Jan Vilém Pexider (1874–1914)

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PEXIDER'S CONTRIBUTIONS TO INSURANCE MATHEMATICS

Tomáš Cipra

Jan Vilém Pexider is also the author of three works devoted to insurance mathematics ([P14], [P16], [P17]); their origin was obviously related to Pexider's private lectureship in Bern.

The insurance in Switzerland and Germany at the end of the nineteenth century and at the beginning of the twentieth century developed both on the commercial level and on the social level. Insurance mathematics was lectured at many universities. It is necessary to remind that also Prague has such a tradition: The first course of insurance mathematics dates to academic year 1895/96 at the Czech Technical University (the lecturer was Professor Matyáš Lerch, a world-known mathematician that worked simultaneously in the Country Insurance Fond[1]). It is doubtless that the stay in Switzerland with many banks and insurance companies evoked in Pexider the interest in mathematics of life insurance, which was applied here on a high level in commercial insurance institutions and pension funds. Moreover, one can see that these works of Pexider, formally exact and with a detailed analysis of relationships among insurance variables, are influenced by the Swiss and German school.

The exactness, formally correct and nearly "encyclopedic" description of all relationships that are in the given context possible among considered insurance variables are the main eligibilities of Pexider's works, since they are not distinct in any way from the usual average of that period and often only formalize known results. Although there are no revolutionary ideas pushing the actuarial science forward, they could have been useful in Pexider's period, because they contain formally correct instructions for many situations important for practical actuarial calculations. Even nowadays, a reader awakes to various connections when studying these works. In this way, Pexider's contributions to insurance mathematics were also accepted by specialists community in the times of their origin; for example, Pexider's contemporary Dr. Oster from Mannheim wrote in his review of the work [P17]: *One must evaluate more the exact way of presentation than new results in this work.* Moreover, Dr. Oster criticized that Pexider's symbols differ sometimes from international conventions. In this point, the reviewer is probably right, since the Second International Actuarial Congress in London (1898) has accepted some rules for actuarial symbols. These rules are in principle respected by Pexider, but sometimes he uses "exotic" symbols of the type ${}_{n\sim T}a_i(x)$ or $a^{t[}(x)$.

The literal translation of the title of the work [P14] is *Fundamental Relationships among Insurance Premiums in Insurance of Life, of Invalidity and for the Case of Death.* However, if we use the contemporary terminology and judge the content of the paper, we can see that the paper deals with relationships among values of claims (exactly among present values of claims) for various types of pension and capital life insurance (the present value of the claim corresponds to the one-off net premium). More precisely, Pexider reacted in this paper to relations of the type "the value of life annuity is equal to the value of claims of an active insured for active and invalid pension," which have been used as approximative ones without verifying a size of error of such approximations.

For example, Pexider wrote the previous relation as

$$a_i(x) = a(x) - a_a(x), \tag{1}$$

where

$$a(x) = \sum_x^{\omega} D(\cdot)/D(x)$$

is the value of claims of a man at the age of x for a unit life annuity (with $D(x) = l(x)v^x$),

$$a_a(x) = \sum_x^{\omega} D_a(\cdot)/D_a(x)$$

is the value of claims of an active man at the age of x for a unit active pension (with $Da(x) = la(x)v^x$),

$$a_i(x) = \sum_x^{\omega} D_{ai}(\cdot + 1) \cdot a_u(x+1)/D_a(x)$$

is the value of claims of an active man at the age of x for a unit invalid pension (with $D_{ai}(x)$ being the discounted number of men changing from active to invalid state at the age of x) and

$$a_u(x) = \sum_x^{\omega} D_u(\cdot)/D_u(x)$$

is the value of claims of an invalid man at the age of x for a unit invalid pension (with $D_u(x) = l_u(x)v^x$; the letter u is due to the German word *Unfähigkeit*).

In the preceding formulas, one can recognize the contemporary symbols for decrement orders l and commutation numbers D (see e.g. [Bo]).

Using nontrivial formal treatment, Pexider replaced the approximation (1) by the exact formula

$$a_i(x) = a(x) - a_a(x) + \frac{l_i(x)}{l_a(x)}(a(x) - a_u(x)), \tag{1'}$$

which he also justified in a heuristic way.

Pexider applied a similar approach in the case of approximation

$$A_i(x) = A(x) - A_a(x), \tag{2}$$

where

$$A(x) = \sum_{x}^{\omega} C(\cdot)/D(x)$$

is the value of claims of a man at the age of x for a unit sum insured in the insurance for the case of death (with $C(x) = d(x)v^x$; remind that nowadays, this commutation number is commonly used as $C(x) = d(x)v^{x+1}$),

$$A_a(x) = \sum_{x}^{\omega} C_a(\cdot)/D_a(x)$$

is the value of claims of an active man at the age of x for a unit sum insured in the insurance for the case of death (with $C_a(x) = da(x)v^x$),

$$A_i(x) = \sum_{x}^{\omega} D_{ai}(\cdot + 1) \cdot A_u(x+1)/D_a(x)$$

is the value of claims of an active man at the age of x for a unit sum insured in the insurance for the case of death payed only with preceding invalidity (where $D_{ai}(x)$ has the same meaning as above) and

$$A_u(x) = \sum_{x}^{\omega} C_i(\cdot)/D_i(x)$$

is the value of claims of an invalid man at the age of x for a unit sum insured in the insurance for the case of death (with $C_i(x) = d_i(x)v^x$).

Using analogical formal treatment as in the case of pension insurance, Pexider obtained the following exact formula for the capital insurance:

$$A_i(x) = A(x) - A_a(x) + \frac{l_i(x)}{l_a(x)}(A(x) - A_u(x)). \qquad (2')$$

The applicability of both approximations (1) and (2) with respect to the exact formulas (1′) and (2′) has been evaluated by Pexider numerically by means of two life tables that were used in that time in some pension funds in Germany and Switzerland. Pexider constructed also graphs for this purpose. One must appreciate the numerical effort that was undoubtedly necessary for such calculations in that time. Therefore Pexider belongs to the actuaries who have been capable, using only mechanical calculators, to perform calculations with a high level of accuracy that cannot be labeled even nowadays as simple ones.

Pexider's final conclusion was correct and is fully valid in the sense of conclusions that are accepted nowadays: The approximations (1) and (2) cannot be recommended for practical applications, since their error is high in comparison with the financial quantities appearing here. Pexider presented an example in [P14]: He took advantage of the life table of the Swiss pension fund of railway employees (here is e.g. $l_{21} = 99072$ and $l_{22} = 98188$, so that the probability of death for a man at the age of 21 in this age is $q_{21} = 8.923\,‰$, which is much more than $q_{21} = 0.958\,‰$ for men in the Czech Republic in 2005). According to Pexider's results, if the pension fund uses the approximation (1) instead of the exact formula (1′) for calculating the invalid pension of 1 000 CHF, then its loss will be 198.60 CHF for a man at the age of 50 (the one-off premium would be less for 5.4 %) and 438.10 CHF for a man at the age of 60 (the one-off premium would be less for 11.5 %) each year.

As late as during the review process, Pexider learned that was not the only mathematician dealing with this problem. Ch. Moser (professor at the University of Bern and simultaneously the director of the pension fund (Direktor des eidgenössischen Versicherungsamtes in Bern) gave him a discreet notice that he had derived a more general formula than the one in (1′) in his work *Untersuchungen und Materialien zur Beurteilung der sechs Entwürfe für eine Hülfskasse des Personals der eidgenössischen Verwaltungen* (note the Swiss form of German). Moreover, Moser's formula takes into account also the possibility of transfer from the invalid state back to the active state. Pexider's reaction was an apology in the addendum to this paper.

In the work [P16] written still during his stay in Bern, Pexider dealt with another problem that was important for contemporary applied insurance, namely the interest rate payable m times a year.

In practice, the insurance premium and the pension payments are usually paid with a higher frequency than once a year, e.g. quarterly with $m = 4$, or monthly with $m = 12$.

If $a(x)$ denotes again the value of claims of a man at the age of x for a unit life annuity paid yearly (as a unity always at the beginning of the year), then the corresponding life annuity $a^{(m)}(x)$ is paid m times a year (always as $\frac{1}{m}$ at the beginning of the mth part of the year).

In Pexider's times, the exact relationship between $a^{(m)}(x)$ and $a(x)$, namely

$$a^{(m)}(x) = \alpha \cdot a(x) + \beta, \tag{3}$$

where

$$\alpha = \frac{1}{m^2} \sum_{k=0}^{m-1} (m + ik) \cdot v^{\frac{k}{m}}, \tag{4}$$

$$\beta = -\frac{1}{vm^2} \sum_{k=0}^{m-1} k \cdot v^{\frac{k}{m}}, \tag{5}$$

(i is the interest rate p.a. and $v = \frac{1}{(1+i)}$ is the corresponding discount factor) was already known.

These formulas were too complicated for practical applications in banks and insurance companies. On the other hand, the usual Geer's approximation

$$a^{(m)}(x) \sim a(x) - \frac{m-1}{2m} \tag{6}$$

was too crude according to the opinion of many actuaries.

In his work [P16], Pexider suggested an improvement of the approximation (6) that was acceptable even for practical calculations. Using methods which are nowadays considered as elementary (Taylor's expansion etc.), he obtained the approximation

$$\alpha \sim 1 + \frac{i^2}{12} \cdot \left(1 - \frac{1}{m^2}\right) \tag{4'}$$

or more precisely

$$\alpha \sim 1 + \frac{i^2}{12} \cdot (1 - i) \cdot \left(1 - \frac{1}{m^2}\right) + \frac{i^3}{6m^3}, \tag{4''}$$

and also

$$\beta \sim -\frac{m-1}{2m} \cdot \left(1 + \frac{i}{3} \cdot \frac{m+1}{m}\right) \tag{5'}$$

or more precisely

$$\beta \sim -\frac{m-1}{2m} \cdot \left(1 + \frac{i}{3} \cdot \left(1 - \frac{i}{4}\right) \cdot \frac{m+1}{m}\right). \tag{5''}$$

Moreover, Pexider also derived the limit relationships for $m \to \infty$ corresponding to the so-called continuous interest rates. For example, the limit forms of the exact formulas (4) and (5) are

$$\alpha(\infty) \sim v \cdot \left(\frac{i}{j}\right)^2, \qquad \beta(\infty) = -\frac{i-j}{j^2},$$

where $j = \log(1+i)$ is the so-called interest intensity.

At the end of the paper, Pexider gave tables of numerical values of the coefficients α and β for various approximations and various interest rates i p.a. For example, if $i = 4\,\%$, then the approximations (4″) and (5″) give the following values of α and β:

m	α	β
2	1.0000973	−0.254950
4	1.0001202	−0.381187
6	1.0001249	−0.423083
12	1.0001271	−0.464887
24	1.0001278	−0.485754
52	1.0001279	−0.496983
∞	1.0001280	−0.506600

With slight exceptions, all tabled values are correct when compared with high-precision computer calculations. At the time of their publication, the results of this paper were undoubtedly very important for practice.

The work [P17] is a very extensive paper (33 pages). In accordance with Dr. Oster (see above), one can summarize that the work contains a very extensive survey of formulas from invalid insurance (including their derivation) structured in a logical way. Although these relationships were already known in Pexider's time, they had not been available in such a compact form. One must stress that despite their complexity, these formulas are correct from contemporary point of view.

Pexider's contributions to insurance mathematics (according to the works [P14], [P16] and [P17]) can be summarized thus:

– Pexider's works do not represent any break-through that would push the actuarial science more notably forward; they stay rather at the standard level of the works published in his period.
– Pexider possessed a good knowledge of insurance mathematics. His texts are elaborate in all details, they have a logic structure and are almost without errors; moreover, he was apparently a skilled arithmetician.
– Pexider must have known the actuarial problems also from the practical point of view, since he responded to the needs of practice; on contrary and without exaggerating, his works must have been useful for practice.

Notes

[1]) Matyáš Lerch wrote the paper *Weak spots of insurance theory* (in Czech) published in *Přehled* 6 (1907/08), pp. 569-570 (no. 34 from 15th May 1908) and pp. 588-590 (no. 35 from 22nd May 1908). Josef Beneš had a critical comment to it in a paper with the same heading (see *Přehled* 6 (1907/08), pp. 621-622, no. 37). Matyáš Lerch published later another paper *Weak spots of insurance theory II* (see *Přehled* 6 (1907/08), p. 692, no. 41 from 3rd July 1908 and pp. 861–862, no. 51 from 11th September 1908), where he faces to Beneš critisism.

References

[Bo] N. L. Bowers, *Actuarial Mathematics*, Society of Actuaries, Itasca, Illinois, 1986.