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CONNECTEDNESS DEGREE OF G-V FUZZY MATROIDS AND ITS UNIFICATION WITH FUZZIFYING MATROIDS

Xiu Xin and Huan Lian

This paper studies the connected degree of G-V fuzzy matroids. The concept is generalized from the concept of matroid connectedness through plausible reasoning and has good expansibility and practicability. It is proved that a G-V fuzzy matroid M is disconnected if and only if its connected degree is the minimum (that is, $Con(M) = 0$), that is to say, it is connected if and only if $Con(M) \neq 0$. It is also proved that the connected degree $Con(M(\widetilde{G})) = 1$ of the fuzzy cycle matroid $M(\widetilde{G})$ of a fuzzy graph $\widetilde{G}$ if and only if the fuzzy graph $\widetilde{G}$ is 2-connected. Finally, combined with the existing connected degree theory of fuzzifying matroids, and relying on the transformation relationship between closed G-V fuzzy matroids and fuzzifying matroids, the consistency of the definitions of the connected degrees of the two types of matroids is demonstrated, revealing the inherent unity of different fuzzy matroid models in connectedness analysis.

Keywords: fuzzy graph, G-V fuzzy matroid, perfect $[0, 1]$-matroid, connectedness, connected degree, fuzzifying matroid

Classification: 03E70, 05B35, 52B40

1. INTRODUCTION

As a core branch of discrete mathematics, matroid theory, first proposed by Whitney in 1935 [29], has become a unified framework for characterizing cycle structures in graph theory and linear dependencies in matrices. Its axiomatic system not only provides a theoretical foundation for combinatorial optimization (such as the validity proof of greedy algorithms) but also demonstrates powerful application potential in network design, coding theory, and other fields [4, 22].

In 1965, Zadeh proposed the fuzzy set theory [37], whose core is to extend classical two-valued logic to the $[0, 1]$ interval via membership functions, laying the foundation for addressing uncertainty problems and realizing the fuzzification of matroids. In 1988, Goetschel and Voxman introduced this theory into the field of matroids and proposed G-V fuzzy matroids [5], the first combination of matroids and fuzzy theory, providing a new tool for analyzing element correlations in uncertain environments. In 2009, Shi proposed fuzzifying matroids [26], pioneering a new path for the fuzzification of matroids. In 2010,

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Shi further proposed $[0,1]$-matroids and bi-fuzzy matroids [27]. Among them, [0,1]-matroids extend G-V fuzzy matroids to more general fuzzy structures, and it is proved that perfect $[0,1]$-matroids are equivalent to G-V fuzzy matroids [11]. By introducing the concept of fuzzy cardinality of fuzzy sets, this theory first established a one-to-one correspondence between closed and perfect $[0,1]$-matroids (which essentially coincide with G-V fuzzy matroids) and fuzzy rank functions [33, 34], which differ from the fuzzy rank function defined in [10], laying a rigorous mathematical foundation for subsequent connectedness analysis. For the sake of simplicity in subsequent discussions, we will treat perfect $[0,1]$-matroids and G-V fuzzy matroids as theoretically identical constructs and no longer distinguish them strictly.

Connectedness, as a core concept in matroid theory, essentially measures the integrity of structures. Among the fundamental classes of matroids, cycle matroids establish an inherent link between graph theory and matroid theory: for a crisp graph $G = (V, E)$, the cycle matroid $M(G) = (E, \mathcal{I}_G)$ takes the edge set of G as its ground set, and a subset of edges is independent in $M(G)$ (i. e., belongs to $\mathcal{I}_G$) if and only if it does not contain any cycle of G [13]. Notably, the 2-connectedness of a crisp graph is equivalent to the connectedness of its cycle matroid [22], the key conclusion provides important inspiration for subsequent fuzzy extension studies. The theoretical development of fuzzy matroid connectedness has progressed in phases:

- **2010 (Foundational Framework):** Shi [27] proposed [0,1]-matroids and bi-fuzzy matroids, and demonstrated that fuzzy graphs can induce [0,1]-matroids (theoretical predecessors of fuzzy cycle matroids), building a bridge between fuzzy graphs and fuzzy matroids.

- **Parallel (Refined Studies):** Li and collaborators [15] analyzed the connectedness of refined G-V fuzzy matroids, while independently characterizing fuzzy graph structural properties [16] laying dual theoretical foundations for the correlation between fuzzy graphs and fuzzy matroids.

- **2021-2023 (Recent Advances):** Shabna and Sameena [24] defined graphic fuzzy matroids (fuzzy isomorphic with fuzzy cycle matroids); Xin et al. [33] established connected degree theory for fuzzifying matroids (filling the gap in quantitative analysis for this model); Shabna and Sameena [25] later characterized graphic fuzzy matroid connectedness via equivalence relations.

Notably, the 2025 fuzzy graphic binary matroid approach has been successfully applied to power grid communication network analysis [14], highlighting the practical value of fuzzy matroid theory in complex engineering scenarios. These studies have collectively enriched the theoretical system of fuzzy matroids and fuzzy graphs, but existing results have not yet established a direct quantitative correlation between 2-connectedness of fuzzy graphs and connectedness of G-V fuzzy matroids, and the connectedness theories of the two types of fuzzy matroids (G-V fuzzy matroids and fuzzifying matroids) remain fragmented. This raises our thinking: can the 2-connectedness of fuzzy graphs be quantitatively characterized by the connectedness degree of their fuzzy cycle matroids? Is there an inherent unity between the connectedness definitions of the two types of fuzzy matroids?

This paper aims to construct a quantitative analysis framework for G-V fuzzy matroid connectedness by introducing the connected degree concept—with good expansibility, it can form a unified framework with fuzzifying matroid connectedness theory (two core paths of matroid fuzzification) and generalize to broader models like bi-fuzzy matroids. Specifically, we first prove that a G-V fuzzy matroid is disconnected if and only if its connected degree is zero; second, establish the equivalence between a fuzzy cycle matroid's connected degree of 1 and the fuzzy graph's 2-connectedness; finally, reveal the consistency of connected degree definitions between closed G-V fuzzy matroids and fuzzifying matroids via structural transformations, highlighting the inherent unity of different fuzzy matroid models in connectedness analysis and extending the existing theoretical system.

To elaborate on the above research content, the structure of this paper is arranged as follows: Section 2 reviews the basic definitions and core properties of matroids and G-V fuzzy matroids; Section 3 defines connected G-V fuzzy matroids and analyzes their key characteristics; Section 4 proposes the connected degree of G-V fuzzy matroids and verifies its core properties; Section 5 unifies the connected degree theories of G-V fuzzy matroids and fuzzifying matroids; Section 6 summarizes the research conclusions and discusses future research directions.

2. PRELIMINARIES

Throughout this paper, the ground set E of all matroids and fuzzy matroids is assumed to be a finite non-empty set, which is consistent with the foundational framework of classical matroid theory [13, 22, 28] and fuzzy matroid theory [5, 26, 27].

We begin with the definition of a matroid.

Definition 2.1. (Welsh [28]) A matroid M is an order pair $(E,\mathcal{I})$, where $\mathcal{I}$ satisfies the following conditions:

(I1) $\emptyset \in \mathcal{I}$;

(I1) If $B \in \mathcal{I}$ and $A \subseteq B$, then $A \in \mathcal{I}$;

(I2) If $A, B \in \mathcal{I}$ and $|A| < |B|$, then there exists $e \in B - A$ such that $A \cup \{e\} \in \mathcal{I}$.

The members of $\mathcal{I}$ are the independent sets of M. A base of M is a maximal independent set in M, the collection of bases is denoted $\mathcal{B}_{\mathcal{I}}$. Then $\{E - B : B \in \mathcal{B}_{\mathcal{I}}\}$ is the set of bases of a matroid $M^* = (E,\mathcal{I}^*)$, which is called the dual of M. A subset of E not belonging to $\mathcal{I}$ is called dependent. A circuit of M is a minimal dependent set in M, the collection of circuits is denoted by $\mathcal{C}_{\mathcal{I}}$. The rank function of $(E,\mathcal{I})$ is a map $R_{\mathcal{I}} : 2^E \to \mathbb{N}$ defined by $R_{\mathcal{I}}(A) = \{|B| : B \subseteq A, B \in \mathcal{I}\}$ $(\forall A \in 2^E)$, where $\mathbb{N}$ denote the set of all natural numbers.

We will use the following notation in the course of this paper. If X be a set, then a fuzzy set A on X is a mapping $A : X \to [0, 1]$. We denote the set of all fuzzy sets on X by $[0, 1]^X$. If $A \in [0, 1]^X$ and $a \in [0, 1]$, define

$$\begin{aligned} &\mathrm{supp}A = \{x \in X : A(x) > 0\}, \quad m(A) = \inf\{A(x) : x \in \mathrm{supp}A\} \\ &A_{[a]} = \{x \in X : A(x) \geq a\}, \quad A_{(a)} = \{x \in X : A(x) > a\}. \end{aligned}$$

If $\mathcal{I} \subseteq [0,1]^X$ and $a \in [0,1]$, define

$$\mathcal{I}[a] = \{A_{[a]} : A \in \mathcal{I}\},\ \mathcal{I}(a) = \{A_{(a)} : A \in \mathcal{I}\}.$$

For $a \in [0,1]$ and $A \subseteq X$, define a fuzzy sets $a \wedge A$ as follows:

$$(a \wedge A)(x) = \left\{ \begin{array}{l} a,\ x \in A; \\ 0,\ x \notin A. \end{array} \right.$$

Then $1 \wedge A$ is the characteristic function of A, denoted by χ_A.

If $\mathbb{N}$ is the set of all natural numbers, then a fuzzy natural number is an antitone map $\lambda : \mathbb{N} \to [0,1]$ satisfying

$$\lambda(0) = 1, \bigwedge_{n \in \mathbb{N}} \lambda(n) = 0.$$

The set of all fuzzy natural numbers is denoted by $\mathbb{N}([0,1])$.
For any $\lambda, \mu \in \mathbb{N}([0,1])$, the addition $\lambda + \mu$ of λ and μ is defined by

$$\forall n \in \mathbb{N}, (\lambda + \mu)(n) = \bigvee_{n=k+l} (\lambda(k) \wedge \mu(l)).$$

If $a \in [0,1]$, then $(\lambda + \mu)_{[a]} = \lambda_{[a]} + \mu_{[a]}$ and $(\lambda + \mu)_{(a)} = \lambda_{(a)} + \mu_{(a)}$ [26, 33].

If E is a nonempty finite set and $A \in [0,1]^E$, then its fuzzy cardinality $|A|$ is a fuzzy natural number defined by

$$\forall n \in \mathbb{N},\ |A|(n) = \bigvee \{a \in [0,1] : |A_{[a]}| \geq n\}.$$

A G-V fuzzy matriod on E was defined as follows.

Definition 2.2. (Goetschel and Voxman [5]) A pair $(E, \mathcal{I})$ is a G-V fuzzy matroid if and only if $\mathcal{I} \subseteq [0,1]^E$ is a nonempty set family of fuzzy sets satisfying:

(FI1) If $A \in \mathcal{I}, B \in [0,1]^E$,, and $B \leq A$, then $B \in \mathcal{I}$;

(FI2) If $A, B \in \mathcal{I}$ and $|\mathrm{supp}A| < |\mathrm{supp}B|$, then there exists $C \in \mathcal{I}$ such that

(i) $A < C \leq A \vee B$,

(ii) $\mathrm{m}(C) \geq \min\{\mathrm{m}(A),\mathrm{m}(B)\}$.

Example 2.3. Let $E = \{1,2,3\}$ and $\mathcal{I} = \{A \in [0,1]^E \mid A(2) < 0.5\}$. This is a G-V fuzzy matroid [6], with (FI1) and (FI2) verified as follows:

(FI1) For any $A \in \mathcal{I}$ and $B \in [0,1]^E$ with $B \leq A$, $B(2) \leq A(2) < 0.5$, so $B \in \mathcal{I}$;

(FI2) Take $A, B \in \mathcal{I}$ such that $|\mathrm{supp}A| < |\mathrm{supp}B|$. Construct C as:

$$C(x) = \begin{cases} \min\{m(A), m(B)\}, & A(x) = 0, \\ A(x) \vee B(x), & A(x) > 0. \end{cases}$$

- $C \in \mathcal{I}$: when $A(2) = 0$, $C(2) = \min\{m(A), m(B)\} < 0.5$; when $A(2) > 0$, $C(2) = A(2) \vee B(2) < 0.5$;
- (i) $A < C \leq A \vee B$: when $A(x) = 0$, $C(x) > 0 = A(x)$; when $A(x) > 0$, $C(x) \geq A(x)$, so $A < C \subseteq A \vee B$;
- (ii) $m(C) \geq \min\{m(A), m(B)\}$: when $A(x) = 0$, $C(x) = \min\{m(A), m(B)\}$; when $A(x) > 0$, $C(x) = A(x) \vee B(x) \geq A(x) \geq m(A) \geq \min\{m(A), m(B)\}$.

Therefore, $(E, \mathcal{I})$ is a G-V fuzzy matroid.

A $[0,1]$-matroid and a perfect $[0,1]$-matroid on E were defined as follows.

Definition 2.4. (Shi [27], Xin and Shi [33]) A pair $(E, \mathcal{I})$ is a $[0,1]$-matroid if and only if $\mathcal{I} \subseteq [0,1]^E$ is a nonempty set family of fuzzy sets satisfying:

(LI1) $A \in [0,1]^E, B \in \mathcal{I}, A \leq B \Rightarrow B \in \mathcal{I}$;

(LI2) If $A, B \in \mathcal{I}$ and $b = |B|(n) \not\leq |A|(n)$ for some $n \in \mathbb{N}$, then there exists $x \in F(A, B)$ such that $(b \wedge A_{[b]}) \vee x_b \in \mathcal{I}$, where

$$F(A, B) = \{x \in E : b \leq B(x), b \not\leq A(x)\}.$$

A $[0,1]$-matroid is called a perfect $[0,1]$-matroid, if it satisfies:

(LI3) If $A \in [0,1]^E$, $a \wedge A_{[a]} \in \mathcal{I}$ for all $a \in (0,1]$, then $A \in \mathcal{I}$. If $\mathcal{I}$ is a family of independent L-fuzzy sets on E, then the pair $(E, \mathcal{I})$ is called an L-matroid.

As noted in the Introduction, a G-V fuzzy matroid on E is equivalent to a perfect $[0,1]$-matroid (see [11, 32]), so we uniformly refer to it as a G-V fuzzy matroid throughout the paper.

Theorem 2.5. (Goetschel and Voxman [5], Shi [27]) Let $(E, \mathcal{I})$ be a G-V fuzzy matroid, then each of the following holds:

(1) $\forall a \in (0,1]$, $(E, \mathcal{I}[a])$ is a matroid.

(2) $\forall a \in [0,1)$, $(E, \mathcal{I}(a))$ is a matroid.

Theorem 2.6. (Goetschel and Voxman [5]) Let $(E, \mathcal{I})$ be a G-V fuzzy matroid. Then there is a finite sequence $0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$ such that

(1) If $a_i < a, b < a_{i+1}$, then $\mathcal{I}[a] = \mathcal{I}[b]$, $0 \leqslant i \leqslant n-1$;

(2) If $a_i < a < a_{i+1} < b < a_{i+2}$, then $\mathcal{I}[a] \supset \mathcal{I}[b]$, $0 \leqslant i \leqslant n-2$;

(3) If $0 < a < b \leq 1$, then $\mathcal{I}[a] \supseteq \mathcal{I}[b]$.

The sequence $a_0, a_1, \ldots, a_n$ is called the fundamental sequence for $(E, \mathcal{I})$.

A G-V fuzzy matroid $(E, \mathcal{I})$ with fundamental sequence $a_0, a_1, \ldots, a_n$ is said to closed if whenever $a_{i-1} < a \leq a_i$ $(1 \leq i \leq n)$, then $\mathcal{I}[a] = \mathcal{I}[a_i]$. It is said to regular if whenever $0 < a < b \leq 1$, then for each basis B of $(E, \mathcal{I}[a])$, there is a basis A of $(E, \mathcal{I}[b])$ such that $A \subseteq B$. It is said to refined if whenever $0 < a < b \leq 1$, then for each circuit of $(E, \mathcal{I}[a])$ is a circuit of $(E, \mathcal{I}[b])$.

The following is 'partial converse' of Theorem 2.6.

Theorem 2.7. (Goetschel and Voxman [5]) Let $0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$ be a finite sequence. Suppose that $(E, \mathcal{I}_{a_1})$, $(E, \mathcal{I}_{a_2}), \cdots, (E, \mathcal{I}_{a_n})$ is a sequence of matroids on E such that $\mathcal{I}_{a_{i+1}} \subset \mathcal{I}_{a_i}$ $(1 \leq i \leq n-1)$. For each $a \in (\mathcal{I}_{a_{i-1}}, \mathcal{I}_{a_i}]$ $(1 \leq i \leq n)$, let $\mathcal{I}_a = \mathcal{I}_{a_i}$. If $\mathcal{I} = \{A \in [0,1]^E : \forall a \in (0,1), A_{[a]} \in \mathcal{I}_a\}$, then $(E, \mathcal{I})$ is a G-V fuzzy matroid.

The fuzzy basis was defined as follows. A G-V fuzzy matroid may has no basis in general. But if it is closed, then its bases exist.

Definition 2.8. (Goetschel and Voxman [6]) A fuzzy base for a G-V fuzzy matroid $(E, \mathcal{I})$ is a maximal member B in $\mathcal{I}$ (where B is said to be maximal in $\mathcal{I}$ if whenever $A \in \mathcal{I}$ and $B \leq A$, then $B = A$.)

The fuzzy circuit was defined as follows.

Definition 2.9. (Goetschel and Voxman [7]) Let $(E, \mathcal{I})$ be a G-V fuzzy matroid. $\mu \in [0,1]^E$ is called a fuzzy circuit of $(E, \mathcal{I})$ if $\mu \notin \mathcal{I}$ and $\mu \setminus \setminus x \in \mathcal{I}$ for each $x \in$suppμ, where

$$(\mu \setminus \setminus x)(y) = \begin{cases} \mu(y), & y \neq x; \\ 0, & y = x. \end{cases}$$

Theorem 2.10. (Goetschel and Voxman [7]) Suppose that $(E, \mathcal{I})$ is a G-V fuzzy matroid and $\mu \in [0,1]^E$. Let $R^+(\mu) = \{b_1, b_2, \ldots, b_k\}$, where $b_1 < b_2 < \cdots < b_k$. Then μ is a fuzzy circuit in $(E, \mathcal{I})$ if and only if $\mu_{[b_1]}$ is a circuit in $(E, \mathcal{I}[b_1])$ and $\mu_{[b_i]} \in \mathcal{I}[b_i], 2 \leq i \leq k$.

The fuzzy rank function was defined as follows.

Definition 2.11. (Shi [27]) Let $(E, \mathcal{I})$ be a G-V fuzzy matroid. The mapping $R_{\mathcal{I}} : [0,1]^E \to \mathbb{N}([0,1])$ defined by $\forall A \in [0,1]^E, R_{\mathcal{I}}(A) = \bigvee\{|B| : B \leqslant A, B \in \mathcal{I}\}$ is called the fuzzy rank function for $(E, \mathcal{I})$.

Theorem 2.12. (Shi [27], Xin and Shi [33]) Let $(E, \mathcal{I})$ be a G-V fuzzy matroid. We have

(1) $R_{\mathcal{I}}(A)_{(a)} = R_{\mathcal{I}(a)}(A_{(a)})$ $(\forall A \in [0,1]^E, a \in [0,1))$.

(2) If $(E, \mathcal{I})$ is closed, then $R_{\mathcal{I}}(A)_{[a]} = R_{\mathcal{I}[a]}(A_{[a]})$ $(\forall A \in [0,1]^E, a \in (0,1])$.

G-V fuzzy matroids quantify the fuzzy correlations of elements through membership functions, which not only are compatible with the core axioms of classical matroids and the stability of level structures but also can handle uncertain scenarios that cannot be covered by traditional models [5, 27]. Notably, their compatibility with greedy algorithms (a key tool for classical matroid applications) was first established in foundational work [8], laying the theoretical groundwork for practical optimizations. Specially, graphic fuzzy matroids (a typical subclass of G-V fuzzy matroids) have been applied in practical fields such as social network analysis, environmental reliability monitoring, and geographic information systems [24, 25].

3. CONNECTED G-V FUZZY MATROIDS

The concept of matroid connectivity was first proposed by Whitney in 1935 [29], and its equivalent formulations and properties were systematically summarized in [28]:

Definition 3.1. (Welsh [28], Whitney [29]) A matroid $(E,\mathcal{I})$ is said to be connected if for every proper subset A of E, $R_\mathcal{I}(A)+R_\mathcal{I}(E-A)>R_\mathcal{I}(E)$.

Theorem 3.2. (Welsh [28]) A matroid $(E,\mathcal{I})$ is connected if and only if its dual $(E,\mathcal{I}^*)$ is connected.

Theorem 3.3. (Welsh [28]) Let $(E,\mathcal{I})$ be a matroid, then the following conditions are equivalent:

(1) $(E,\mathcal{I})$ is disconnected;

(2) there is a nonempty proper subset A of E, such that for every circuit C of $(E,\mathcal{I})$, $C\subseteq A$ or $C\subseteq E-A$;

(3) there is a nonempty proper subset A of E, such that $R_\mathcal{I}(A)+R_\mathcal{I}(E-A)=R_\mathcal{I}(E)$;

(4) there is a nonempty proper subset A of E, such that $R_{\mathcal{I}^*}(A)+R_{\mathcal{I}^*}(E-A)=R_{\mathcal{I}^*}(E)$.

Definition 3.4. (Welsh [28]) A nonempty proper subset A of E satisfying (2), (3) or (4) in Theorem 3.3 is called a separator of $(E,\mathcal{I})$.

The above theorem shows that $(E,\mathcal{I})$ is disconnected if and only if there exists a proper subset A of E, such that A is a separator.

To better illustrate the concepts of connected matroids, disconnected matroids, and separators, we present the following concrete examples of matroids:

Example 3.5. (1) Let $M=(E,\mathcal{I})$ be a matroid with ground set $E=\{x,y,z\}$ and independent set family $\mathcal{I}=\{\emptyset,\{x\},\{y\},\{z\},\{x,z\},\{y,z\}\}$. Its rank function is $R_\mathcal{I}(\emptyset)=0$, $R_\mathcal{I}(\{x\})=R_\mathcal{I}(\{y\})=R_\mathcal{I}(\{z\})=1$, $R_\mathcal{I}(\{x,y\})=R_\mathcal{I}(\{x,z\})=R_\mathcal{I}(\{y,z\})=2$, $R_\mathcal{I}(E)=2$, and we can verify that $R_\mathcal{I}(A)+R_\mathcal{I}(E-A)>R_\mathcal{I}(E)$ holds for every nonempty proper subset $A\subset E$ (e.g., $A=\{x\}$ gives $1+2=3>2$, $A=\{x,z\}$ gives $2+1=3>2$), so M is connected by Definition 3.1.

(2) Let $M=(E,\mathcal{I})$ be a matroid with ground set $E=\{x,y,z\}$ and independent set family $\mathcal{I}=\{\emptyset,\{x\},\{y\},\{x,y\}\}$. Its rank function is $R_\mathcal{I}(\emptyset)=0$, $R_\mathcal{I}(\{x\})=R_\mathcal{I}(\{y\})=1$, $R_\mathcal{I}(\{z\})=0$, $R_\mathcal{I}(\{x,y\})=2$, $R_\mathcal{I}(\{x,z\})=R_\mathcal{I}(\{y,z\})=1$, $R_\mathcal{I}(E)=2$; its collection of circuits is $\mathcal{C}_\mathcal{I}=\{\{z\}\}$. Take the proper subset $A=\{x,y\}$: the circuit $\{z\}$ is contained in $E-A$ (Theorem 3.3(2)), $R_\mathcal{I}(A)+R_\mathcal{I}(E-A)=2+0=2=R_\mathcal{I}(E)$ (Theorem 3.3(3)), and $A=\{x,y\}$ is a separator (Definition 3.4), so M is disconnected.

The obvious fuzzy analog of a separator is the following:

Definition 3.6. Let $(E,\mathcal{I})$ be a G-V fuzzy matroid, $A\in 2^E$. If for every fuzzy circuit μ of $(E,\mathcal{I})$, $\mu\leq\chi_A$ or $\mu\leq\chi_E-\chi_A$, then χ_A is called a separator of $(E,\mathcal{I})$.

Obviously, $\chi_\emptyset$ and χ_E are separators of $(E,\mathcal{I})$.

Theorem 3.7. Let $(E,\mathcal{I})$ be a G-V fuzzy matroid. Then χ_A is a separator of $(E,\mathcal{I})$ if and only if A is a separator of $(E,\mathcal{I}[a])$ for each $a\in(0,1]$.

Proof. Necessity. Let $a\in(0,1]$ and C be a circuit of $(E,\mathcal{I}[a])$, then $a\wedge C$ is a fuzzy circuit of $(E,\mathcal{I})$ by Theorem 2.10. Since χ_A is a separator of $(E,\mathcal{I})$, $a\wedge C\le\chi_A$ or $a\wedge C\le\chi_E-\chi_A$. Thus $C\subseteq A$ or $C\subseteq E-A$. This implies that A is a separator of $(E,\mathcal{I}[a])$ for each $a\in(0,1]$. Sufficiency. Let μ be a fuzzy circuit of $(E,\mathcal{I})$, then $\mu_{[m(\mu)]}$ is a circuit of $(E,\mathcal{I}[m(\mu)])$ by Theorem 2.10. Since A is a separator of $(E,\mathcal{I}[m(\mu)])$, $\mu_{[m(\mu)]}\subseteq A$ or $\mu_{[m(\mu)]}\subseteq E-A$. Thus $\mu\le\chi_A$ or $\mu\le\chi_E-\chi_A$. This implies that χ_A is a separator of $(E,\mathcal{I})$. □

An immediate consequence of Theorem 3.7 characterizes a separator of the closed G-V fuzzy matroid.

Theorem 3.8. Let $(E,\mathcal{I})$ be a closed G-V fuzzy matroid with fundamental sequence $0=a_0<a_1<a_2<\cdots a_n=1$. Then χ_A is a separator of $(E,\mathcal{I})$ if and only if A is a separator of $(E,\mathcal{I}[a_i])$ $(i=1,2,\ldots,n)$.

The following result also characterizes a separator of the closed G-V fuzzy matroid.

Theorem 3.9. Let $(E,\mathcal{I})$ be a closed G-V fuzzy matroid. Then the following conditions are equivalent:

(1) χ_A is a separator of $(E,\mathcal{I})$;

(2) $R_{\mathcal{I}}(\chi_A)+R_{\mathcal{I}}(\chi_E-\chi_A)=R_{\mathcal{I}}(\chi_E)$;

(3) A is a separator of $(E,\mathcal{I}(a))$ for each $a\in[0,1)$.

Proof. By Definition 3.4 and Theorem 3.7, we have

$$\begin{aligned}
&\chi_A \text{ is a separator of } (E,\mathcal{I})\\
\Leftrightarrow\ & A \text{ is a separator in } (E,\mathcal{I}[a]) \text{ for each } a\in(0,1]\\
\Leftrightarrow\ & R_{\mathcal{I}[a]}(A)+R_{\mathcal{I}[a]}(E-A)=R_{\mathcal{I}[a]}(E) \text{ for each } a\in(0,1]\\
\Leftrightarrow\ & R_{\mathcal{I}}(\chi_A)_{[a]}+R_{\mathcal{I}}(\chi_E-\chi_A)_{[a]}=R_{\mathcal{I}}(\chi_E)_{[a]} \text{ for each } a\in(0,1]\\
\Leftrightarrow\ & [R_{\mathcal{I}}(\chi_A)+R_{\mathcal{I}}(\chi_E-\chi_A)]_{[a]}=R_{\mathcal{I}}(\chi_E)_{[a]} \text{ for each } a\in(0,1]\\
\Leftrightarrow\ & R_{\mathcal{I}}(\chi_A)+R_{\mathcal{I}}(\chi_E-\chi_A)=R_{\mathcal{I}}(\chi_E)\\
\Leftrightarrow\ & [R_{\mathcal{I}}(\chi_A)+R_{\mathcal{I}}(\chi_E-\chi_A)]_{(a)}=R_{\mathcal{I}}(\chi_E)_{(a)} \text{for each} a\in[0,1)\\
\Leftrightarrow\ & R_{\mathcal{I}}(\chi_A)_{(a)}+R_{\mathcal{I}}(\chi_E-\chi_A)_{(a)}=R_{\mathcal{I}}(\chi_E)_{(a)} \text{ for each } a\in[0,1)\\
\Leftrightarrow\ & R_{\mathcal{I}(a)}(A)+R_{\mathcal{I}(a)}(E-A)=R_{\mathcal{I}(a)}(E) \text{ for each } a\in[0,1)\\
\Leftrightarrow\ & A \text{ is a separator in } (E,\mathcal{I}(a)) \text{ for each } a\in[0,1).
\end{aligned}$$

Therefore, (1)⇔(2)⇔(3). □

Remark 3.10. Let $(E,\mathcal{I})$ be a closed regular G-V fuzzy matroid with a fundamental sequence $0=a_0<a_1<a_2<\cdots a_n=1$, $\mathcal{B}\subseteq[0,1]^E$ be the family of fuzzy bases of $(E,\mathcal{I})$. If $\mathcal{B}^*=\{\chi_E-B:B\in\mathcal{B}\}$, then $\mathcal{B}^*=\{\chi_E-B:B\in\mathcal{B}\}$ forms the family of fuzzy bases for a closed regular G-V fuzzy matroid $(E,\mathcal{I}^*)$ with a fundamental sequence is $0=b_n<b_{n-1}<\cdots b_1<b_0=1$, where $b_i=1-a_i, i=0,1,\cdots,n$. $(E,\mathcal{I}^*)$ is called the dual of $(E,\mathcal{I})$, and $\mathcal{I}^*[b_i]=(\mathcal{I}[a_{n-i}])^*,\ (i=0,1,2\cdots,n)$. Moreover, we have $(\mathcal{I}^*)^*=\mathcal{I}$ (see [9, 30, 31]).

Theorem 3.11. Let $(E,\mathcal{I})$ be a closed regular G-V fuzzy matroid. Then the following conditions are equivalent:

(1) χ_A is a separator of $(E,\mathcal{I})$;

(2) χ_A is a separator of $(E,\mathcal{I}^*)$;

(3) $R_{\mathcal{I}^*}(\chi_A) + R_{\mathcal{I}^*}(\chi_E - \chi_A) = R_{\mathcal{I}^*}(\chi_E)$.

Proof. (1)⇒(2). If χ_A is a separator of $(E,\mathcal{I})$, then A is a separator of $(E,\mathcal{I}[a_i])$ for each $i = 1, 2, \cdots, n$ by Theorem 3.8, thus A is a separator of $(E,\mathcal{I}^*[b_i]) = (E, (\mathcal{I}[a_{n-i}])^*)$ for each $i = 0, 1, \cdots, n-1$. Since $0 = b_n < b_{n-1} < \cdots b_1 < b_0 = 1$ is the fundamental sequence of $(E,\mathcal{I}^*)$, χ_A is a separator of $(E,\mathcal{I}^*)$ by Theorem 3.8.
(2)⇒(1) holds since $\mathcal{I} = (\mathcal{I}^*)^*$ and (1)⇒(2).
(2)⇔(3) is obvious by Theorem 3.9. □

The following result characterizes a separator of the refined G-V fuzzy matroid. For the definition and properties of the refined G-V fuzzy matroid, refer to Literature [18].

Theorem 3.12. Let $(E,\mathcal{I})$ be a refined G-V fuzzy matroid. Then χ_A is a separator of $(E,\mathcal{I})$ if and only if A is a separator of $(E,\mathcal{I}[1])$.

Proof. Necessity: If χ_A is a separator of $(E,\mathcal{I})$, then A is a separator of $(E,\mathcal{I}[a])$ for each $a \in (0,1]$ by Theorem 3.7, thus A is a separator of $(E,\mathcal{I}[1])$. Sufficiency: Let $a \in (0,1]$ and C is a circuit of $(E,\mathcal{I}[a])$, then C is a circuit of $(E,\mathcal{I}[1])$ by $(E,\mathcal{I})$ is refined. Since A is a separator of $(E,\mathcal{I}[1])$, $C \subseteq A$ or $C \subseteq E - A$. Thus A is a separator of $(E,\mathcal{I}[a])$ for each $a \in (0,1]$, hence χ_A is a separator of $(E,\mathcal{I})$ by Theorem 3.7. □

The obvious fuzzy analog of a disconnected matroid is the following:

Definition 3.13. A G-V fuzzy matroid $(E,\mathcal{I})$ is said to be disconnected if there exists a nonempty proper subset A of E, such that χ_A is a separator of $(E,\mathcal{I})$.

Based on the above definition, a G-V fuzzy matroid is either connected or not. It is reasonable to define a connected G-V fuzzy matroid as follows:

Definition 3.14. A G-V fuzzy matroid $(E,\mathcal{I})$ is said to be connected if it is not disconnected, i. e. for every nonempty proper subset A of E, χ_A is not a separator of $(E,\mathcal{I})$.

The following result is the fuzzy analog of Definition 3.1, based on Theorem 3.9 and Definition 3.14. We provide a detailed proof below.

Theorem 3.15. A closed G-V fuzzy matroid $(E,\mathcal{I})$ is connected if and only if for every nonempty proper subset A of E, $R_{\mathcal{I}}(\chi_A) + R_{\mathcal{I}}(\chi_E - \chi_A) > R_{\mathcal{I}}(\chi_E)$.

Proof. First, by the submodularity rank function [33] (For any $A, B \in [0,1]^E$, $R_{\mathcal{I}}(A) + R_{\mathcal{I}}(B) \geq R_{\mathcal{I}}(A \vee B) + R_{\mathcal{I}}(A \wedge B)$), the following inequality holds for any subset $A \subseteq E$,

$$R_{\mathcal{I}}(\chi_A) + R_{\mathcal{I}}(\chi_E - \chi_A) \geq R_{\mathcal{I}}(\chi_E).$$

Necessity: Suppose $(E,\mathcal{I})$ is connected. According to Definition 3.14, there exists no nonempty proper subset A of E such that χ_A is a separator. Combining with the equivalent characterization of separator in Theorem 3.9, there is no nonempty proper subset A satisfying $R_{\mathcal{I}}(\chi_A) + R_{\mathcal{I}}(\chi_E - \chi_A) = R_{\mathcal{I}}(\chi_E)$. Thus, by the aforementioned inequality, for all nonempty proper subsets A of E, we must have $R_{\mathcal{I}}(\chi_A) + R_{\mathcal{I}}(\chi_E - \chi_A) > R_{\mathcal{I}}(\chi_E)$.

Sufficiency: Assume that for every nonempty proper subset A of E, $R_{\mathcal{I}}(\chi_A)+R_{\mathcal{I}}(\chi_E-\chi_A) > R_{\mathcal{I}}(\chi_E)$ holds. From the aforementioned inequality, there is no nonempty proper subset A such that the equality holds. By Theorem 3.9, $\mathcal{M}$ has no separator, and hence $(E,\mathcal{I})$ is connected by Definition 3.14.

In conclusion, a closed G-V fuzzy matroid $(E,\mathcal{I})$ is connected if and only if $R_{\mathcal{I}}(\chi_A)+R_{\mathcal{I}}(\chi_E - \chi_A) > R_{\mathcal{I}}(\chi_E)$ for every nonempty proper subset A of E. □

The following result is the fuzzy analog of Theorem 3.2. It is obvious by Theorem 3.11 and Definition 3.14.

Theorem 3.16. A closed regular G-V fuzzy matroid $(E,\mathcal{I})$ is connected if and only if its dual $(E,\mathcal{I}^*)$ is connected.

Proof. Assume $(E,\mathcal{I})$ is connected, then $(E,\mathcal{I})$ has no nonempty proper separators, and by the equivalence in Theorem 3.11, $(E,\mathcal{I}^*)$ also has no nonempty proper separators. Thus, $(E,\mathcal{I}^*)$ is connected by Definition 3.14.

For the reverse direction, suppose $(E,\mathcal{I}^*)$ is connected. By Remark 3.10, $(\mathcal{I}^*)^* = \mathcal{I}$. Applying the equivalence in Theorem 3.11 to $(E,\mathcal{I}^*)$, we know $(E,\mathcal{I}^*)$ has no nonempty proper separators if and only if $(E,(\mathcal{I}^*)^*) = (E,\mathcal{I})$ has no nonempty proper separators. Hence, $(E,\mathcal{I})$ is connected by Definition 3.14. □

The following result relates the connectedness of a G-V fuzzy matroid to that of its level matroids, based on Theorem 3.4, Theorem 3.9 and Definition 3.13.

Theorem 3.17. Let $(E,\mathcal{I})$ be a G-V fuzzy matroid. If $(E,\mathcal{I})$ is disconnected, then $(E,\mathcal{I}[a])$ is disconnected for each $a \in (0,1]$. That is, if $(E,\mathcal{I}[a])$ is connected for some $a \in (0,1]$, then $(E,\mathcal{I})$ is connected. But the converse is not true. This can be seen from Example 3.18.

Proof. If $(E,\mathcal{I})$ is disconnected, then there exists a nonempty proper subset A of E, such that χ_A is a separator of $(E,\mathcal{I})$ by Definition 3.13, thus A is a separator of $(E,\mathcal{I}[a])$ for each $a \in (0,1]$ by Theorem 3.9, hence $(E,\mathcal{I}[a])$ is disconnected for each $a \in (0,1]$ by Theorem 3.4. □

Example 3.18. Let $(E,\mathcal{I})$ be the G-V fuzzy matroid constructed as in [10, Example 4.2], with the ground set $E = \{x,y,z,u,v\}$. Specifically, $(E,\mathcal{I}_{\frac{1}{3}})$, $(E,\mathcal{I}_{\frac{2}{3}})$ and $(E,\mathcal{I}_1)$ are matroids with the families of circuits $\mathcal{C}_{\frac{1}{3}} = \{\{x,z,u,v\}\}$, $\mathcal{C}_{\frac{2}{3}} = \{\{y,z\},\{x,y,u\},\{x,z,u\}\}$, $\mathcal{C}_1 = \{\{y,z\},\{y,u\},\{y,v\},\{z,u\},\{z,v\},\{u,v\}\}$, respectively. Define $\mathcal{I} = \{A \in [0,1]^E : \forall a \in (0,1], A_{[a]} \in \mathcal{I}_a\}$, where

$$\mathcal{I}_a = \begin{cases} \mathcal{I}_{\frac{1}{3}}, & a \in (0,\frac{1}{3}]; \\ \mathcal{I}_{\frac{2}{3}}, & a \in (\frac{1}{3},\frac{2}{3}]; \\ \mathcal{I}_1, & a \in (\frac{2}{3},1]. \end{cases}$$

By Theorem 2.7, $(E,\mathcal{I})$ is a G-V fuzzy matroid. It is straightforward to verify that none of the three level matroids is connected. However, $(E,\mathcal{I})$ itself is connected. In fact, the separators of $(E,\mathcal{I}_{\frac{1}{3}})$ are $A=\{x,z,u,v\}$ and $A=\{y\}$, but they are not the separators of $(E,\mathcal{I}_{\frac{2}{3}})$. Hence there is not a nonempty proper subset A of E, such that A is a separator of $(E,\mathcal{I}[a])$ for each $a\in(0,1]$. This implies that there is not a nonempty proper subset A of E, such that χ_A is a separator of $(E,\mathcal{I})$, i. e. $(E,\mathcal{I})$ is connected.

The following result characterizes the connectedness of a refined G-V fuzzy matroid in terms of that of 1-level matroid. It shows that our definition of connectedness coincides with the definition of Li [15] for a refined G-V fuzzy matroid.

Theorem 3.19. A refined G-V fuzzy matroid $(E,\mathcal{I})$ is connected if and only if $(E,\mathcal{I}[1])$ is connected.

Proof. By Theorem 3.4, Theorem 3.12 and Definition 3.13, a refined G-V fuzzy matroid $(E,\mathcal{I})$ is disconnected if and only if there exists a nonempty proper subset A of E, such that χ_A is a separator of $(E,\mathcal{I})$ if and only if there exists a nonempty proper subset A of E, such that A is a separator of $(E,\mathcal{I}[1])$ if and only if $(E,\mathcal{I}[1])$ is disconnected. That is, a refined G-V fuzzy matroid $(E,\mathcal{I})$ is connected if and only if $(E,\mathcal{I}[1])$ is connected. □

But our definition of a connected G-V fuzzy matroid is not equivalent with that of Li [15]. This can be seen from Example 3.20.

Example 3.20. Let $E=\{x,y,z\}$, define

$$\mathcal{I}_a=\{\emptyset,\{x\},\{y\},\{z\}\}\ (0<a\leq\frac{1}{2}),\mathcal{I}_a=\{\emptyset,\{x\},\{y\}\}\ (\frac{1}{2}<a\leq 1).$$

Then $(E,\mathcal{I})$ is a G-V fuzzy matroid by Theorem 2.7, where $\mathcal{I}=\{A\in[0,1]^E: A_{[a]}\in\mathcal{I}_a, a\in(0,1]\}$. Since $(E,\mathcal{I}[\frac{1}{2}])$ is connected, $(E,\mathcal{I})$ is connected by Theorem 3.17. For every fuzzy circuit μ of $(E,\mathcal{I})$, $x\notin\mu_{[1]}$ or $z\notin\mu_{[1]}$, i. e. $x_1\vee z_1\nleq\mu$, hence $(E,\mathcal{I})$ is not connected of Li [15].

Remark 3.21. The connectedness for G-V fuzzy matoids is the fuzzy analog of that for matroids. But a connected G-V fuzzy matroid $(E,\mathcal{I})$ does not satisfy that there exists an a-level matroid $(E,\mathcal{I}[a])$ such that it is connected. Hence the notion of connectedness for G-V fuzzy matroids does not correspond to the idea of connectedness for fuzzy graphs. In order to introduce a notion of connectedness for G-V fuzzy matroids that extends the corresponding notion for fuzzy graphs, we will consider fuzziness degree of connectedness in the next section.

4. CONNECTED DEGREE OF G-V FUZZY MATROIDS

To define the notion of the connected degree of G-V fuzzy matroids, we need the following definition and results.

Definition 4.1. Let $(E, \mathcal{I})$ be a G-V fuzzy matroid and $A \in 2^E$, define

$$Sep(\chi_A) = \bigwedge\{a \in (0,1] : A \text{ is not a separator of } (E, \mathcal{I}[a])\}.$$

Then $Sep(\chi_A)$ is said to be the degree to χ_A which is a separator of $(E, \mathcal{I})$.

Obviously, $Sep(\chi_\emptyset) = 1$ and $Sep(\chi_E) = 1$.

The following theorem shows that Definition 4.1 is the generalization of Definition 3.6.

Theorem 4.2. Let $(E, \mathcal{I})$ be a closed G-V fuzzy matroid and $A \in 2^E$. $Sep(\chi_A) = 1$ if and only if χ_A is a separator of $(E, \mathcal{I})$.

Proof. Since $(E, \mathcal{I})$ is a closed G-V fuzzy matroid, and by Theorem3.7, we have

$$\begin{aligned} Sep(\chi_A) = 1 \quad &\Leftrightarrow \quad \bigwedge\{a \in (0,1] : A \text{ is not a separator of } (E, \mathcal{I}[a])\} = 1 \\ &\Leftrightarrow \quad \{a \in (0,1] : A \text{ is not a separator of } (E, \mathcal{I}[a])\} = \emptyset \\ &\Leftrightarrow \quad A \text{ is a separator of } (E, \mathcal{I}[a]) \text{ for each } a \in (0,1] \\ &\Leftrightarrow \quad \chi_A \text{ is a separator of } (E, \mathcal{I}). \end{aligned}$$

□

Theorem 4.3. Let $(E, \mathcal{I})$ be a closed G-V fuzzy matroid and $A \in 2^E$. Then $Sep(\chi_A) = 0$ if and only if A is not a separator of $(E, \mathcal{I}(0))$.

Proof. Let $0 = a_0 < a_1 < a_2 < \cdots a_n = 1$ be the fundamental sequence of $(E, \mathcal{I})$. $\forall A \in \mathcal{I}[a_1]$, then there exists $B \in \mathcal{I}$ such that $A = B_{[a_1]} \subseteq B_{(0)} \in \mathcal{I}(0)$, hence $A \in \mathcal{I}(0)$. Conversely, $\forall A \in \mathcal{I}(0)$, then there exists $B \in \mathcal{I}$ such that $A = B_{(0)} = B_{[r]} \in \mathcal{I}[r] = \mathcal{I}[a_1]$, where $r = \min\{m(B), a_1\}$. Hence $\mathcal{I}(0) = \mathcal{I}[a_1]$. $Sep(\chi_A) = 0$ if and only if $\bigwedge\{a \in (0,1] : A$ is not a separator of $(E, \mathcal{I}[a])\} = 0$ if and only if A is not a separator of $(E, \mathcal{I}[a_1])$ if and only if A is not a separator of $(E, \mathcal{I}(0))$. □

Definition 4.4. Let $M = (E, \mathcal{I})$ be a G-V fuzzy matroid. Define

$$Con(M) = \bigwedge\{[1 - Sep(\chi_A)] : A \neq \emptyset, A \neq E, A \in 2^E\}.$$

Then $Con(M)$ is said to be the connectedness degree of M.

The following theorem shows that Definition 4.4 is the generalization of Definition 3.13.

Theorem 4.5. Let $M = (E, \mathcal{I})$ be a closed G-V fuzzy matroid. Then $Con(M) = 0$ if and only if M is disconnected, i. e. $Con(M) \neq 0$ if and only if M is connected.

Proof. $Con(M) = 0$ if and only if $\bigwedge\{[1 - Sep(\chi_A)] : A \neq \emptyset, A \neq E, A \in 2^E\} = 0$ if and only if there is a proper subset A of E, such that $Sep(\chi_A) = 1$ if and only if there is a nonempty proper subset A of E, such that χ_A is a separator of M if and only if M is disconnected. □

Theorem 4.6. Let $M = (E, \mathcal{I})$ be a closed G-V fuzzy matroid. If $|E| \geq 2$, then $Con(M) = 1$ if and only if $(E, \mathcal{I}(0))$ is connected.

Proof. Since $|E| \geq 2$, $\{[1 - Sep(\chi_A)] : A \neq \emptyset, A \neq E, A \in 2^E\} \neq \emptyset$. Thus $Con(M) = 1$ if and only if $\bigwedge\{[1 - Sep(\chi_A)] : A \neq \emptyset, A \neq E, A \in 2^E\} = 1$ if and only if $Sep(\chi_A) = 0$ for every proper subset A of E if and only if A is not a separator of $(E, \mathcal{I}(0))$ for every proper subset A of E if and only if $(E, \mathcal{I}(0))$ is connected. □

We shall recall the connectedness for fuzzy graphs, and show the connected degree $Con(M) = 1$ for G-V fuzzy matroids corresponds directly to the idea of 2-connectedness for fuzzy graphs.

Definition 4.7. (Rosenfeld [23]) A fuzzy graph is defined as follows: let V be a set of nodes, $\sigma : V \to [0, 1]$ (vertex fuzzy set) and $\mu : V \times V \to [0, 1]$ (edge fuzzy set) satisfy $\forall x, y \in V, \mu(x, y) \leq \sigma(x) \wedge \sigma(y)$, then $\widetilde{G} = (V, \sigma, \mu)$ is called a fuzzy graph.

Let $\sigma \equiv 1$, we denote $\widetilde{G} = (V, \mu)$ for simplicity. Its basic graph is defined as $G = (V, E)$ where $E = \{e = v_i v_j : \mu(v_i, v_j) > 0\}$. Note we will assume that, unless otherwise specified, the underlying graph of any fuzzy graph here is finite and undirected.

Rosenfeld generalized graph connectedness to fuzzy graphs: $\widetilde{G}$ is connected if and only if its basic graph G is connected. In this paper, $\widetilde{G} = (V, \mu)$ is called 2-connected if G is a 2-connected graph (i. e. any two distinct edges are contained in a cycle).

The G-V fuzzy matroids derived from the fuzzy graphs in the following way.

Definition 4.8. (Shi [27]) For a fuzzy graph $\widetilde{G} = (V, \mu)$, define $\mathcal{I}_{\widetilde{G}} \subseteq [0, 1]^E$ by $\mathcal{I}_{\widetilde{G}} = \{A \in [0, 1]^E : A \leq \mu \text{ and } \forall a \in (0, 1], A_{[a]} \text{ does not contain any cycle of } (V, \mu_{[a]})\}$. Then $(E, \mathcal{I}_{\widetilde{G}})$ is a $[0, 1]$-matroid. And we can check that $(E, \mathcal{I}_{\widetilde{G}})$ is a closed and perfect $[0, 1]$-matroid, i. e. a closed G-V fuzzy matroid. We call it the fuzzy cycle matroid of $\widetilde{G}$. It is denoted by $M(\widetilde{G})$.

In order to prove the following theorem, we need the following lemma firstly.

Lemma 4.9. If $G = (V, E)$ is the basic graph of a fuzzy graph $\widetilde{G} = (V, \mu)$, then $\mathcal{I}_G = \mathcal{I}_{\widetilde{G}}(0)$.

Proof. $\forall A_{(0)} \in \mathcal{I}_{\widetilde{G}}(0)$, $A \in \mathcal{I}_{\widetilde{G}}$, then $A \leq \mu$ and $A_{[r]}$ does not contain any cycle of $(V, \mu_{[r]})$, where $r = \min\{m(A), m(\mu)\}$. Since $A_{(0)} = A_{[r]}$ and $\mu_{(0)} = \mu_{[r]}$, $A_{(0)} \subseteq \mu_{(0)}$ and $A_{(0)}$ does not contain any cycle of $(V, \mu_{(0)})$, i. e. $A_{(0)} \in \mathcal{I}_G$. This implies that $\mathcal{I}_{\widetilde{G}}(0) \subseteq \mathcal{I}_G$. Conversely, assume that $A \in \mathcal{I}_G$, i. e. $A \subseteq \mu_{(0)}$ and A does not contain any cycle of $(V, \mu_{(0)})$. Let $r = m(\mu)$, then $r \wedge A \leq \mu$, $(r \wedge A)_{[a]} = A$ for every $a \leq r$ and $(r \wedge A)_{[a]} = \emptyset$ for every $a > r$. Thus $(r \wedge A)_{[a]}$ does not contain any cycle of $(V, \mu_{[a]})$ for every $a \in (0, 1]$, i. e. $r \wedge A \in \mathcal{I}_{\widetilde{G}}$. Hence $A = (r \wedge A)_{(0)} \in \mathcal{I}_{\widetilde{G}}(0)$. This implies that $\mathcal{I}_G \subseteq \mathcal{I}_{\widetilde{G}}(0)$. □

Theorem 4.10. Let $\widetilde{G} = (V, \mu)$ be a fuzzy graph. If its basic graph $G = (V, E)$ is a loopless graph without isolated vertices, and has at least three vertices, then $Con(M(\widetilde{G})) = 1$ if and only if $\widetilde{G}$ is a 2-connected fuzzy graph.

Proof. By Definition 4.8, $M(\tilde{G})$ is a closed G-V fuzzy matroid. Since G is loopless, has no isolated vertices, and $|V| \geq 3$, we have $|E| \geq 2$ (a necessary structural condition for non-trivial connectedness analysis).

Necessity: Suppose $Con(M(\tilde{G})) = 1$, then by Theorem 4.6, $(E, \mathcal{I}_{\tilde{G}}(0))$ is connected. By Lemma 4.9, $\mathcal{I}_{\tilde{G}}(0) = \mathcal{I}_G$ (the independent set family of G's cycle matroid $M(G)$), so $M(G)$ is connected. By the classic graph-matroid correspondence (Section 1), G is 2-connected, hence $\tilde{G}$ is 2-connected by Definition 4.7.

Sufficiency: Assume $\tilde{G}$ is 2-connected, then G is 2-connected by Definition 4.7. By the classic graph-matroid correspondence (Section 1), $M(G)$ is connected. By Lemma 4.9, $\mathcal{I}_{\tilde{G}}(0) = \mathcal{I}_G$, so $(E, \mathcal{I}_{\tilde{G}}(0))$ is connected. Since $M(\tilde{G})$ is closed and $|E| \geq 2$, Theorem 4.6 implies $Con(M(\tilde{G})) = 1$.

Thus the equivalence holds. □

5. ASSOCIATION WITH CONNECTED DEGREE OF FUZZIFYING MATROIDS

Based on the systematic study of the connected degree of G-V fuzzy matroids in the previous section, this section focuses on the connectedness theory of another important model — the fuzzifying matroids. The connected degree of fuzzifying matroids, along with concrete examples (e. g., fuzzifying uniform matroids, paving matroids, and cycle matroids induced by fuzzy graphs) and their calculation methods via the separator degree method, has already been thoroughly discussed and verified in existing literature [35]. This provides solid concrete support for revealing the inherent connection between the connected degree definitions of G-V fuzzy matroids and fuzzifying matroids through structural transformations and property correspondences in this section.

Definition 5.1. (Shi [26], Shi [27]) A fuzzifying matroid is a pair $(E, \mathcal{I})$, where $\mathcal{I}$: $2^E \to [0,1]$ is a mapping that satisfies three independent sets axioms: $\forall A, B \in 2^E$,

(FI1) $\mathcal{I}(\emptyset) = 1$;

(FI2) $A \supseteq B \Rightarrow \mathcal{I}(A) \leq \mathcal{I}(B)$;

(FI3) $|A| < |B| \Rightarrow \bigvee_{e \in B-A} \mathcal{I}(A \cup \{e\}) \geq \mathcal{I}(A) \wedge \mathcal{I}(B)$.

Theorem 5.2. (Shi [27], Yao and Shi [36]) If $\mathcal{I} : 2^E \to [0,1]$ is a mapping, then

(1) $(E, \mathcal{I})$ is a fuzzifying matroid if and only if $(E, \mathcal{I}_{[a]})$ is a matroid for each $a \in (0,1]$ if and only if $(E, \mathcal{I}_{(a)})$ is a matroid for each $a \in [0,1)$.

(2) For any fuzzifying matroid $(E, \mathcal{I})$, there exists a finite sequence $0 = a_0 < a_1 < \cdots < a_n = 1$ such that $\mathcal{I}_{[a]}$ and $\mathcal{I}_{(a)}$ are constant within intervals $(a_i, a_{i+1}]$ and $[a_i, a_{i+1})$, respectively, with $\mathcal{I}_{[a]}$ strictly decreasing and $\mathcal{I}_{(a)}$ strictly decreasing across adjacent intervals.

Definition 5.3. (Xin et al. [35], Yao and Shi [36]) For a fuzzifying matroid $M = (E, \mathcal{I})$. Its circuit-map $\mathcal{C}_{\mathcal{I}} : 2^E \to [0,1]$ is defined by $\forall C \in 2^E$, $\mathcal{C}_{\mathcal{I}}(C) = 1 - \bigwedge\{a \in [0,1) | \ C \in \mathcal{C}_{\mathcal{I}_{(a)}}\}$ and satisfies two circuit-map axioms: (FC1) $\forall a \in [0,1), \mathrm{Min}(\mathcal{C}_{\mathcal{I}})_{[a]} = \mathcal{C}_{\mathcal{I}_{(1-a)}}$; (FC2) $\forall C \in 2^E$, if $\mathcal{C}(C) = a \neq 0$, then $C \in \mathrm{Min}(\mathcal{C}_{\mathcal{I}})_{[a]}$. The separator degree of a subset $A \subseteq E$ is defined as: $Sep(A) = \bigwedge\{1 - \mathcal{C}_{\mathcal{I}}(C) \mid C \subseteq E, C \nsubseteq A \text{ and } C \nsubseteq E - A\}$, and $Con(M) = \bigwedge\{1 - Sep(A) \mid A \in 2^E - \{\emptyset, E\}\}$ is called the connected degree of M.

As shown in [35, Theorem 6 (2)], for a fuzzifying matroid $M = (E, \mathcal{I})$, $A \in 2^E$ and $a \in [0, 1)$, $Sep_M(A) = a$ implies A is not a separator of $(E, \mathcal{I}_{(a)})$.

The following lemma provides an equivalent description of the above separator degree.

Lemma 5.4. Let $M = (E, \mathcal{I})$ be a fuzzifying matroids and $A \in 2^E$, then $Sep(A) = \bigwedge\{a \in (0, 1] : A \text{ is not a separator in } (E, \mathcal{I}_{[a]})\}$.

P r o o f. Let $0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$ be the fundamental sequence of $(E, \mathcal{I})$. In the one hand, suppose that $Sep(A) = b \in [a_{i-1}, a_i)$, then A is not a separator of $(E, \mathcal{I}_{(b)})$. Since $0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$ be the fundamental sequence of $(E, \mathcal{I})$, for any $a \in (a_{i-1}, a_i])$ we have $\mathcal{I}_{(b)} = \mathcal{I}_{[a]}$. Hence $\bigwedge\{a \in (0, 1] : A$ is not a separator of $(E, \mathcal{I}_{[a]})\} \leq a_{i-1} \leq b \leq Sep(A)$. In the other hand, assume that $\bigwedge\{a \in (0, 1] : A$ is not a separator of $(E, \mathcal{I}_{[a]})\} = b$, then there exists $i \in \{1, 2, \cdots, n\}$ such that $b = a_{i-1}$ and A is not a separator of $(E, \mathcal{I}_{[a]})$ $(\forall a \in (a_{i-1}, a_i])$, thus A is not a separator of $(E, \mathcal{I}_{(a_{i-1})})$. By Theorem 3.3 and Definition 3.4, there exists $C \in \mathcal{C}_{\mathcal{I}_{(a_{i-1})}}$, such that $C \not\subseteq A$ and $C \not\subseteq E - A$, hence $\mathcal{C}_{\mathcal{I}}(C) = 1 - \bigwedge\{a \in [0, 1) \mid C \in \mathcal{C}_{\mathcal{I}_{(a)}}\} \geq 1 - a_{i-1}$, thus $Sep(A) = \bigwedge\{1 - \mathcal{C}_{\mathcal{I}}(C) \mid C \in 2^E \text{ with } C \not\subseteq A, C \not\subseteq E - A\} \leq a_{i-1} = b = \bigwedge\{a \in (0, 1] : A$ is not a separator of $(E, \mathcal{I}_{[a]})\}$. In summary, $Sep(A) = \bigwedge\{a \in (0, 1] : A$ is not a separator of $(E, \mathcal{I}_{[a]})\}$. □

In the following two theorems, both G-V fuzzy matroids and fuzzifying matroids are involved, so subscripts are added to distinguish the separator degrees and connected degrees.

Theorem 5.5. Let $M = (E, \mathcal{I})$ be a fuzzifying matroid. Define $\tau(\mathcal{I}) = \{A \in [0, 1]^E : \forall a \in (0, 1], A_{[a]} \in \mathcal{I}_{[a]}\}$, then

(1)([32]) $\tau(M) = (E, \tau(\mathcal{I}))$ is a closed G-V matroid and $\tau(\mathcal{I})[a] = \mathcal{I}_{[a]}$ $(\forall a \in (0, 1])$;

(2) $Sep_{\tau(M)}(\chi_A) = Sep_M(A)$;

(3) $Con(\tau(M)) = Con(M)$.

P r o o f. By Definition 4.1, (1) and Lemma 5.4, $Sep_{\tau(M)}(\chi_A) = \bigwedge\{a \in (0, 1] : A$ is not a separator of $(E, \tau(\mathcal{I})[a])\} = \bigwedge\{a \in (0, 1] : A$ is not a separator of $(E, \mathcal{I}_{[a]})\} = Sep_M(A)$. This implies (2) holds.

By Definition 4.4, (2) and Definition 5.3, $Con(\tau(M)) = \bigwedge\{1 - Sep_{\tau(M)}(\chi_A) : A \neq \emptyset, A \neq E, A \in 2^E\} = \bigwedge\{1 - Sep_M(A) : A \in 2^E - \{\emptyset, E\}\} = Con(M)$. Hence (3) holds. □

Theorem 5.6. Let $M = (E, \mathcal{I})$ be a closed G-V matroids. Define $\sigma(\mathcal{I})(A) = \bigvee\{a \in (0, 1] : A \in \mathcal{I}[a]\}$, then

(1)([32]) $\sigma(M) = (E, \sigma(\mathcal{I}))$ is a fuzzifying matroid and $\sigma(\mathcal{I})_{[a]} = \mathcal{I}[a]$ $(\forall a \in (0, 1])$;

(2) $Sep_{\sigma(M)}(A) = Sep_M(\chi_A)$;

(3) $Con(\sigma(M)) = Con(M)$.

Proof. By Lemma 5.4, (1) and Definition 4.1, $Sep_{\sigma(M)}(A) = \bigwedge\{a \in (0,1] : A$ is not a separaror of $(E, \sigma(\mathcal{I})_{[a]})\} = \bigwedge\{a \in (0,1] : A$ is not a separaror of $(E, \mathcal{I}[a])\} = Sep_M(\chi_A)$. This implies (2) holds.

By Definition 5.3, (2) and Definition 4.4, $Con(\sigma(M)) = \bigwedge\{1 - Sep_{\sigma(M)}(A) : A \in 2^E - \{\emptyset, E\}\} = \bigwedge\{1 - Sep_M(\chi_A) : A \neq \emptyset, A \neq E, A \in 2^E\} = Con(M)$. Hence (3) holds. $\square$

These theorems establish a bidirectional transformation between G-V fuzzy matroids and fuzzifying matroids, demonstrating that their connected degree definitions are essentially consistent. This unification highlights the generality of the connectedness analysis framework across different fuzzy matroid models.

6. CONCLUSION

In [15], the connectedness of G-V fuzzy matroids was introduced via an equivalence relation on all fuzzy points of the ground set. However, such connected G-V fuzzy matroids fail to preserve some basic properties of classical connected matroids, leading to focused research on the connectedness of a class of refined G-V fuzzy matroids.

This study proposes a new definition of connectedness for G-V fuzzy matroids, serving as a fuzzy analog of matroid connectedness. It is shown that for refined G-V fuzzy matroids, this definition aligns with that in [15], but differs from the concept of connectedness in fuzzy graphs. To bridge this gap and extend the framework to fuzzy graphs, the connected degree is introduced, with key properties revealed:

Quantitative Discriminability: A G-V fuzzy matroid M is disconnected if and only if its connected degree $Con(M) = 0$, and connected otherwise ($Con(M) \neq 0$), providing a numerical characterization of connectedness.

Fuzzy Graph Correspondence: For the fuzzy cycle matroid $M(\tilde{G})$ of a fuzzy graph $\tilde{G}$, $Con(M(\tilde{G})) = 1$ if and only if $\tilde{G}$ is 2-connected, establishing a direct link between fuzzy matroid connectedness and fuzzy graph 2-connecedness.

Model Unification: Through bi-directional transformations between closed G-V fuzzy matroids and fuzzifying matroids, the consistency of connected degree definitions across these models is demonstrated, highlighting their inherent unity in connectedness analysis.

This research enriches the theoretical system of fuzzy matroids via rigorous mathematical proofs, offering theoretical support for their application in uncertainty modeling. It also paves the way for interdisciplinary research between fuzzy graph theory and combinatorial optimization. Future research can prioritize extending the connected degree concept proposed in this paper to intuitionistic fuzzy matroids, m-polar fuzzy matroids, and Pythagorean fuzzy matroids [1, 17, 19, 20, 21]. By integrating the foundational framework of intuitionistic fuzzy matroids, the generalized results of G-V intuitionistic fuzzy matroids, and the characterization conclusions of their base properties, a systematic connectedness theory for intuitionistic fuzzy matroids can be established; exploring the adaptability of this connected degree framework to the multi-dimensional polarity characteristics of m-polar fuzzy matroids and the relaxed membership constraints of Pythagorean fuzzy matroids can further expand the application scope of this research. Additionally, the connected degree can be extended to antimatroids (including shelling types) [3] and greedoids [12], leveraging their structural compatibility with matroids. For other fuzzy set structures such as Picture Fuzzy Sets, HyperFuzzy Sets, and Plithogenic Sets, their integration with matroid theory remains in the initial stage. Nevertheless, the research on SVN-matroids derived from single-valued neutrosophic fuzzy graphs [2] has provided a feasible extension paradigm. Future studies can refer to the axiomatic methods of existing fuzzy matroids to gradually extend the connected degree concept to new models, thereby supporting the solution of optimization problems in uncertain networks. Fuzzy graphic binary matroid theory has been applied to power grid communication network optimization [14], which provides a practical reference for the engineering application of our connected degree theory. It should be noted that this paper only focuses on the theoretical construction of the connected degree for classical G-V fuzzy matroids, and its practical application in complex engineering optimization scenarios still needs to be further verified by case studies. This limitation also points out a clear direction for the in-depth development of subsequent research.

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