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“The crisis of intuition” - Austrian-Hungarian contributions in the quest of defining the mathematical term “Dimension” from the 1850’s to the 1920’s

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"THE CRISIS OF INTUITION"
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AUSTRIAN-HUNGARIAN CONTRIBUTIONS IN THE QUEST OF DEFINING THE MATHEMATICAL TERM "DIMENSION" FROM THE 1850's TO THE 1920's

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Abstract: When Hans Hahn gave his talk *The Crisis of Intuition* on November 30th, 1932 to a broad Viennese audience, the quest to define the mathematical term "dimension" had already been solved by Menger and Urysohn. This paper will give a brief outline of the mathematical approaches towards "dimension". As a start, we will look at Georg Cantor's work showing that a square can be mapped to a line segment with one-to-one correspondence. Whereas Cantor's result had merely put the previously unquestioned concept of dimension for the first time in doubt, Giuseppe Peano's space-filling curves gave it a severe blow. Around the turn of the century, Poincaré proposed a recursive definition of dimension, which was soon taken up by the Hungarian Frigyes Riesz. However, long before Poincaré, Bernhard Bolzano dealt with the problem in the first half of the 19th century with a useful dimension concept, formulated in the best way possible in his times.

1 Introduction

1.1 "The Crisis of Intuition"

I have adapted the title from a talk given by Hans Hahn (1879–1934), an Austrian mathematician who was also one of the leading figures in the Vienna Circle.[1] The lecture by Hahn was part of series entitled *Crisis and Reconstruction in the Exact Sciences*[2] which was organized by Karl Menger, a former student of Hahn, who by the 1930's was himself already a professor of geometry at the University of Vienna.[3]

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[1] For biographical notes on Hans Hahn see Karl Mayrhofer: *Nachruf auf Hans Hahn*, Monatshefte für Mathematik und Physik 41(1934), 221–238, and Karl Sigmund: *A Philosopher's Mathematician – Hans Hahn and the Vienna Circle*, The Mathematical Intelligencer 17(1995), 16–29. For Hahn's role in the Vienna Circle see: Friedrich Stadler: *Studien Zum Wiener Kreis. Ursprung, Entwicklung und Wertung des Logischen Empirismus im Kontext*, Frankfurt, 1997.

[2] Hans Hahn: *Die Krise der Anschauung*, In *Krise und Neuaufbau in den Exakten Wissenschaften. Fünf Wiener Vorträge*, Wien, 1933, 41–64.

[3] The Author is currently writing an intellectual biography of Karl Menger. For biographical notes on Karl Menger see Seymour Kass: *Karl Menger*, Notices of the AMS 43 (1996), no. 5, 558–561, Karl Sigmund: *Karl Menger and Vienna's Golden Autumn*, In B. Schweizer K. Sigmund, A. Sklar, P. Gruber, E. Hlawka, L. Reich and L. Schmetterer (eds.), *Karl Menger. Selecta Mathematica*, Springer, Wien, 2002, 7–21. As well as Menger's posthumously published Memoirs: Karl Menger: *Reminiscences of the Vienna Circle and the Mathematical Colloquium*, In Brian McGuiness Louise Golland, Abe Sklar (eds.), *Vienna Circle Collection*, vol. 20, Dortrecht, 1994.

In the inter-war period, due to the financial crisis, the highly talented scientific youth had difficulties getting employed at university. Thus, they had to go into business rather than staying in the academy. Therefore, Menger tried to raise funds for the younger scientific generation within this series of five lectures in the late fall of 1932.[4] Although the tickets for the talks within Menger's series were not cheap at all, they soon ran out of them. For the price of an opera ticket the audience could get an inside view of the problems which scientists from various fields are facing currently.[5] Even though Hahn's talk could be seen as an instance of "popular science" according to Ludwik Fleck's theory,[6] his presentation is truly more than just popularizing mathematics to a broader audience. Moreover, Hahn gave an introduction into mathematical thinking and reasoning. This thought can be underlined by the fact that Friedrich Waismann,[7] another Vienna Circle member, one-to-one adapted Hahn's talk in his book *Einführung in das mathematische Denken* [Introduction to Mathematical Thinking].[8]

1.2 The Quest of defining the mathematical term "Dimension" under the light of Lakatos' theory

The development of dimension theory through the period covered in this article could be seen as a case study of Lakatos' theory.[9] The Hungarian Philosopher of Science Imre Lakatos (1922–1974)[10] wrote after his emigration to London (via Austria) his dissertation, *Proofs and refutations – The logic of mathematical discovery*[11] at the London School of Economics. Until his death he kept working on his theory, in which he tried to construct a theory of how mathematical knowledge increases and posthumously his expanded thoughts have been published.[12]

Simply speaking, for Lakatos mathematical progress is a result of a permanent quest to improve existing conjectures and theorems through proofs and refutations. At the start, a mathematician formulates a raw version of a theorem or states simply a conjecture, which he later tries to prove. While proving this conjecture several counterexamples may arise. These counterexamples force a revision of the previous statements. By then the

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[4] Karl Menger: *Selected Papers in Logic and Foundations, Didactics, Economics*, Dordrecht, 1979, 17.
[5] Ibid. 17.
[6] Ludwik Fleck: *Entstehung Und Entwicklung einer Wissenschaftlichen Tatsache. Einführung in die Lehre vom Denkstil und Denkkollektiv*, In Lothar Schäfer und Thomas Schnelle (eds.), *Suhrkamp Taschenbuch – Wissenschaft*, Frankfurt am Main, 1980, vol. 312.
[7] For biographical notes on Waismann and his role in the Vienna Circle see: F. Stadler: *Studien zum Wiener Kreis. Ursprung, Entwicklung und Wertung des Logischen Empirismus im Kontext*, Frankfurt, 1997.
[8] Friedrich Waismann: *Einführung in das Mathematische Denken. Die Begriffsbildung der Modernen Mathematik. Mit einem Vorwort von Karl Menger*, Wien, 1936.
[9] Dale M. Johnson already had pointed out the usefulness of Lakatos's reasoning towards the development of the invariance problem; see: Dale M. Johnson: *The Problem of the Invariance of Dimension in the Growth of Modern Topology. Part 1*, Archive for History of Exact Sciences 20(1979), 97–188, Dale M. Johnson: *The Problem of the Invariance of Dimension in the Growth of Modern Topology. Part 2*, Archive for History of Exact Sciences 25(1981).
The author currently works on an article wherein Karl Menger's first topological contributions should be seen under the light of Lakatos' theory outlined in Imre Lakatos: *Proofs and Refutations. The Logic of Mathematical Discovery*, Cambridge, 1976.
[10] For information on Lakatos' life and work see: George Kampis (ed.): *Appraising Lakatos. Mathematics, Methodology, and the Man* vol. 1, *Vienna Circle Institute Library*, Dordrecht, 2002.
[11] Lakatos: *Proofs and Refutations. The Logic of Mathematical Discovery*.
[12] Elie Zahar, John Worral: *Vorwort der Herausgeber*, In I. Lakatos: *Beweise Und Widerlegungen*, Detlef D. Spalt (ed.), Braunschweig: 1979, V.

new version of the previous statement will incorporate to some extent the counterexamples and the circle of proofs and refutations starts again.

1.3 "Dimension" between Mathematics and Philosophy

According to Hahn's talk I will first point out, why mathematicians in the late 19th century felt a crisis had appeared concerning the term "dimension". Although in this article most of the mathematical details are left out, you will get a brief insight of the quest of defining "dimension" and the thoughts that came from Austrian-Hungarian mathematicians towards the topic.

In accordance with Lakatos' theory, counterexamples play an important role in the story of dimension. Thus, we must look at how the mathematical community dealt with these examples that had shattered our intuitive idea of "dimension". Since a satisfying mathematical solution to the question could not be found in the pre-topological era, some of the most interesting attempts came from outside mathematics – namely from philosophy. One of the first contributions to the topic came from the Austrian-Hungarian mathematician Bernhard Bolzano's (1781–1848), who tried to hit the problem from a philosophical point of view. Whereas Henri Poincaré's (1854–1912) philosophical ideas towards dimension[13] were well known within the mathematical community, Bolzano's contribution was not public at that time. However, Bolzano had a clear understanding of the problem of "dimension" and how it might be attacked already almost 50 years before Poincaré.[14]

Around the turn of the Century topology, both point-set and algebraic, slowly developed and within this expansion of the field new solutions to the question of dimension had been made possible. One of them came from the Hungarian mathematician Frigyes Riesz (1880–1956), who's dimension theoretical contribution, likewise the one of Bolzano, had been widely unnoticed.[15]

Finally, we will look at the so called Menger-Urysohn or small inductive dimension definition, which brought an end to the quest of defining "dimension" and was a result of combining philosophical ideas with topological tools.

2 The Problem of "Dimension" and Austrian-Hungarian contributions

2.1 The Unquestioned use of "Dimension" from Euclid up to the Crisis in late 19th Century

Although everybody has an intuitive idea of what a "curve" is, around the turn of the century the mathematical community was in need for an exact definition, and consequently of "dimension". Several outstanding mathematicians shed more light on the

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[13] Poincaré's thoughts on dimension had been published in several articles in the French journal *Revue de Metaphysique et de Morale*. Later his articles had been reissued in the following two books: Henri Poincare: *Letzte Gedanken*, Leipzig, 1913, Henri Poincare: *Wissenschaft Und Hypothese*, Autorisierte Deutsche Ausgabe mit erläuternden Anmerkungen von F. und L. Lindemann, Zweite verbesserte Auflage ed., Leipzig, 1906.

[14] *Tony Crilly with the Assistance of Dale Johnson : The Emergence of Topological Dimension Theory*, In Ioan M. James (ed.): *History of Topology*, Amsterdam, 1999, 3.

[15] Laura Regina Rodrìguez Hernàndez obtained her PhD in mathematics with a dissertation on Frigyes Riesz' contributions to abstract spaces. While looking at Riesz' work on topology Rodrìguez Hernàndez showed the influences of the mathematical communities of Paris and Göttingen on Riesz: Laura Regina Rodrìguez Hernàndez: *Friedrich Riesz' Beiträge zur Herausbildung des modernen mathematischen Konzepts abstrakter Räume. Synthesen Intellektueller Kulturen in Ungarn, Frankreich und Deutschland*, Mainz, 2006.

topic, but could only construct directions from which the problem might be solved. The question of "dimension" can be already found in the writings of the ancient Greek philosophers. In the most influential mathematical textbook of all time, Euclid's elements, Euclid gave the very first distinction between objects of different dimension, like point, line, square and solid. According to Euclid, *a point is that which has no part, a line is breadth less length, and a surface is that which has length and breadth only (Book I). A solid is that which has length, breadth and depth (Book XI).*[16]

Following Euclid's footsteps a rudimentary and unquestioned idea of dimension based on quantity set place in the mathematical thinking. Thus there was no doubt that a square includes more points than a line and so on. These ancient thoughts influenced also the works of 19th century mathematicians Bernhard Riemann (1826–1866) and Hermann Helmholtz (1821–1894). Thus, their informal theory of continuous manifolds dimension was based on the number of coordinates that are necessary to locate a point in space, square or on a line; again quantity set the pattern for dimension.[17]

The important shift that brought an end to the connection of dimension and quantity came from Georg Cantor (1845–1918). Whereas Cantor was comparing two sets for the notion of equality, he discovered, through mappings and correspondence to the real numbers, cardinality. Since the combination of mappings and correspondence had been fruitful, he adopted them for dimensional matters.[18]

Thus he asked in letter to Dedekind, if a *surface (perhaps a square including its boundary) can be put into a one-to-one correspondence with a line (perhaps a straight line segment including its endpoints) so that to each point of the surface there corresponds a point of the line and conversely to each point of the line there corresponds a point of the surface?*[19]

By 1878 Cantor could show in his article *Ein Beitrag zur Mannigfaltigkeitslehre*[20] that a continuous square (or even a cube) could be one-to-one mapped onto a continuous line segment. Although one must admit that this mapping is discontinuous, Cantor had started connecting geometric ideas with set-theoretical views.[21]

At first the mathematical community didn't want to accept Cantor's counter-intuitive example. According to Lakatos' theory they were so to say "Monster banners", as Johnson had pointed out.[22] Thus, immediately after Cantor's publication five mathematicians tried to save the old theory of dimension, while proving that a one-to-one continuous mapping between different dimensional object was simply impossible [that is the Invariance Theorem]. However, these early proofs of the Invariance Theorem were

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[16] Cited after Dale M. Johnson: *The Emergence of Topological Dimension Theory,* 2.
[17] Ibid. 7.
[18] Dale M. Johnson: *The Problem of the Invariance of Dimension in the Growth of Modern Topology. Part 1*, 132.
[19] Cited after Ibid. 132.
[20] Georg Cantor: *Ein Beitrag zur Mannigfaltigkeitslehre*, Journal für die Reine und die Angewandte Mathematik 84(1878), 242–258.
[21] Dale M. Johnson:*The Emergence of Topological Dimension Theory*, 7.
[22] Dale M. Johnson: *The Problem of the Invariance of Dimension in the Growth of Modern Topology. Part 1,* 145.

weakened by the use of calculus and geometrical reasoning, which couldn't give proofs beyond the third dimension.[23]

Even though a general proof was still missing, by the 1880's the mathematical community believed that at least one of the five mathematicians had solved the question of the invariance of dimension and consequently had solved the old concept of dimension.[24]

But then by 1890 the Italian mathematician Guiseppe Peano (1858–1932) gave a severe blow to the concept while publishing an only four page long article, *Sur une courbe, qui remplit une aire plane*.[25] In his paper Peano had constructed in purely analytically way (without the use of any diagrams) a curve which covers all point of a unit square (space-filling curves).[26] Like Cantor's first paradox example Peano's curve lacks some important mathematical properties:

Firstly, the creating function of the space-filling curve is not injective, which means that some points of the unit square are covered several times by the function. Secondly, the Peano's curve is not derivable.[27]

In fall of the same year at the Annual Meeting of the German naturalists and medical doctors David Hilbert (1862–1943) presented a visualisation of a space-filling curve that to become a classic.[28]

With the works of Cantor, Peano and Hilbert the "crisis of intuition" and the quest of defining the mathematical term "dimension" had begun. By the turn of the century several outstanding mathematicians tried to rethink the concept of dimension, some of them will be covered in the next section.

2.2 Bernhard Bolzano (1781–1848) – "Paradoxes of the Infinite"

Besides his mathematical work, Bolzano is most famous for the Bolzano-Weierstrass theorem, he was also a free-thinker and strong bohemian nationalist who fought for social justice and equality for the Czech speaking people within the Habsburg Empire. Due to his political thoughts a publication ban was imposed to him. Thus, most of his writings became only public after his death.[29] In 1851, one of his former students, namely Fr. Prihonsky published for the first time Bolzano's *Paradoxes of the Infinite*, which have

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[23] An excellent analysis of some early efforts to prove the Invariance of Dimension can be found in Ibid. 146–162.

[24] Dale M. Johnson: *The Emergence of Topological Dimension Theory*, 9.

[25] Giuseppe Peano: *Sur une courbe, qui remplit une aire plane*, Mathematische Annalen 36(1890), 157–160.

[26] Further examples of space-filling curves can be found in Hans Sagan: *Space-Filling Curves*, New York, 1994.

[27] Dale M. Johnson: *The Problem of the Invariance of Dimension in the Growth of Modern Topology. Part 1*, 171. Additionally, the author likes to thank Peter Schmitt for his remarks on the Peano curve.

[28] The first diagramms of Hilbert's space-filling can be found in David Hilbert: *Über die stetige Abbildung einer Linie auf ein Flächenstück* (In: 63. Versammlung Der Deutschen Naturforscher Und Ärzte Zu Bremen 1890, ed. O. Lassar, Leipzig, 1891, 11–12) which have been later published (with only marginal changes in the text) in David Hilbert: *Über die stetige Abbildung einer Linie auf ein Flächenstück*, Mathematische Annalen 38(1891), 459–460, and David Hilbert: *Über die stetige Abbildung einer Linie auf ein Flächenstück*, Prace Matematyczno--Fizyczne 5(1894), 13–14.

[29] For biographical notes on Bolzano see e.g.: Eduard Winter: *Bernhard Bolzano, Ein Lebensbild*, In Eduard Winter (ed.): *Bernhard-Bolzano-Gesamtausgabe*, Stuttgart, 1969.

been reissued and added with mathematical notes by Hans Hahn in 1920.[30] Nearly 20 years before Georg Cantor had started his thoughts towards dimension, Bolzano realized that the unquestioned usage of "dimension" had to be precised.

In the 40th paragraph of his treatise entitled *Paradoxien des Raumes* [Paradox of the Space], Bolzano adapted again the classical idea of expansion. Whereas Riemann and Helmboltz had connected dimension number with the quantity of parameters necessary to locate a point in a geometric object, Bolzano went into another direction. In the following paragraph he extended his philosophical thoughts on dimension number via various forms of expansion and comparing them with their neighbouring objects:

Let's call a single expansion or line, if in an arbitrary small distance of all points of these expansion are one or arbitrary many neighbours located. But none of these neighbouring objects are an expansion themselves. Likewise, let's call a double expansion or square if in an arbitrary small distance of all points of the expansion have lines (of points) as their neighbouring objects. Finally, let's denote a triple expansion or solid if in an arbitrary small distance of all points of this expansion have squares as their neighbouring objects.[31]

In his mathematical remarks on Bolzano's thoughts Hahn stated, that *although Bolzano's approach towards dimension number cannot be seen as ultimate, it is very astonishing, because it shows that Bolzano's developed already precise definitions*.[32]

In comparison with the final solution opposed by Urysohn and Menger, Bolzano's early attempt included already the idea of "neighbourhood of points" and their "boundaries"; both notions that became key definitions in topology.

However, when Bolzano wrote his mathematical-philosophical paper on infinity, topological tools, that paved the way to the final solution by Menger/Urysohn, have not been invented yet.

2.3 Frigyes Riesz (1880–1956) – "Sur les ensembles discontinus"

Whereas Riesz also attacked the problem of dimension (likewise Poincaré) in a mathematical philosophical way in his longer article *Genesis des Raumbegriffes* [Genesis of the space concept][33] he wrote almost a year before a paper on the same topic.

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[30] Bernhard Bolzano: *Paradoxien des Unendlichen. Herausgegeben aus dem schriftlichen Nachlasse des Verfassers von Dr. Fr. Prihonsky. Mit Anmerkungen versehen von Hans Hahn*, Die Philosophische Bibliothek, vol. 99, Leipzig, 1920.

[31] Translation by the author himself – Original taken from Ibid. 80: *So sage ich* [Bolzano], *ein räumlich Ausgedehntes sei einfach ausgedehnt, oder eine Linie, wenn jeder Punkt für jede hinlänglich kleine Entfernung einen oder mehrere, keinesfalls aber so viele Nachbarn hat, dass deren Inbegriff für sich allein schon ein Ausgedehntes wäre; ich sage ferner, ein räumlich ausgedehntes sei doppelt ausgedehnt, oder eine Fläche, wenn jeder Punkt für jede hinlänglich kleine Entfernung eine ganze Linie von Punkten zu seinen Nachbarn hat; ich sage endlich, ein räumlich Ausgedehntes sei dreifach ausgedehnt oder ein Körper, wenn jeder Punkt für jede hinlänglich kleine Entfernung eine ganze Fläche voll Punkte zu seinen Nachbarn hat.*

[32] Translation by the author himself – Original taken from Ibid. 149 – *Die* [...] *gegebene Definition der Dimensionenzahl ist – wenn sie auch nicht als endgültig anerkannt werden kann – insofern sehr bemerkenswert, als sie zeigt, wie weit B. In seinen exakten Begriffsbildungen vorgeschritten war.*

[33] For a detailed analyis of *Genesis des Raumbegriffes* see: Hernàndez: *Friedrich Riesz' Beiträge zur Herausbildung des modernen mathematischen Konzepts abstrakter Räume. Synthesen Intellektueller Kulturen in Ungarn, Frankreich und Deutschland*, Mainz 2006.

In his article *Sur les ensembles discontinus*[34] Riesz applied topological tools towards the quest of dimension. After trying to prove a theorem by Zoretti,[35] he tried to build up a new concept of dimension in the second part of his paper. Whereas the definitions Riesz had made in the beginning of his paper are quite interesting from a modern point of view, his theory attempt must be seen as a dead end street. On the contrary, with respect to the Menger/Urysohn dimension theory, one must admit that Riesz' disconnected sets are 0-dimensional in the framework of Menger and Urysohn according to Johnson,[36] which underlines Riesz profound thinking on the topic.

Thus we focus only at the very first definitions, which Riesz used as a starting point of his theory: the definitions of connected and disconnected sets.

Riesz calls *a (bounded) set of points in the plane or Euclidean n-space (implicitly given the usual topology) 'connected' ('d'un seul tenant') if it cannot be separated into two (nonempty) subsets having neither points nor limit points in common.*[37] Once he had defined connectedness he could easily define *a (bounded) set that has no connected subset as a 'discontinuous set'.*[38]

Later Karl Menger (1902–1985) likewise Riesz, whose contributions were unknown to him, built up his first dimension theoretical thoughts on disconnected sets,[39] which should have been the starting point of a recursive dimension definition. However, Menger's approach via disconnected sets was already put into doubt by a counterexample published by the Polish mathematician Sierpinski.[40] As a result of Sierpinski's counterexample, Menger was forced to rethink his definition.

2.4 Menger/Urysohn – "Small Intuitive Dimension" (Early 1920's)

At the end of our glance at Riesz' approach towards dimension we already encountered the contribution of Menger. By the time when Menger and Urysohn had made their contributions to the topic the Austrian-Hungarian Empire did not exist anymore. Although Menger did not know Bolzano's work, [41] some "traces" of the thoughts of the Bohemian mathematician could be found in the Menger/Urysohn definition:

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[34] Frigyes Riesz: *Sur les ensembles discontinus*, Comptes Rendus 142(1906), 763–764.
[35] Dale M. Johnson: *The Problem of the Invariance of Dimension in the Growth of Modern Topology. Part 1,* 112.
[36] Ibid. 112.
[37] Cited after Ibid. 112.
[38] Cited after Ibid. 112; additionally see the review of Riesz paper: *Vivanti: Besprechung zu Frigyes Riesz "Sur les ensembles Discontinus"*, Comptes Rendus 141, Jahrbuch über die Fortschritte der Mathematik 36(1906), 103.
[39] Karl Menger: *Zur Dimensions- und Kurventheorie. Unveröffentlichte Aufsätze aus den Jahren 1921–1923*, Monatshefte für Mathematik und Physik 36(1929), 416.
[40] For the actual counterexample see: Waclaw Sierpinski: *Sur les ensembles connexes et non connexes*, Fundamenta Mathematicae 2(1921), 81–95. Menger himself gives some remarks on the issue in *Menger: Zur Dimensions- und Kurventheorie. Unveröffentlichte Aufsätze aus den Jahren 1921–1923*, 112–418.
[41] Whereas I don't know anything about Urysohn's knowlegde of Bolzano's theory, Menger seemed not to know anything about the dimensional thoughts of Bolzano. In Karl Menger: *My Memories of L. E. J. Brouwer*, In Karl Menger (ed.): *Selected Papers in Logic and Foundations, Didactics, Economics*, Dordrecht, 1979, 251 he states the following: *That Hahn failed to mention this fact* [Bolzano's remark about curves and dimension] *to me is particulary odd since he was about to publish a new edition of Bolzano's booklet.*

A set S of points of the space is at most n-dimensional if each point of S lies in arbitrarily small neighbourhoods whose boundaries have at most (n-1)-dimensional intersections with S. The set S is n-dimensional if it is at most n-dimensional but not at most (n-1)-dimensional. The empty set, called -1-dimensional, is the starting point of the recursive definition.[42]

Whereas Menger and Urysohn had found this definition independently and almost at the same time, after Urysohn's death a priority conflict between Menger and Brouwer arose. But that's another story to be told.

3 Conclusion

Although everybody has an intuitive idea of what a "curve" is, in the late 19th century the mathematical community was in need for an exact definition, and consequently of "dimension". While explaining the counter-intuitive examples of Cantor and Peano's space-filling curve I have tried to shed light on the problem the mathematical community was facing. Due to the inappropriate mathematical tools of that time various mathematical attempts towards the quest stranded. Thus, some mathematicians like Bolzano or Poincaré tried to hit the problem from a philosophical viewpoint. Even though Bolzano's contribution towards dimension theory was almost unknown to his colleagues at that time, he had already developed a profound thinking on the topic, which dealt with the use of "neighbourhood" of points and their "boundaries". With the rise of topology, mathematicians tried to treat the problem with topological methods. Among the very first who applied topological tools towards dimension theory was the Hungarian mathematician Frigyes Riesz. Although his concept of dimension was not promising, some of his results fit into the modern framework built up by Menger and Urysohn in the early 1920's.

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