_{\Gamma_0})$ is bounded as a measure from Corollary 4.5, we obtain that $\dot u_\tau$ is uniformly bounded in $BV([0,T];H^{-1}_{\Gamma_0})$. This uniform (in τ) BV–in–time bound implies, by a standard BV computation, the time–translation estimate

$$\|\dot u_\tau(\cdot+h)-\dot u_\tau(\cdot)\|_{L^2(0,T-h;H^{-1}_{\Gamma_0})} \le C\,h^{1/2} \quad \text{for all } h\in(0,T),$$

hence Simon's time–translation condition in Theorem 7.1 (Case 1) holds; see [12, Thm. 5]. By applying Theorem 7.1 with

$$H^1 \hookrightarrow\hookrightarrow L^2 \hookrightarrow H^{-1}_{\Gamma_0},$$

we obtain (23d).

Strong compactness of γ_τ in $L^q(0,T;L^q)$ (23e):
Since $\alpha\ge q$, applying the Gagliardo–Nirenberg inequality recalled in the problem setting to $\gamma_\tau(t)$ for every $t\in(0,T)$, and combining this with (19), we obtain

$$\|\gamma_\tau\|_{L^\infty(0,T;L^\alpha)} \le C, \qquad \|\gamma_\tau\|_{L^\infty(0,T;W^{1,\alpha})} \le C.$$

Moreover, combining (20) with $\theta\ge\theta_0$, we obtain

$$\int_0^T \|\dot\gamma_\tau(s)\|_{L^1}\,\mathrm{d}s \le \theta_0^{-1}\int_0^T R(\dot\gamma_\tau(s))\,\mathrm{d}s \le C,$$

so that $\{\gamma_\tau\}$ is bounded in $BV([0,T];L^1)$. Applying the same compactness principle as above with

$$W^{1,\alpha} \hookrightarrow\hookrightarrow L^\alpha \hookrightarrow L^1,$$

we conclude, up to a subsequence, that

$$\gamma_\tau \to \gamma \quad \text{in } L^p\big(0,T;L^\alpha\big) \quad \text{for every } 1 \le p < \infty.$$

Since $\alpha \ge q$ and Ω is bounded, the embedding $L^\alpha(\Omega) \hookrightarrow L^q(\Omega)$ is continuous. In particular, choosing $p = q$, we obtain (23e).

By (20) and the lower bound $\theta \ge \theta_0 > 0$, the sequence $\{\gamma_\tau\}$ is uniformly bounded in $BV([0,T];L^1(\Omega))$. Since Ω is bounded, (23e) implies $\gamma_\tau \to \gamma$ in $L^1(0,T;L^1(\Omega))$. Hence, by the lower semicontinuity of the total variation, we obtain

$$\gamma \in BV([0,T];L^1(\Omega)).$$

Weak-$*$ convergence of $\dot\gamma_\tau$ in $\mathscr{M}((0,T)\times\Omega)$ (23f):
Moreover, $\dot\gamma_\tau(t,x)$ is uniformly bounded in $L^1\big((0,T)\times\Omega\big)$ and hence in total variation in

$$\mathscr{M}((0,T)\times\Omega) \;=\; \big(C_0((0,T)\times\Omega)\big)^*.$$

By the Banach–Alaoglu theorem, there exists a subsequence (not relabeled) and $\mu \in \mathscr{M}((0,T)\times\Omega)$ such that

$$\dot\gamma_\tau \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathscr{M}((0,T)\times\Omega).$$

By (23e), for any $\varphi \in C_c^\infty((0,T)\times\Omega)$ we have

$$\iint \varphi\,\mathrm{d}(\dot\gamma_\tau) = -\iint \gamma_\tau\,\dot\varphi\,\mathrm{d}x\mathrm{d}t \;\longrightarrow\; -\iint \gamma\,\dot\varphi\,\mathrm{d}x\mathrm{d}t.$$

On the other hand, by the weak-$*$ convergence above, the left-hand side converges to $\iint \varphi\,\mathrm{d}\mu$. Hence, $\mu = \dot\gamma$ in the sense of distributions. Therefore, (23f) follows.

In particular, the measure $\dot\gamma$ identified in (23f) is exactly the distributional time derivative of the limit slip γ, and it will be used in Step 4 to identify the measure representation of the $\mathcal{D}$-variation away from jump times.

Pointwise-in-time strong convergence of u_τ in H^1 (23b):
The gradient operator with respect to u,

$$A_\gamma := \partial_u E(t,\cdot,\gamma)\colon H^1_{\Gamma_0} \to H^{-1}_{\Gamma_0},$$

satisfies the so-called property of type (S^+). As mentioned in [8] (p. 152), the condition (S^+) takes the following form in our setting:

$$\left.\begin{aligned} &u_\tau(t) \rightharpoonup u(t) \text{ in } H^1,\;\; \gamma_\tau(t) \to \gamma(t) \text{ in } L^q,\\ &\limsup_{\tau\to 0}\, \langle \partial_u E\left(t,u_\tau,\gamma_\tau\right) - \partial_u E\left(t,u,\gamma_\tau\right),\, u_\tau - u\rangle \le 0 \end{aligned}\right\} \Rightarrow u_\tau(t) \to u(t) \text{ in } H^1.$$

As shown below, in our problem this implication can be obtained in an integral-in-time form of the second condition, which yields the strong convergence of u_τ in the Bochner space $L^2(0,T;H^1)$. From this, the required almost everywhere strong convergence follows. However, this argument requires the strong convergence of γ_τ in $L^2\big(0,T;L^2\big)$ (see (23e)).

Indeed, the operator is expressed in weak form for $p, q \in H^1_{\Gamma_0}$ as

$$\langle A_\gamma(p), q\rangle = \int_\Omega \mathbb{C}(x)\big(e(p) - \gamma S\big) : e(q)\,\mathrm{d}x - \int_\Omega f(t)\cdot q\,\mathrm{d}x - \int_{\Gamma_1} g(t)\cdot q\,\mathrm{d}S. \tag{25}$$

Hence, for all $p, q \in H^1_{\Gamma_0}$,

$$\big\langle A_\gamma(p) - A_\gamma(q),\, p-q\big\rangle = \int_\Omega \mathbb{C}(x) e(p-q) : e(p-q)\,\mathrm{d}x \;\geq\; \lambda_0 \|e(p-q)\|^2_{L^2}.$$

By Korn's inequality, $\|p-q\|_{H^1} \leq C_K \|e(p-q)\|_{L^2}$,

$$\big\langle A_\gamma(p) - A_\gamma(q),\, p-q\big\rangle \;\geq\; c\,\|p-q\|^2_{H^1}, \qquad c := \lambda_0 / C_K^2 > 0.$$

This estimate does not depend on γ. Therefore, if

$$\limsup_{\tau\to 0} \int_0^T \big\langle A_\gamma(\overline{u}_\tau) - A_\gamma(u),\, \overline{u}_\tau - u\big\rangle\,\mathrm{d}t \leq 0, \tag{26}$$

then it follows that

$$\limsup_{\tau\to 0} \int_0^T c\,\|\bar{u}_\tau - u\|^2_{H^1}\,\mathrm{d}t \leq\, 0 \,\leq \liminf_{\tau\to 0} \int_0^T c\,\|\bar{u}_\tau - u\|^2_{H^1}\,\mathrm{d}t.$$

Thus, we have $\bar{u}_\tau \to u$ in $L^2(0,T;H^1)$. Since also $\overline{u}_\tau \to u_\tau$ in $L^2(0,T;H^1)$, it follows that $u_\tau \to u$ in $L^2(0,T;H^1)$. Hence, after extracting a subsequence (not relabeled),

$$u_\tau(t) \to u(t) \quad \text{in } H^1 \quad \text{for a.e. } t \in [0,T].$$

Since the convergence in (26) does not depend on γ, if we establish (26) for

$$A := A_{\overline{\gamma}_\tau(t-\tau)} = \partial_u E(t,\cdot,\overline{\gamma}_\tau(t-\tau))\,, \tag{27}$$

then the desired strong pointwise convergence is obtained. This choice is made in order to make use of the (discrete) weak momentum balance.

In the following, we denote by $\bar{f}_\tau$ and $\bar{g}_\tau$ the piecewise constant interpolants with respect to the time partition, namely,

$$\bar{f}_\tau(t) := f(t_k), \qquad \bar{g}_\tau(t) := g(t_k) \quad \text{for } t \in [t_k, t_{k+1}), \quad k = 0,\dots,N-1,$$

and $\bar{f}_\tau(T) := f(T)$, $\bar{g}_\tau(T) := g(T)$. Then

$$\bar{f}_\tau \to f, \qquad \bar{g}_\tau \to g \quad \text{strongly in } L^2\big(0,T;L^2\big). \tag{28}$$

Let $\phi \in H^1\big(0,T;H^1_{\Gamma_0}\big)$, and set $\phi^k := \phi(t_k) \in H^1_{\Gamma_0}$ for $k = 0,\dots,N-1$. We denote by ϕ_τ the piecewise affine interpolant of ϕ with respect to the same time partition, namely,

$$\phi_\tau(t) := \phi^k + \frac{t-t_k}{\tau}\,(\phi^{k+1} - \phi^k) \quad \text{for } t \in [t_k, t_{k+1}), \quad k = 0,\dots,N-1,$$

and by $\overline{\phi}_\tau$ the piecewise constant interpolant, namely,

$$\overline{\phi}_\tau(t) := \phi^k \quad \text{for } t \in [t_k, t_{k+1}), \quad k = 0, \ldots, N-1, \qquad \overline{\phi}_\tau(T) := \phi^N.$$

Testing (3) with ϕ^{k+1} yields

$$\begin{aligned}
&\frac{1}{\tau^2} \int_\Omega \rho \left(u^{k+1} - 2u^k + u^{k-1}\right) \cdot \phi^{k+1} \,\mathrm{d}x + \int_\Omega \mathbb{C}\big(e(u^{k+1}) - \gamma^k S\big) : e(\phi^{k+1}) \,\mathrm{d}x \\
&\quad + \frac{1}{\tau} \int_\Omega \mathbb{D}\, e(u^{k+1} - u^k) : e(\phi^{k+1}) \,\mathrm{d}x \\
&\quad = \int_\Omega f(t_{k+1}) \cdot \phi^{k+1} \,\mathrm{d}x + \int_{\Gamma_1} g(t_{k+1}) \cdot \phi^{k+1} \,\mathrm{d}S.
\end{aligned} \tag{29}$$

Summing from $k = 0$ to $k = N-1$ and making use of the identity

$$\begin{aligned}
&\sum_{k=0}^{N-1} \left(\int_\Omega \rho \left(u^{k+1} - 2u^k + u^{k-1}\right) \cdot \phi^{k+1} \,\mathrm{d}x \right) \\
&\qquad = -\sum_{k=1}^{N} \left(\int_\Omega \rho \left(u^{k-1} - u^{k-2}\right) \cdot \left(\phi^k - \phi^{k-1}\right) \,\mathrm{d}x \right) \\
&\qquad\quad + \int_\Omega \rho \left(u^N - u^{N-1}\right) \cdot \phi^N \,\mathrm{d}x - \int_\Omega \rho \left(u^0 - u^{-1}\right) \cdot \phi^0 \,\mathrm{d}x,
\end{aligned}$$

we then arrive at the discrete equation of motion

$$\begin{aligned}
&\int_0^T \left[\int_\Omega \Big(\mathbb{C}\big(e(\overline{u}_\tau(t)) - \overline{\gamma}_\tau(t-\tau) S\big) + \mathbb{D}\, e(\dot{u}_\tau(t)) \Big) : e(\overline{\phi}_\tau(t)) \,\mathrm{d}x \right. \\
&\qquad \left. - \int_\Omega \rho\, \dot{u}_\tau(t-\tau) \cdot \dot{\phi}_\tau(t) \,\mathrm{d}x \right] dt + \int_\Omega \rho\, \dot{u}_\tau(T) \cdot \phi_\tau(T) \,\mathrm{d}x \\
&\quad = \int_\Omega \rho v^0 \cdot \phi_\tau(0) \,\mathrm{d}x + \int_0^T \left(\int_\Omega \bar{f}_\tau(t) \cdot \phi_\tau(t) \,\mathrm{d}x + \int_{\Gamma_1} \bar{g}_\tau(t) \cdot \phi_\tau(t) \,\mathrm{d}S \right) \mathrm{d}t.
\end{aligned} \tag{30}$$

We define v_τ and $\bar{v}_\tau$ as the piecewise affine and piecewise constant interpolants of u with respect to the same time partition, exactly as ϕ_τ and $\bar{\phi}_\tau$ above with ϕ replaced by u, and require that $v_\tau(0) = u^0$. Then

$$v_\tau \to u \quad \text{in } H^1\big(0, T; H^1_{\Gamma_0}\big), \qquad \bar{v}_\tau \to u \quad \text{in } L^2\big(0, T; H^1_{\Gamma_0}\big).$$

Concerning A in (27), from the boundedness of $\mathbb{C}$ and Proposition 4.4, we obtain the uniform boundedness of $A(\overline{u}_\tau)$ as well as $A(\overline{v}_\tau) - A(u) \to 0$ in $L^2(0, T; H^{-1}_{\Gamma_0})$. Since $\overline{v}_\tau \to u$ and $\overline{u}_\tau \rightharpoonup u$ in $L^2(0, T; H^1_{\Gamma_0})$, it follows that

$$\lim_{\tau \to 0} \int_0^T \big\langle A(\overline{u}_\tau) - A(u), \overline{u}_\tau - u \big\rangle - \big\langle A(\overline{u}_\tau) - A(\overline{v}_\tau), \overline{u}_\tau - \overline{v}_\tau \big\rangle \,\mathrm{d}t$$

$$
\begin{aligned}
&= \lim_{\tau\to 0}\int_0^T \big\langle A(\overline{u}_\tau),\, \overline{v}_\tau - u\big\rangle + \big\langle A(\overline{v}_\tau) - A(u),\, \overline{u}_\tau\big\rangle \,+\, \big\langle A(u),\, u\big\rangle - \big\langle A(\overline{v}_\tau),\, \overline{v}_\tau\big\rangle\,\mathrm{d}t\\
&= 0.
\end{aligned}
$$

Therefore, it suffices to show that

$$
\limsup_{\tau\to 0}\int_0^T \langle A(\overline{u}_\tau) - A(\overline{v}_\tau)\,,\, \overline{u}_\tau - \overline{v}_\tau\rangle\ \mathrm{d}t \ \le\ 0.
$$

By employing (30) with $\phi_\tau = v_\tau - u_\tau$ and $\bar{\phi}_\tau = \bar{v}_\tau - \bar{u}_\tau$, and recalling the definition (25) of A, we obtain

$$
\begin{aligned}
&\limsup_{\tau\to 0}\int_0^T \Big\langle A(\overline{u}_\tau) - A(\overline{v}_\tau)\,,\, \overline{u}_\tau - \overline{v}_\tau\Big\rangle\,\mathrm{d}t\\
&\quad= \limsup_{\tau\to 0}\int_0^T \bigg[\Big\langle A(\overline{v}_\tau)\,,\, \overline{v}_\tau - \overline{u}_\tau\Big\rangle - \int_\Omega \rho\dot{u}_\tau(t-\tau)\cdot(\dot{v}_\tau - \dot{u}_\tau)\,\mathrm{d}x\\
&\qquad\qquad - \int_\Omega \mathbb{D}\, e(\dot{u}_\tau) : e(\overline{u}_\tau - \overline{v}_\tau)\,\mathrm{d}x\bigg]\mathrm{d}t \qquad\qquad (31)\\
&\qquad - \int_0^T \bigg(\int_\Omega (\bar{f}_\tau - f)\cdot(\overline{v}_\tau - \overline{u}_\tau)\,\mathrm{d}x + \int_{\Gamma_1}(\bar{g}_\tau - g)\cdot(\overline{v}_\tau - \overline{u}_\tau)\,\mathrm{d}S\bigg)\,\mathrm{d}t\\
&\qquad + \int_\Omega \rho\dot{u}_\tau(T)\cdot(v_\tau(T) - u_\tau(T))\,\mathrm{d}x - \int_\Omega \rho v^0\cdot(v_\tau(0) - u^0)\,\mathrm{d}x.
\end{aligned}
$$

Regarding the first term on the right-hand side, due to the strong convergence of $\mathbb{C}(x)\big(e(\overline{v}_\tau) - \overline{\gamma}_\tau(t-\tau)S\big)$ in $L^2(0,T;H^{-1})$, which follows from (23e) together with the extension $\gamma^{-1} = \gamma^0$ and the continuity of time translations, and the weak convergence of $\overline{v}_\tau - \overline{u}_\tau$ to zero in $L^2(0,T;H^1)$, this term converges to zero.

For the second term, we first estimate the difference between the shifted and unshifted inertial terms for any $\psi \in L^2\big(0,T;L^2\big)$ as follows:

$$
\begin{aligned}
\bigg|\int_0^T \langle \rho\dot{u}_\tau(t-\tau) - \rho\dot{u}_\tau(t),\, \psi(t)\rangle\,\mathrm{d}t\bigg| &\le \bigg|\int_0^T \langle \rho\dot{u}_\tau(s),\, \psi(s+\tau) - \psi(s)\rangle\,\mathrm{d}s\bigg| + \bigg|\int_0^\tau \langle \rho v^0,\, \psi(t)\rangle\,\mathrm{d}t\bigg|\\
&\le \|\rho\dot{u}_\tau\|_{L^\infty(0,T;L^2)}\,\|\psi(\cdot+\tau) - \psi\|_{L^1(0,T;L^2)} + \|\rho v^0\|_{L^2}\,\|\psi\|_{L^1(0,\tau;L^2)}.
\end{aligned}
$$

Here, ψ is extended by 0 outside $[0,T]$ so that the shift $\psi(\cdot+\tau)$ is well-defined and the above estimate holds. Now set $\psi = \dot{v}_\tau - \dot{u}_\tau$. Then

$$
\|\psi(\cdot+\tau) - \psi\|_{L^1(0,T;L^2)} \le 2T^{1/2}\|\dot{v}_\tau - \dot{u}_\tau\|_{L^2(0,T;L^2)},
$$

and

$$
\|\psi\|_{L^1(0,\tau;L^2)} \le \tau^{1/2}\|\dot{v}_\tau - \dot{u}_\tau\|_{L^2(0,T;L^2)}.
$$

Hence, by (23d), the second term converges to zero.

For the third term, by summing

$$\int_\Omega \mathbb{D}\, e(u^k - u^{k-1}) : e(u^k)\,\mathrm{d}x \geq V(u^k) - V(u^{k-1})$$

for $k = 1, \dots, N$, we obtain

$$\begin{aligned}\sum_{k=1}^{N} \tau \int_\Omega \mathbb{D}\, e\Big(\frac{u^k - u^{k-1}}{\tau}\Big) : e(u^k)\,\mathrm{d}x &= \int_0^T \int_\Omega \mathbb{D}\, e(\dot{u}_\tau) : e(\overline{u}_\tau)\,\mathrm{d}x\mathrm{d}t \\ &\geq V(u_\tau(T)) - V(u^0).\end{aligned}$$

Furthermore, by the weak convergence of $\mathbb{D}\, e(\dot{u}_\tau)$ and the strong convergence of $e(\overline{v}_\tau)$ in $L^2(0,T;L^2)$, we have

$$\begin{aligned}&\limsup_{\tau\to 0}\Big(-\int_0^T \int_\Omega \mathbb{D}\, e(\dot{u}_\tau) : e(\overline{u}_\tau - \overline{v}_\tau)\,\mathrm{d}x\mathrm{d}t\Big) \\ &\qquad \leq \limsup_{\tau\to 0}\Big(V(u^0) - V(u_\tau(T)) + \int_0^T \int_\Omega \mathbb{D}\, e(\dot{u}_\tau) : e(\overline{v}_\tau)\,\mathrm{d}x\mathrm{d}t\Big) \\ &\qquad = V(u^0) - \liminf_{\tau\to 0} V(u_\tau(T)) + \int_0^T \int_\Omega \mathbb{D}\, e(\dot{u}) : e(u)\,\mathrm{d}x\mathrm{d}t \\ &\qquad \leq V(u^0) - V(u(T)) + \int_0^T \int_\Omega \mathbb{D}\, e(\dot{u}) : e(u)\,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{32}$$

Here we used $u_\tau(T) \rightharpoonup u(T)$ in H^1 and the weak lower semicontinuity of V. The pointwise weak convergence of u_τ in H^1 can be shown as follows. Fix an arbitrary $t \in [0,T]$. By Proposition 4.4, the set $\{u_\tau(t)\}$ is bounded in H^1. Hence, every sequence (τ_n) admits a subsequence (not relabeled) and a function $v \in H^1$ with

$$u_{\tau_n}(t) \rightharpoonup v \quad \text{in } H^1.$$

Because the embedding $H^1 \hookrightarrow L^2$ is continuous, the same subsequence converges weakly in L^2 to v. On the other hand, by (23a), for the same (arbitrary) t we have the strong convergence

$$u_{\tau_n}(t) \to u(t) \quad \text{in } L^2.$$

Therefore $v = u(t)$. Since every subsequence admits a further subsequence with weak limit $u(t)$, the whole net $\{u_\tau(t)\}_\tau$ converges weakly to $u(t)$ in H^1.

By applying the integration-by-parts formula (Lemma B.5.7 in [8]), the rightmost term in (32) becomes zero, and hence the lim sup of the third term in (31) is estimated from above by zero.

For the external force term in (31), since $\bar{f}_\tau - f \to 0$ strongly in $L^2(0,T;L^2)$ and $\bar{g}_\tau - g \to 0$ strongly in the corresponding trace space, while $\overline{v}_\tau - \overline{u}_\tau \rightharpoonup 0$ in $L^2(0,T;L^2)$ and also weakly in the trace space, it follows that this term converges to zero. From (18), we have $\|\dot{u}_\tau(T)\|_{L^2} \leq C$, and since $v_\tau(T) - u_\tau(T) \to 0$ in L^2 by (23a), the convergence of the second but last term in (31) follows. The last term is zero since $v_\tau(0) - u^0 = 0$.

Step 2: Weak momentum balance

Let $\mathcal{V} := H^1_{\Gamma_0}(\Omega;\mathbb{R}^d)$. For a.e. $t \in (0,T)$, let $\ell(t) \in \mathcal{V}^*$ be defined by $\langle \ell(t), v\rangle_{\mathcal{V}^*,\mathcal{V}} := -L(t,v)$ for all $v \in \mathcal{V}$. By the trace theorem, $\ell(t)$ is well defined and continuous on $\mathcal{V}$, and $\ell \in L^1(0,T;\mathcal{V}^*)$. In addition, by (18), up to a subsequence,

$$\dot u_\tau \overset{*}{\rightharpoonup} \dot u \quad \text{in } L^\infty(0,T;L^2).$$

First, we pass to the limit in (30) for test functions $\phi \in H^1(0,T;\mathcal{V})$ satisfying $\phi(T) = 0$. Choosing the discrete interpolants so that $\phi_\tau(T) = 0$, the terminal term vanishes, and all remaining terms converge by the previously established compactness properties. Hence

$$\int_0^T \Big[\int_\Omega \sigma(t) : e(\phi(t))\,\mathrm{d}x - \int_\Omega \rho\,\dot u(t)\cdot\dot\phi(t)\,\mathrm{d}x\Big]\mathrm{d}t = \int_\Omega \rho\, v^0\cdot\phi(0)\,\mathrm{d}x + \int_0^T \langle \ell(t),\phi(t)\rangle\,\mathrm{d}t \quad (33)$$

for every $\phi \in H^1(0,T;\mathcal{V})$ with $\phi(T) = 0$.

Fix $\zeta \in \mathcal{V}$ and define

$$m_\zeta(t) := \int_\Omega \rho\,\dot u(t)\cdot\zeta\,\mathrm{d}x \qquad \text{for a.e. } t \in (0,T).$$

Testing (33) with $\phi(t) = \eta(t)\zeta$, where $\eta \in H^1(0,T)$ and $\eta(T) = 0$, we obtain

$$-\int_0^T m_\zeta(t)\dot\eta(t)\,\mathrm{d}t = \eta(0)\int_\Omega \rho\,v^0\cdot\zeta\,\mathrm{d}x + \int_0^T \Big(\ell(t)[\zeta] - \int_\Omega \sigma(t) : e(\zeta)\,\mathrm{d}x\Big)\eta(t)\,\mathrm{d}t.$$

Therefore $m_\zeta \in W^{1,1}(0,T)$ and thus admits a continuous representative on $[0,T]$. Since $\dot u \in L^\infty(0,T;L^2)$, this representative satisfies

$$|m_\zeta(T)| \le C\|\zeta\|_{L^2}.$$

As $\mathcal{V}$ is dense in L^2, by the Riesz representation theorem, there exists a unique $v_T \in L^2$ such that

$$m_\zeta(T) = \int_\Omega \rho\,v_T\cdot\zeta\,\mathrm{d}x \qquad \forall\zeta \in \mathcal{V}.$$

We denote this weak L^2-trace by $\dot u(T) := v_T$. The same argument yields the weak L^2-trace at $t = 0$, which coincides with v^0. The continuity of $t \mapsto \int_\Omega \rho\,\dot u(t)\cdot\zeta\,\mathrm{d}x$ extends by approximation from $\zeta \in \mathcal{V}$ to arbitrary L^2-test functions.

Now let $\phi \in H^1(0,T;\mathcal{V})$ be arbitrary. Choose $\eta_* \in H^1(0,T)$ such that $\eta_*(0) = 0$ and $\eta_*(T) = 1$, and write

$$\phi(t) = \psi(t) + \eta_*(t)\phi(T), \qquad \psi(T) = 0.$$

Applying (33) to ψ and using the integration-by-parts formula for $m_{\phi(T)}$ yields

$$\int_0^T \Big[\int_\Omega \sigma(t) : e(\phi(t))\,\mathrm{d}x - \int_\Omega \rho\,\dot u(t)\cdot\dot\phi(t)\,\mathrm{d}x\Big]\mathrm{d}t + \int_\Omega \rho\,\dot u(T)\cdot\phi(T)\,\mathrm{d}x$$

$$= \int_\Omega \rho\, v^0 \cdot \phi(0)\,\mathrm{d}x + \int_0^T \langle \ell(t), \phi(t)\rangle\,\mathrm{d}t.$$

Finally, since $\{\dot u_\tau(T)\}$ is bounded in L^2, let $\dot u_{\tau_n}(T) \rightharpoonup w$ weakly in L^2 along a subsequence. Passing to the limit in (30) for arbitrary $\phi \in H^1(0,T;\mathcal{V})$ gives the same identity with w in place of $\dot u(T)$. Subtracting the two identities and using that $\{\phi(T) : \phi \in H^1(0,T;\mathcal{V})\} = \mathcal{V}$, we obtain

$$\int_\Omega \rho\,(w - \dot u(T)) \cdot \zeta\,\mathrm{d}x = 0 \qquad \forall \zeta \in \mathcal{V}.$$

By density of $\mathcal{V}$ in L^2, $w = \dot u(T)$ in L^2. Since the above argument applies to any weakly convergent subsequence of $\{\dot u_\tau(T)\}$, we conclude that

$$\dot u_\tau(T) \rightharpoonup \dot u(T) \qquad \text{weakly in } L^2,$$

which justifies the convergence of the terminal inertial term.

The previous argument shows that $\dot u$ admits a representative in $C_w([0,T];L^2)$; hence the endpoint values $\dot u(0)$ and $\dot u(T)$ are well defined, and the limit identity extends from test functions with $\phi(T) = 0$ to arbitrary $\phi \in H^1(0,T;\mathcal{V})$. Moreover, the initial endpoint satisfies $\dot u(0) = v^0$.

Step 3: Local Semistability

By testing (4) with a general $\tilde\gamma \in W^{1,\alpha}$ and applying the triangle inequality, we obtain

$$E(t_{k+1}, u^{k+1}, \gamma^{k+1}) \le E(t_{k+1}, u^{k+1}, \widetilde\gamma) + D(\gamma^{k+1}, \widetilde\gamma).$$

Similarly to (14), for $t \in [t_{k+1}, t_{k+2})$ we have

$$E(t, u^{k+1}, \gamma^{k+1}) = E(t_{k+1}, u^{k+1}, \gamma^{k+1}) + \int_{t_{k+1}}^t \partial_s L\big(s, u^{k+1}\big)\,\mathrm{d}s,$$
$$E(t, u^{k+1}, \widetilde\gamma) = E(t_{k+1}, u^{k+1}, \widetilde\gamma) + \int_{t_{k+1}}^t \partial_s L\big(s, u^{k+1}\big)\,\mathrm{d}s.$$

Hence, it follows immediately that

$$E(t, \overline u_\tau, \overline\gamma_\tau) \le E(t, \overline u_\tau, \widetilde\gamma) + D(\overline\gamma_\tau, \widetilde\gamma). \tag{34}$$

The pointwise convergence of the left-hand side of (34) is shown in the same way as in Step 4 (p. 68) of the proof of Theorem 3.2 in [4] as follows. More precisely, by (23b), (23e), and $\bar u_\tau - u_\tau \to 0$ in $L^2(0,T;H^1)$, up to a subsequence there exists a null set $N \subset [0,T]$ such that, for every $t \in [0,T] \setminus N$,

$$\bar u_\tau(t) \to u(t) \quad \text{in } H^1, \qquad \bar\gamma_\tau(t) \to \gamma(t) \quad \text{in } L^q,$$

and hence the passage to the limit below is justified for every fixed $t \in [0,T] \setminus N$; see also Remark 2.1.8 in [8]. From this discrete semistability together with the above pointwise convergence, we have, for a.e. $t \in [0,T]$,

$$
\begin{aligned}
E(t,u(t),\gamma(t)) &= \lim_{\tau\to 0} E\big(t,u(t),\gamma(t)\big) + D\big(\overline{\gamma}_\tau(t),\gamma(t)\big) \\
&= \lim_{\tau\to 0} E\big(t,\overline{u}_\tau(t),\gamma(t)\big) + D\big(\overline{\gamma}_\tau(t),\gamma(t)\big) \\
&\geq \limsup_{\tau\to 0} E\big(t,\overline{u}_\tau(t),\overline{\gamma}_\tau(t)\big).
\end{aligned}
$$

Since the weak lower semicontinuity has already been discussed in Proposition 4.2, it follows from Proposition 4.4 that, possibly after taking a subsequence, the opposite liminf inequality holds. Therefore, the pointwise convergence is obtained.

Moreover, for every $t \in [0,T]\backslash N$, the right-hand side of (34) converges to $E(t,u(t),\widetilde{\gamma})+D(\gamma(t),\widetilde{\gamma})$ again by (23b) and (23e). Passing to the limit in (34), we conclude that

$$
E(t,u(t),\gamma(t)) \leq E(t,u(t),\widetilde{\gamma}) + D(\gamma(t),\widetilde{\gamma}) \qquad \text{for a.e. } t \in [0,T]. \tag{35}
$$

Step 4: Energy Imbalance

From (17), for any $t_1,t_2 \in [0,T]$ with $t_1 \leq t_2$, we can gain

$$
\begin{aligned}
&K(\dot{u}_\tau(\lfloor\tau^{-1}t_2\rfloor\tau-\tau)) + E(\lfloor\tau^{-1}t_2\rfloor\tau,\overline{u}_\tau,\overline{\gamma}_\tau) + \int_{\lfloor\tau^{-1}t_1\rfloor\tau}^{\lfloor\tau^{-1}t_2\rfloor\tau} \Big[2V\big(\dot{u}_\tau(s)\big) + R\big(\dot{\gamma}_\tau(s)\big)\Big]\,\mathrm{d}s \\
&\qquad \leq\ K(\dot{u}_\tau(\lfloor\tau^{-1}t_1\rfloor\tau-\tau)) + E(\lfloor\tau^{-1}t_1\rfloor\tau,\overline{u}_\tau,\overline{\gamma}_\tau) + \int_{\lfloor\tau^{-1}t_1\rfloor\tau}^{\lfloor\tau^{-1}t_2\rfloor\tau} \partial_s L\big(s,\overline{u}_\tau(s)\big)\,\mathrm{d}s.
\end{aligned}
$$

By the similar reasoning as in Step 3 and as in the proof of Proposition 4.4 (Step 2), it follows immediately that

$$
\begin{aligned}
&K(\dot{u}_\tau(t_2-\tau)) + E(t_2,\overline{u}_\tau,\overline{\gamma}_\tau) + \int_{\lfloor\tau^{-1}t_1\rfloor\tau}^{\lfloor\tau^{-1}t_2\rfloor\tau} \Big[2V\big(\dot{u}_\tau(s)\big) + R\big(\dot{\gamma}_\tau(s)\big)\Big]\,\mathrm{d}s \\
&\qquad \leq\ K(\dot{u}_\tau(t_1-\tau)) + E(t_1,\overline{u}_\tau,\overline{\gamma}_\tau) + \int_{t_1}^{t_2} \partial_s L\big(s,\overline{u}_\tau(s)\big)\,\mathrm{d}s + \tau C.
\end{aligned} \tag{36}
$$

To justify the passage to the limit in the kinetic-energy terms, compare also [2], and define

$$
r_\tau(t) := \chi_{[\tau,T]}(t)\,\dot{u}_\tau(t-\tau).
$$

Then

$$
\begin{aligned}
\|r_\tau - \dot{u}\|_{L^2(0,T;L^2)} \leq \|\dot{u}\|_{L^2(0,\tau;L^2)} &+ \|\dot{u}_\tau(\cdot-\tau) - \dot{u}(\cdot-\tau)\|_{L^2(\tau,T;L^2)} \\
&+ \|\dot{u}(\cdot-\tau) - \dot{u}\|_{L^2(\tau,T;L^2)} \to 0,
\end{aligned}
$$

by (23d) and the continuity of time translations in $L^2(0,T;L^2)$. Hence, after extraction of a further subsequence, we have

$$
r_\tau(t) \to \dot{u}(t) \qquad \text{strongly in } L^2 \quad \text{for a.e. } t \in (0,T).
$$

Since $r_\tau(t) = \dot{u}_\tau(t-\tau)$ for every fixed $t > 0$ and all sufficiently small τ, it follows that

$$K(\dot{u}_\tau(t-\tau)) \to K(\dot{u}(t)) \qquad \text{for a.e. } t \in (0,T).$$

Fix $\varepsilon_0 > 0$, and define the positive numbers $\varepsilon_1(\tau)$ and $\varepsilon_2(\tau)$ by

$$\varepsilon_1(\tau) := t_1 - \lfloor \tau^{-1} t_1 \rfloor \tau, \qquad \varepsilon_2(\tau) := t_2 - \lfloor \tau^{-1} t_2 \rfloor \tau.$$

As in the proof of Proposition 4.2, the viscosity term is weakly lower semicontinuous with respect to $e(\cdot)$ in $L^2(t_1, t_2 - \varepsilon_0; L^2)$. Since we are taking the limit as $\tau \to 0$, it suffices to consider sufficiently small τ such that $\varepsilon_2(\tau) < \varepsilon_0$; therefore, by (23c),

$$\int_{t_1}^{t_2-\varepsilon_0} V(\dot{u})\,\mathrm{d}s \;\le\; \liminf_{\tau\to 0} \int_{t_1}^{t_2-\varepsilon_0} V(\dot{u}_\tau)\,\mathrm{d}s \;\le\; \liminf_{\tau\to 0} \int_{t_1-\varepsilon_1(\tau)}^{t_2-\varepsilon_2(\tau)} V(\dot{u}_\tau)\,\mathrm{d}s.$$

By letting $\varepsilon_0 \to 0$ in this expression, we obtain

$$\int_{t_1}^{t_2} V(\dot{u})\,\mathrm{d}s \;\le\; \liminf_{\tau\to 0} \int_{t_1-\varepsilon_1(\tau)}^{t_2-\varepsilon_2(\tau)} V(\dot{u}_\tau)\,\mathrm{d}s \;=\; \liminf_{\tau\to 0} \int_{\lfloor \tau^{-1} t_1 \rfloor \tau}^{\lfloor \tau^{-1} t_2 \rfloor \tau} V(\dot{u}_\tau)\,\mathrm{d}s.$$

For the rate-independent dissipation term, we use the weak-$*$ convergence in (23f). By applying Reshetnyak's theorem to (23f) with $U = (t_1, t_2) \times \Omega$ and $H((s,x),\xi) = \theta(x)|\xi|$, we obtain

$$\int_{(t_1,t_2)\times\Omega} \theta(x)\,\mathrm{d}|\dot{\gamma}|(s,x) \le \liminf_{\tau\to 0} \int_{\lfloor \tau^{-1} t_1 \rfloor \tau}^{\lfloor \tau^{-1} t_2 \rfloor \tau} R(\dot{\gamma}_\tau(s))\,\mathrm{d}s.$$

Moreover, for t_1, t_2 outside the jump set of $t \mapsto \gamma(t)$ in $L^1(\Omega)$, the left-hand side coincides with $\mathrm{Var}_D(\gamma; [t_1, t_2])$. Hence

$$\mathrm{Var}_D(\gamma; [t_1, t_2]) \le \liminf_{\tau\to 0} \int_{\lfloor \tau^{-1} t_1 \rfloor \tau}^{\lfloor \tau^{-1} t_2 \rfloor \tau} R(\dot{\gamma}_\tau(s))\,\mathrm{d}s.$$

Since $\gamma \in BV([0,T]; L^1(\Omega))$, the jump set of $t \mapsto \gamma(t)$ is at most countable, and this restriction does not affect the a.e. statement.

By the strong convergence of u_τ obtained in Step 1, the trace theorem, and the H^1-regularity of f and g, we have $\partial_s L(\cdot, u_\tau) \to \partial_s L(\cdot, u)$ in $L^1(0,T)$.

What remains to show in order to finish the limit process as $\tau \to 0$ in (36), is the pointwise convergence of E for a.e. $t_1, t_2 \in [0,T]$, which has already been proved in Step 3 above.

Consequently, the limit (u, γ) is a local solution with inertia in the sense of Definition 3.1. $\square$

5. UNIQUENESS OF ENERGETIC SOLUTIONS IN THE QUADRATIC CASE

Theorem 4.1 ensures the existence of (at least one) local solution with inertia and viscosity in the sense of Definition 3.1. Uniqueness, however, is not automatic for stability-based energetic formulations: already in the quasistatic theory of rate-independent systems, the combination of (local or global) stability with an energy inequality may give rise to set-valued evolutions unless additional structural assumptions (e. g. convexity/regularity) are imposed; see [6, 7]. In particular, RIS are frequently observed to exhibit multi-valued behavior unless strong structural assumptions are imposed.

In the quasistatic setting, a classical uniqueness mechanism is strict (strong) convexity of the incremental potential

$$(u, \gamma) \;\mapsto\; E(t_k, u, \gamma) + D(\gamma, \gamma_{k-1}),$$

which yields a unique minimizer at each time step and thus uniqueness of the energetic evolution; see [9]. The present model is genuinely dynamic: besides energetic information it incorporates the weak momentum balance through the inertia and the viscous stress. Therefore, quasistatic uniqueness arguments cannot be transferred verbatim to our inertial setting. Nevertheless, in the context of dynamic plasticity, the introduction of hardening and viscosity is known to enhance monotonicity and coercivity, thereby contributing to regularity and uniqueness of solutions (see, e. g., [11]).

Here, in order to simplify the analysis, we assume a quadratic form

$$w(\gamma) = \tfrac{\beta}{2}\gamma^2 \quad (\beta > 0), \qquad \alpha = 2,$$

so that the free energy contains the quadratic gradient term $\kappa|\nabla\gamma|^2$.

Theorem 5.1. Under the above assumptions, and assuming the gentle-start compatibility at $t = 0$, namely that

$$\int_\Omega \mathbb{D}\, e(v^0) : e(\psi)\, \mathrm{d}x + \int_\Omega \mathbb{C}\big(e(u^0) - \gamma^0 S\big) : e(\psi)\, \mathrm{d}x = \int_\Omega f(0)\cdot\psi\, dx + \int_{\Gamma_1} g(0)\cdot\psi\, \mathrm{d}S \qquad \forall\, \psi \in H^1_{\Gamma_0}, \tag{37}$$

the energetic solution constructed in the proof of Theorem 4.1 is unique.

The proof is postponed to Appendix 7.

6. OUTLOOK

Possible directions for future work include extensions to more general slip systems and hardening laws. In particular, an important next step is the extension of the present energetic framework to a finite-strain (fully nonlinear) setting. Further perspectives concern asymptotic investigations such as vanishing regularization and/or vanishing viscosity limits, as well as numerical studies of inertial effects on localization processes suggested by experiments on layered structures.

7. APPENDIX

Theorem 7.1. (Simon [12, Aubin–Lions/Simon Theorem; Thm. 5 and Cor. 4])
Let $X \hookrightarrow\hookrightarrow B \hookrightarrow Y$ be Banach spaces with compact embedding $X \to B$ and continuous embedding $B \to Y$.

1. Let $1 < p < \infty$. If a set $\mathcal{F} \subset L^p(0,T;X)$ is bounded and
$$\lim_{h\to 0} \sup_{f\in\mathcal{F}} \|f(\cdot + h) - f(\cdot)\|_{L^p(0,T-h;Y)} = 0,$$
then $\mathcal{F}$ is relatively compact in $L^p(0,T;B)$.
2. More strongly, if $\mathcal{F} \subset L^\infty(0,T;X)$ is bounded and $\{\partial_t f\}_{f\in\mathcal{F}}$ is bounded in $L^r(0,T;Y)$ for some $r > 1$, then $\mathcal{F}$ is relatively compact in $C([0,T];B)$.

Theorem 7.2. (Ambrosio et al. [1, Reshetnyak's lower semicontinuity theorem; Thm. 2.38])

Let $U \subset \mathbb{R}^N$ be an open set, and let $\mu_k, \mu \in \mathscr{M}(U;\mathbb{R}^m)$ be finite Radon measures such that $\mu_k \overset{*}{\rightharpoonup} \mu$ weakly* in $\mathscr{M}(U;\mathbb{R}^m)$ (that is, in the dual of $C(U;\mathbb{R}^m)$). Suppose that $H : U \times \mathbb{R}^m \to [0,\infty]$ is lower semicontinuous in (x,ξ), convex and positively 1-homogeneous in the second variable. Then
$$\int_U H\left(x, \frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(x)\right) \mathrm{d}|\mu|(x) \;\leq\; \liminf_{k\to\infty} \int_U H\left(x, \frac{\mathrm{d}\mu_k}{\mathrm{d}|\mu_k|}(x)\right) \mathrm{d}|\mu_k|(x).$$

Proof of Theorem 5.1. We now sketch the proof of Theorem 5.1 in the notation and functional framework of the present single-slip model. The proof follows the strategy of [8, Proposition 5.1.11], adapted to the present single-slip setting.

P r o o f. Let
$$\mathcal{V} := H^1_{\Gamma_0}(\Omega;\mathbb{R}^d), \qquad \mathcal{W} := W^{1,2}(\Omega).$$
By the assumptions, the stored energy $\mathcal{E}$,
$$\mathcal{E}(u,\gamma) := \frac{1}{2}\int_\Omega \mathbb{C}\big(e(u) - \gamma S\big) : \big(e(u) - \gamma S\big)\,\mathrm{d}x \;+\; \frac{\beta}{2}\int_\Omega \gamma^2\,\mathrm{d}x \;+\; \kappa\int_\Omega |\nabla\gamma|^2\,\mathrm{d}x,$$
is a quadratic form. In particular, $\partial\mathcal{E} := (\partial_u\mathcal{E}, \partial_\gamma\mathcal{E})$ is a bounded linear symmetric operator $\partial\mathcal{E} \in \mathrm{Lin}\big(\mathcal{V}\times\mathcal{W}; \mathcal{V}^*\times\mathcal{W}^*\big)$. Each component is expressed as follows: For any test functions $\varphi \in \mathcal{V}$ and $\psi \in \mathcal{W}$,
$$\langle \partial_u\mathcal{E}(t,u,\gamma),\, \varphi\rangle = \int_\Omega \mathbb{C}\big(e(u) - \gamma S\big) : e(\varphi)\,\mathrm{d}x,$$
$$\langle \partial_\gamma\mathcal{E}(t,u,\gamma),\, \psi\rangle = \int_\Omega \Big(-\,\mathbb{C}\big(e(u) - \gamma S\big) : (S\,\psi) + \beta\,\gamma\,\psi + 2\kappa\,\nabla\gamma\cdot\nabla\psi\Big)\,\mathrm{d}x.$$

The next estimate is standard and follows the argument in [8, pp. 485–486], with only straightforward adaptations to the present notation and to the single-slip setting. More precisely, one combines the discrete momentum balance with the discrete variational inequality for γ_τ, tests them by the corresponding second backward differences, and uses the convexity of $\mathcal{E}$ together with a discrete Gronwall argument. Since no essentially new idea is needed here, we omit the details and record only the resulting estimate.

Proposition 7.3. (Enhanced uniform τ–bounds) Under the assumptions of 5.1, there exists a constant $C > 0$, independent of τ, such that

$$\|\ddot{u}_\tau\|_{L^\infty(0,T;L^2)} \le C, \qquad \|\nabla \dot{u}_\tau\|_{L^\infty(0,T;L^2)} \le C, \tag{38}$$

$$\|\dot{\gamma}_\tau\|_{L^\infty(0,T;L^2)} \le C, \qquad \|\nabla \dot{\gamma}_\tau\|_{L^\infty(0,T;L^2)} \le C, \tag{39}$$

$$\|e(\ddot{u}_\tau)\|_{L^2(0,T;L^2)} \le C. \tag{40}$$

From this, in particular, we may conclude that a solution (u, γ) constructed in the proof of Theorem 4.1 satisfies $\ddot{u} \in L^\infty(0,T;L^2)$ and $\dot{\gamma} \in L^\infty(0,T;\mathcal{W})$. Hence $\gamma \in W^{1,\infty}(0,T;\mathcal{W}) \subset AC([0,T];L^1(\Omega))$, and, using the 1-homogeneity and convexity of R, for every $[t_1, t_2] \subset [0,T]$ we have

$$\mathrm{Var}_D(\gamma;[t_1,t_2]) = \int_{t_1}^{t_2} R(\dot{\gamma}(s))\,\mathrm{d}s.$$

Furthermore, the regularity $\ddot{u} \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))$ allows us to apply the following Lemma.

Lemma 7.4. Suppose that

$$u \in H^1\big(0,T;\mathcal{V}\big), \qquad \ddot{u} \in L^\infty(0,T;L^2), \qquad \gamma \in L^2\big(0,T;\mathcal{W}\big),$$

satisfy the momentum balance in a weak sense in time, i. e., for all $\phi \in H^1\big(0,T;\mathcal{V}\big)$. Then, for almost every $t \in (0,T)$,

$$\int_\Omega \rho\,\ddot{u}(t)\cdot\psi\,\mathrm{d}x + \int_\Omega \mathbb{D}\,e(\dot{u}(t)) : e(\psi)\,\mathrm{d}x + \int_\Omega \mathbb{C}\big(e(u(t)) - \gamma(t)S\big) : e(\psi)\,\mathrm{d}x$$
$$= \int_\Omega f(t)\cdot\psi\,\mathrm{d}x + \int_{\Gamma_1} g(t)\cdot\psi\,\mathrm{d}S, \tag{41}$$

for all $\psi \in \mathcal{V}$.
Equivalently,

$$\rho\,\ddot{u}(t) - \operatorname{div}\sigma = f(t) \quad \text{in } \mathcal{V}^*,$$

with natural boundary condition

$$\sigma n = g(t) \quad \text{in } H^{-\frac{1}{2}}(\Gamma_1), \qquad u(t) = 0 \text{ on } \Gamma_0.$$

P r o o f. If we define $F(t) \in \mathcal{V}^*$ for $t \in [0,T]$ by

$$\langle F(t),\psi\rangle := \int_\Omega \mathbb{D}\,e(\dot{u}(t)) : e(\psi)\,\mathrm{d}x + \int_\Omega \mathbb{C}\big(e(u(t)) - \gamma(t)S\big) : e(\psi)\,\mathrm{d}x$$

$$- \int_\Omega f(t) \cdot \psi \,\mathrm{d}x - \int_{\Gamma_1} g(t) \cdot \psi \,\mathrm{d}S, \tag{42}$$

then $F(t)$ belongs to $L^1(0,T;\mathcal{V}^*)$.
Insert test functions of the form $\phi(t,x) = \eta(t)\psi(x)$ with $\eta \in C_c^1((0,T))$ and $\psi \in \mathcal{V}$ into the weak balance. The definition yields

$$-\int_0^T \Big(\int_\Omega \rho\, \dot u(t) \cdot \psi \,\mathrm{d}x \Big) \dot\eta(t) \,\mathrm{d}t + \int_0^T \langle F(t), \psi\rangle\, \eta(t) \,\mathrm{d}t = 0.$$

Define $m_\psi(t) := \int_\Omega \rho\, \dot u(t) \cdot \psi \,\mathrm{d}x \in L^1(0,T)$. Since $\langle F(\cdot), \psi\rangle \in L^1(0,T)$, this distributional identity shows that m_ψ is absolutely continuous and

$$\dot m_\psi(t) = -\langle F(t), \psi\rangle \quad \text{for a.e. } t \in (0,T).$$

On the other hand, since $\ddot u \in L^\infty(0,T;L^2)$, the map $t \mapsto \dot u(t)$ is absolutely continuous as an L^2-valued function, and thus

$$\dot m_\psi(t) = \int_\Omega \rho\, \ddot u(t) \cdot \psi \,\mathrm{d}x \quad \text{for a.e. } t \in (0,T),$$

Hence, for every $\psi \in \mathcal{V}$,

$$\int_\Omega \rho\, \ddot u(t) \cdot \psi \,\mathrm{d}x = -\langle F(t), \psi\rangle,$$

which is exactly (41). □

Fix t such that local semi-stability (35) holds. Since $L(t,u(t))$ is independent of γ, using the 1-homogeneity and convexity of R, this is equivalent to

$$\langle -\partial_\gamma \mathcal{E}(u(t),\gamma(t)), \varphi\rangle \;\le\; R(\varphi) \qquad \forall\, \varphi \in \mathcal{W},$$

i. e.

$$-\partial_\gamma \mathcal{E}\big(u(t),\gamma(t)\big) \;\in\; \partial R(0) \subset \mathcal{W}^*. \tag{43}$$

This is the standard subgradient optimality condition for the minimization of $\mathcal{E}(u(t),\cdot) + R(\cdot - \gamma(t))$ at $\gamma(t)$.
For a.e. t we can take $\psi = \dot u(t) \in H^1_{\Gamma_0}$ in (41) to obtain the following identity for every $0 \le t_1 \le t_2 \le T$:

$$\begin{aligned} K\big(\dot u(t_2)\big) - K\big(\dot u(t_1)\big) &+ \int_{t_1}^{t_2}\int_\Omega \mathbb{D}\, e(\dot u) : e(\dot u) \,\mathrm{d}x\mathrm{d}t \\ &+ \Big[\int_\Omega \tfrac12\, \mathbb{C}\big(e(u) - \gamma S\big) : \big(e(u) - \gamma S\big) \,\mathrm{d}x \Big]_{t_1}^{t_2} \\ = \int_{t_1}^{t_2} \Big(\int_\Omega f \cdot \dot u \, dx &+ \int_{\Gamma_1} g \cdot \dot u \,\mathrm{d}S \Big) \,\mathrm{d}t \;-\; \int_{t_1}^{t_2}\int_\Omega \big(\mathbb{C}(e(u) - \gamma S) : S \big)\, \dot\gamma \,\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{44}$$

Adding the identity

$$\int_{t_1}^{t_2} \langle \beta\gamma - 2\kappa\Delta\gamma, \dot\gamma\rangle \,\mathrm{d}t = \int_\Omega \Big(\tfrac{\beta}{2}\gamma(t_2)^2 + \kappa|\nabla\gamma(t_2)|^2 - \tfrac{\beta}{2}\gamma(t_1)^2 - \kappa|\nabla\gamma(t_1)|^2 \Big) \,\mathrm{d}x$$

to (44), and using the definition of L, yields the energy–power balance

$$K\big(\dot u(t_2)\big) + \mathcal{E}\big(u(t_2),\gamma(t_2)\big) + L\big(t_2,u(t_2)\big) - \Big(K\big(\dot u(t_1)\big) + \mathcal{E}\big(u(t_1),\gamma(t_1)\big) + L\big(t_1,u(t_1)\big)\Big) + \int_{t_1}^{t_2}\int_\Omega \mathbb{D}\, e(\dot u) : e(\dot u)\,\mathrm{d}x\mathrm{d}t = \int_{t_1}^{t_2} \partial_t L(s,u(s))\,\mathrm{d}s + \int_{t_1}^{t_2} \big\langle \partial_\gamma \mathcal{E}\big(u(s),\gamma(s)\big),\ \dot\gamma(s)\big\rangle\,\mathrm{d}s. \tag{45}$$

Subtract (45) from the energy inequality. For all $0 \le t_1 \le t_2 \le T$ we obtain

$$\int_{t_1}^{t_2} \Big(R(\dot\gamma(s)) + \big\langle \partial_\gamma \mathcal{E}\big(u(s),\gamma(s)\big),\ \dot\gamma(s)\big\rangle \Big)\,\mathrm{d}s \ \le\ 0. \tag{46}$$

On the other hand, for a.e. t,

$$R\big(\dot\gamma(t)\big) + \big\langle \partial_\gamma \mathcal{E}\big(u(t),\gamma(t)\big),\ \dot\gamma(t)\big\rangle \ \ge\ 0.$$

Combining this pointwise nonnegativity with (46) forces

$$R\big(\dot\gamma(t)\big) + \big\langle \partial_\gamma \mathcal{E}\big(u(t),\gamma(t)\big),\ \dot\gamma(t)\big\rangle \ =\ 0 \quad \text{for a.e. } t \in (0,T).$$

By the 1-homogeneity of R and the Fenchel–Young equality, the conjunction of (43) and the last equality is equivalent to

$$-\partial_\gamma \mathcal{E}\big(u(t),\gamma(t)\big) \ \in\ \partial R\big(\dot\gamma(t)\big) \quad \text{for a.e. } t. \tag{47}$$

Let (u_i,γ_i), $i = 1,2$, be two pairs obtained as limits in the time-discrete construction used in the proof of Theorem 4.1, corresponding to the same initial data, which in particular satisfy, for a.e. $t \in (0,T)$,

$$\text{(momentum balance in } \mathcal{V}^*) \quad \rho \ddot u_i(t) + \partial V\big(\dot u_i(t)\big) + \partial_u \mathcal{E}\big(u_i(t),\gamma_i(t)\big) = f(t) \ \text{ in } \mathcal{V}^*, \tag{48}$$

$$\text{(flow rule in } \mathcal{W}^*) \quad 0 \in \partial R(\dot\gamma_i(t)) + \partial_\gamma \mathcal{E}\big(u_i(t),\gamma_i(t)\big) \quad \text{in } \mathcal{W}^*. \tag{49}$$

Set $u_{12} := u_1 - u_2$, $\gamma_{12} := \gamma_1 - \gamma_2$, take $\xi_i(t) \in \partial R(\dot\gamma_i(t))$ such that

$$0 = \xi_i(t) + \partial_\gamma E(u_i(t),\gamma_i(t)), \tag{50}$$

and define $\xi_{12} := \xi_1 - \xi_2$. Subtract the momentum balances for $i = 1$ and 2:

$$\rho \ddot u_{12} + \mathbb{D}\, e(\dot u_{12}) + \partial_u \mathcal{E}(u_{12},\gamma_{12}) = 0 \quad \text{in } \mathcal{V}^* \ \text{ a.e.}$$

Testing by $\dot u_{12} \in \mathcal{V}$ and using symmetry of $\mathbb{D}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} K(\dot u_{12}) + \int_\Omega \mathbb{D}\, e(\dot u_{12}) : e(\dot u_{12})\,\mathrm{d}x + \big\langle \partial_u \mathcal{E}(u_{12},\gamma_{12}),\, \dot u_{12}\big\rangle = 0 \quad \text{a.e.} \tag{51}$$

Subtracting (50) for $i = 1$ and 2, using that $\partial_\gamma \mathcal{E}$ is linear in (u, γ), and testing by $\dot\gamma_{12} = \dot\gamma_1 - \dot\gamma_2 \in \mathcal{W}$, we obtain for a.e. t:

$$\big\langle \partial_\gamma \mathcal{E}(u_{12}, \gamma_{12}),\, \dot\gamma_{12} \big\rangle = -\big\langle \xi_{12},\, \dot\gamma_{12} \big\rangle. \tag{52}$$

Since $\mathcal{E}$ is quadratic, the chain rule holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \mathcal{E}(u_{12}, \gamma_{12}) = \big\langle \partial_u \mathcal{E}(u_{12}, \gamma_{12}),\, \dot u_{12} \big\rangle + \big\langle \partial_\gamma \mathcal{E}(u_{12}, \gamma_{12}),\, \dot\gamma_{12} \big\rangle.$$

Adding (51) and (52) and invoking the chain rule yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(K(\dot u_{12}) + \mathcal{E}(u_{12}, \gamma_{12})\Big) + \int_\Omega \mathbb{D}\, e(\dot u_{12}) : e(\dot u_{12})\, \mathrm{d}x \;= -\big\langle \xi_{12},\, \dot\gamma_{12} \big\rangle \quad \text{a.e.} \tag{53}$$

By monotonicity of ∂R one has $\langle \xi_{12}, \dot\gamma_{12} \rangle \geq 0$ a.e.

Integrating (53) from 0 to t and using the fact that the initial data are identical, we obtain

$$K\big(\dot u_{12}(t)\big) + \mathcal{E}\big(u_{12}(t), \gamma_{12}(t)\big) + \int_0^t \int_\Omega \mathbb{D}\, e(\dot u_{12}) : e(\dot u_{12})\, \mathrm{d}x\mathrm{d}s \;\leq\; 0.$$

Since all terms on the left are nonnegative, they must vanish identically. In particular, $\dot u_{12} \equiv 0$ and $\mathcal{E}(u_{12}, \gamma_{12}) \equiv 0$ a.e. in $[0, T]$. By using that the two solutions share the same initial data $u_1(0) = u_2(0)$, $\gamma_1(0) = \gamma_2(0)$, and by the positive definiteness of the hardening terms $\beta > 0$, it follows that $u_{12} \equiv 0$ and $\gamma_{12} \equiv 0$. $\square$

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