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GLOBAL ATTRACTORS OF ONE–DIMENSIONAL PARABOLIC EQUATIONS: SIXTEEN EXAMPLES

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ABSTRACT. We list all global attractors of dissipative equations (1.1) involving up to nine hyperbolic equilibria.

1. Global attractors and permutations

Mathematicians can rarely resist the temptations of classification. For example consider one-dimensional parabolic equations

$$u_t = u_{xx} + f(x,u,u_x), \quad -1 < x < 1,\ u \in \mathbb{R}, \tag{1.1}$$

with Neumann boundary conditions $u_x = 0$ at $x = \pm 1$. Suppose the equation is dissipative so that there exists a compact global attractor $\mathcal{A}$ in the sense of [Hal88], [Lad91], [BV89]. Let $E \subseteq \mathcal{A}$ denote the set of equilibria. Let us assume that E consists of e distinct equilibria,

$$E = \{v_1, \ldots, v_e\}, \tag{1.2}$$

all hyperbolic. How many "different" attractors with e equilibria exist? We investigate this question for $e \leq 9$. For $e = 9$, we find 16 different examples. The pictures are in Section 3. All results are based on forthcoming joint work with Carlos Rocha, see [FR94].

For a dissipative setting of $u(t,\cdot) \in X$, let X denote the Sobolev space H^2 with Neumann boundary conditions and consider $f \in C^2$ such that

$$f(x,u,0)\cdot u < 0 \quad \text{and} \quad f_x(x,u,p)\cdot p + f_u(x,u,p)\cdot p^2 < 0 \tag{1.3}$$

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holds, uniformly for $|u| + |p|$ large. Due to a gradient-like structure [Zel68], [Mat78] the global attractor consists of equilibria and heteroclinic orbits between them:

$$\mathcal{A} = E \cup \bigcup_{v,w \in E} C(v,w). \tag{1.4}$$

Here $C(v,w)$ indicates the heteroclinic connecting orbits $u(t,\cdot)$ which are defined for all real times t and converge to v, w, respectively, as $t \to -\infty$, $t \to +\infty$. Note that $C(v,w)$ is the intersection of the unstable manifold $W^u(v)$ with the stable manifold $W^s(w)$. Suppose the equilibria E are known. By (1.4), we then have to understand the connecting sets $C(v,w)$ well enough in order to describe $\mathcal{A}$.

Describing E is an ODE problem. Describing $C(v,w)$ is a PDE problem which appeared stagnant after some initial progress by [CI74], [CS80]. These early results were restricted to the case of cubic nonlinearities $f = f(u)$. See also [Hen81] for a survey. It was only after Matano's insight into the importance of nodal properties of solutions, [Mat82], that significant further progress concerning the sets $C(v,w)$ was made, see [Hen85], [Ang86], [AF88], [BF88], [BF89], [FR91], [Roc92], [FR94] and the many references there. The basic observation is that

$$t \mapsto z\big(u_1(t,\cdot) - u_2(t,\cdot)\big) \tag{1.5}$$

is nonincreasing with t, for any two solutions u_1, u_2, if z denotes the number of strict sign changes of the corresponding x-profile. This observation essentially goes back to Sturm; for a recent account see [Ang88].

The nodal structure (1.5) is much finer than just monotonicity or PDE comparison principles. However, it only works in one space dimension and for scalar equations; see [FP90], [Pol94]. For monotone feedback time delay equations, a similar structure has been discovered and used; see, e.g., [MP88], [FMP89a], [Mis87]. A related line of research is directed at describing the possible ω-limit sets of single trajectories; see, e.g., [FMP89b], [Nad90], [CM89], [BPS92], [FS92], [Ter93].

Let us now return to (1.1) and, at first, its equilibrium solutions $v_1, \dots, v_e$. Let us order the equilibria v by increasing v-value at the left end point $x = -1$ of the x-interval,

$$v_1 < v_2 < \cdots < v_e, \quad \text{at} \quad x = -1. \tag{1.6}$$

At $x = +1$, these orders may have changed. Specifically, define the permutation $\pi \in S_e$ of e elements by

$$v_{\pi(1)} < v_{\pi(2)} < \cdots < v_{\pi(e)}, \quad \text{at} \quad x = +1. \tag{1.7}$$

We call π the *permutation associated to* f. The dynamic importance of π was first realized by [FR91].

To describe the dynamic significance of π, let $i(v)$ denote the unstable dimension (the Morse Index) of the equilibrium v, that is, $i(v) = \dim W^u(v)$. For nodal considerations, the zero numbers $z(v-w)$ will also be relevant. We can express these quantities explicitly, in terms of π. The results in 1.1–1.5 below are taken from [FR94].

PROPOSITION 1.1.

(i) *For any equilibrium v_m, the Morse index $i(v_m)$ is given by*

$$i(v_m) = \sum_{j=1}^{m-1} (-1)^{j+1} \operatorname{sign}\big(\pi^{-1}(j+1) - \pi^{-1}(j)\big)$$

(ii) *For any pair v_m, v_n of equilibria, $m < n$, the zero number $z(v_n - v_m)$ is given by*

$$z(v_n - v_m) = i(v_m) + \frac{1}{2}\big[(-1)^n \operatorname{sign}\big(\pi^{-1}(n) - \pi^{-1}(m)\big) - 1\big] + \sum_{j=m+1}^{n-1} (-1)^j \operatorname{sign}\big(\pi^{-1}(j) - \pi^{-1}(m)\big).$$

As usual, empty sums are zero.

Equipped with the permutation π, let us now return to the global attractor $\mathcal{A}$ and the connecting orbits $C(v,w)$. We say that v connects to w, $v \searrow w$, if $C(v,w)$ is nonempty. The existence of such connections, a PDE problem, is completely encoded in π, an ODE information.

THEOREM 1.2. *Given any two equilibria v, w, the permutation π determines, explicitly and constructively, whether or not v connects to w.*

To be a little more explicit and constructive, we need three more principles: *cascading, blocking,* and *liberalism.*

THEOREM (cascading) 1.3. *Let v, w denote any two equilibria such that $i(v) = i(w) + n > i(w)$. Then $v \searrow w$ if, and only if, there exists a sequence $e_0 = w, \ldots, e_n = v$ of equilibria such that $i(e_k) = i(w) + k$, and*

$$v = e_n \searrow e_{n-1} \searrow \cdots \searrow e_1 \searrow e_0 = w\,. \tag{1.8}$$

In other words, v connects to w iff there is a cascade of connecting orbits along which the Morse index successively drops by one.

Note that the "only if" part fails in a general variational context. Already the gradient flow of the height function on a 2–sphere is a counterexample.

The "if" part, a transitivity result, holds for general Morse–Smale systems; see [Oli92], [PS70]. The nodal property (1.5) forces the connection set $C(v,w)$ to be a manifold of dimension

$$\dim\ C(v,w) = i(v) - i(w)\,; \tag{1.9}$$

see [Hen85], [Ang86]. The sets $C(v,w)$ define a cell decomposition of the attractor $\mathcal{A}$. The one-dimensional steps in the cascade (1.8), together with the equilibria, define the 1-skeleton of the attractor $\mathcal{A}$. To prove Theorem 1.2, it therefore remains to describe the sets $C(v,w)$ in case $i(v) = i(w) + 1$. It was observed in [BF89] that $C(v,w)$ consists of (at most) a single trajectory, in that case. In certain cases it is easy to see that a connection does not exist, cf. [BF88], [BF89].

PROPOSITION (blocking) 1.4. *Suppose v, w are equilibria such that $i(v) = i(w) + 1$. Then v does not connect to w if at least one of the following two conditions is satisfied:*

a) $z(v - w) \neq i(w)$, *or*

b) there exists an equilibrium $\bar{w}$ such that $\bar{w}$ is between v and w, at $x = -1$, and

$$z(v - \bar{w}) = z(w - \bar{w})\,.$$

We call case a) a Morse blocking and case b) a zero number blocking.

The final ingredient to our description of $\mathcal{A}$ is a touch of liberalism: anything that is not explicitly blocked actually goes. Or, more formally

THEOREM (liberalism) 1.5. *Suppose v, w are equilibria such that $i(v) = i(w) + 1$. If the connection from v to w is not blocked, then v connects to w.*

Using results 1.1–1.5 we can now completely decide whether or not v connects to w, for any Morse indices, in terms of the permutation π. We can also attempt to enumerate attractors with a given number e of equilibria; see Section 2. In Section 3 we give a complete list, and the pictures, of the sixteen distinct cases which arise for $e = 9$.

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2. Constructing and counting attractors

Any classification is based on a notion of equivalence. So, when do we consider two attractors $\mathcal{A}_1$, $\mathcal{A}_2$ to be "the same"? For example, we could ask whether $\mathcal{A}_1$ and $\mathcal{A}_2$ are homeomorphic sets or not. This ignores the dynamics on $\mathcal{A}_1$, $\mathcal{A}_2$. Ever since [Mor34], questions of equivalence in the calculus of variations are based on the homotopy type or the homology of sublevel sets of the underlying (Ljapunov) functional. Although our results would allow for such an analysis, in terms of Conley index and connection matrices, we take a somewhat more radical approach here.

The strongest notion which we have in mind is *flow equivalence* of $\mathcal{A}_1$, $\mathcal{A}_2$: does there exist a homeomorphism

$$h:\ \mathcal{A}_1 \to \mathcal{A}_2 \tag{2.1}$$

which conjugates the (semi-) flows on $\mathcal{A}_1$ and $\mathcal{A}_2$? Let f_1, f_2 be the associated nonlinearities in (1.1). Then, for example, $\mathcal{A}_1$ is flow equivalent to $\mathcal{A}_2$ if

$$f_2(x,u,p) = -f_1(x,-u,-p)\,, \tag{2.2}$$

or also if

$$f_2(x,u,p) = f_1(-x,u,-p)\,, \tag{2.3}$$

and similarly if

$$f_2(x,u,p) = -f_1(-x,-u,p)\,. \tag{2.4}$$

Indeed the corresponding conjugating homeomorphisms h are linear involutions given by

$$\begin{aligned} \big(h(v)\big)(x) &= -v(x)\,; \\ \big(h(v)\big)(x) &= v(-x)\,; \\ \big(h(v)\big)(x) &= -v(-x)\,; \end{aligned} \tag{2.5}$$

in the respective cases. Denoting the permutations associated to f_1, f_2 by π_1, π_2 we see that

$$\begin{aligned} \pi_2 &= \tau\pi_1\tau^{-1}\,; \\ \pi_2 &= \pi_1^{-1}\,; \\ \pi_2 &= \tau\pi_1^{-1}\tau\,; \end{aligned} \tag{2.6}$$

in the respective cases (2.2)–(2.4). Here τ corresponds to a reflection of indices,

$$\tau = \begin{pmatrix} 1 & 2 & \dots & e-1 & e \\ e & e-1 & \dots & 2 & 1 \end{pmatrix} \tag{2.7}$$

or in short $\tau = \{e, e-1, \dots, 1\}$. In particular, we will see later that attractors with different permutations π can be flow equivalent.

As an aside we note here that f may possess symmetries which allow us to take $f_1 = f_2 = f$ in one (or all) of the equations (2.2)–(2.4). By (2.6), the associated permutation π inherits these symmetries. In particular, suppose $f = f(u)$. Then $\pi = \pi^{-1}$, by (2.3) and (2.6). Hence π can only consist of cycles of length at most two.

We conjecture that $\mathcal{A}_1$ is flow equivalent to $\mathcal{A}_2$ if $\pi_1 = \pi_2$. Since we cannot prove this conjecture, we proceed with a weakened notion of equivalence. We call $\mathcal{A}_1, \mathcal{A}_2$ *connection equivalent* if there exists a bijection

$$\sigma\colon\ E_1 \to E_2$$

between the sets of equilibria in $\mathcal{A}_1, \mathcal{A}_2$ such that the following two conditions hold, for all equilibria v, w in E_1

$$i\big(\sigma(v)\big) = i(v), \quad \text{and} \tag{2.8}$$

$$\sigma(v) \searrow \sigma(w) \ \Leftrightarrow\ v \searrow w\,. \tag{2.9}$$

By definition, flow equivalence implies connection equivalence. By Theorem 1.2, attractors belonging to the same permutation π are connection equivalent.

Note that, even if the conjecture were true, there still could exist connection equivalent attractors which are not flow equivalent. Fortunately we do not know of any such example.

Having set our notion of connection equivalence, we need an enumeration of possible cases next. First we fix a number e of "equilibria". By dissipativeness, e must be odd. Indeed, consider a generic homotopy of f to $f \equiv -u$; then E is generated from $E = \{0\}$ by a sequence of saddle-node bifurcations which do not change the parity of e.

We enumerate Jordan permutations π next. This is motivated by a shooting method for the stationary boundary value problem (1.1). Indeed, let us solve the second order system

$$\begin{aligned} v_x &= \eta\,, \\ \eta_x &= -f(x, v, \eta)\,, \end{aligned} \tag{2.10}$$

with initial conditions $v = v_0$, $\eta = 0$, at $x = -1$. At $x = +1$, this defines a Jordan curve

$$S: \; a \mapsto \big(v(x=1;a),\; \eta(x=1;a)\big)$$

in the (v,η)-plane. The intersection points of S with the v-axis provide us with the stationary solutions satisfying Neumann boundary conditions. Numbering the intersection points along the v-axis, and then along S, immediately provides us with the permutation π. See Figure 2.1 for an example. Note that $\pi(1) = 1$, $\pi(e) = e$, by dissipativeness. More generally, a *Jordan permutation* $\pi \in S_e$ arises from any Jordan curve S, not necessarily defined via a differential equation (2.10), which strictly crosses the v-axis in precisely e points. We again require $\pi(1) = 1$, $\pi(e) = e$, for definiteness.

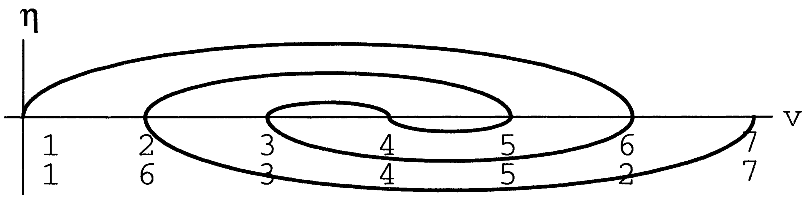

FIGURE 2.1. An S–curve with $\pi = \{1,6,3,4,5,2,7\}$.

Let $j(e)$ denote the number of Jordan permutations in S_e. They are related to the *meandering river numbers* M_n by

$$j(e) = M_{e-2}\,;$$

see A r n o l'd in [AV89]. See Table 2.1 for numerical values obtained by the author after minor programming headaches with Mathematica on an Apple Quadra 800.

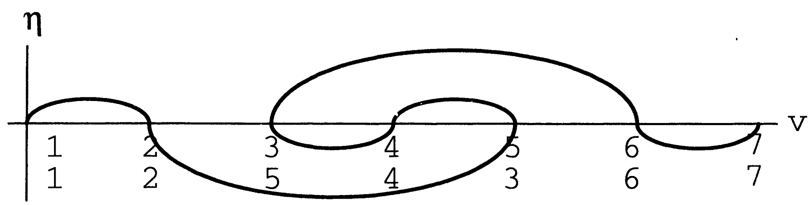

FIGURE 2.2. An impossible S–curve, $\pi = \{1,2,5,4,3,6,7\}$.

TABLE 2.1. Counting attractors with e equilibria, up to connection equivalence.

e	$j(e)$	$m(e)$	$\tilde{c}(e)$	$c(e)$
1	1	1	1	1
3	1	1	1	1
5	2	2	2	2
7	8	7	5	5
9	42	32	18	16
11	262	175	75	56
13	1828	1083	383	?
15	13820	7342	2250	?
17	110954	53372	14984	?

Not all Jordan permutations can arise through a dissipative second order equation. For example consider Figure 2.2. A quick computation of the Morse vector $i(v_k)$, $k = 1, \ldots, e$, according to Proposition 1.1 yields

$$\big(i(v_k)\big)_k = (0,1,0,-1,0,1,0)\,.$$

For differential equations, all $i(v_k)$ must be nonnegative. Therefore Figure 2.2 has to be discarded, in our context. The entries $m(e)$ in Table 2.1 count only those permutations for which all $i(v_k)$ are nonnegative. We see that Figure 2.2 is the first, and only, example which has to be discarded at levels $e \leq 7$.

Let us ask the inverse question: given a Jordan permutation π such that all $i(v_k) \geq 0$, does there exist an f leading to π? For $e \leq 15$, we have convinced ourselves that the answer is yes; we have used successive inspection of the corresponding non-pitchforkable S–curves, together with the arguments in [FR91]. We conjecture that $i(v_k) \geq 0$ is a general necessary and sufficient condition in order to construct f from π. We call π *Morse* if indeed all Morse indices $i(v_k)$ are nonnegative.

The remaining two columns, $\tilde{c}(e)$ and $c(e)$ count attractors taking into account connection equivalence. The column $\tilde{c}(e)$ counts attractors up to trivial equivalence of the permutations in the sense of (2.6). The simplest example of a Morse permutation π such that

$$\pi \neq \tau\pi\tau^{-1}$$

requires seven equilibria:

$$\pi = \{1,4,5,6,3,2,7\} = (2\ 4\ 6)\ (3\ 5)\,. \tag{2.11}$$

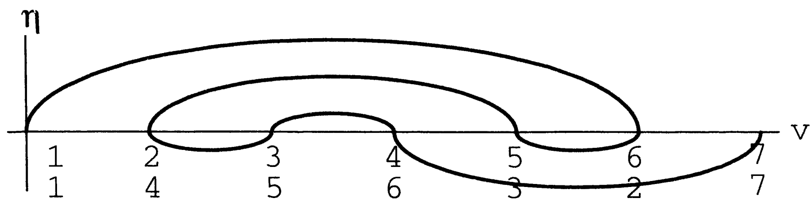

FIGURE 2.3. An S–curve for $\pi = \{1,4,5,6,3,2,7\} = (2\ 4\ 6)(3\ 5)$.

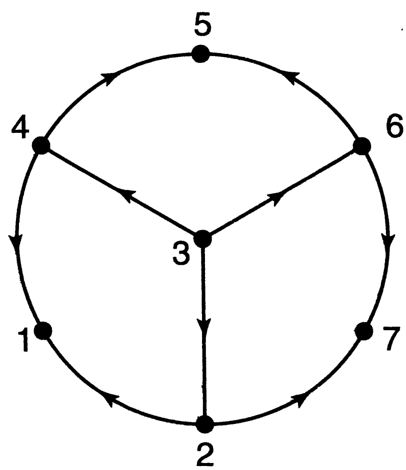

FIGURE 2.4. The "peace" attractor for $\pi = \{1,4,5,6,3,2,7\}$.

This is also the simplest Morse example for which $\pi \neq \pi^{-1}$. However, $\tau\pi\tau^{-1} = \pi^{-1}$. This is how $\tilde{c}(7) = 5$ arises from $m(7) = 7$. For the S-curve corresponding to (2.11) see Figure 2.3; the corresponding 1-skeleton of the attractor is graphed in Figure 2.4. The simplest example for which all four permutations π_1, $\tau\pi_1\tau^{-1}$, π_1^{-1}, $\tau\pi_1^{-1}\tau^{-1}$ are distinct requires nine equilibria and is given by

$$\pi_1 = \{1,4,5,8,7,6,3,2,9\} = (2\ 4\ 8)(3\ 5\ 7)\,. \tag{2.12}$$

It arises as Case 2.11 in Section 3. The same permutation is also involved in one of the two examples, up to $e = 9$, of a nontrivial equivalence: it is nontrivially equivalent to the permutation

$$\pi_2 = \{1,6,7,8,5,4,3,2,9\} = (2\ 6\ 4\ 8)(3\ 7)\,. \tag{2.13}$$

For π_2 see Case 2.9 in Section 3. Note that π_1 and π_2 are not conjugate because the lengths of their respective cycles differ. The bijection σ of their attractor graphs in the sense of (2.8), (2.9) is given by

$$\sigma = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ 7\,6\,5\,8\,9\,4\,3\,2\,1 \end{pmatrix}. \tag{2.14}$$

The other nontrivial equivalence is given in Cases 2.2 and 2.8 in Section 3. This explains why $c(e)$ is two less than $\tilde{c}(e)$ for $e = 9$ in Table 2.1.

3. The sixteen attractors with nine equilibria

In this section we list all attractors $\mathcal{A}$ of dissipative equations (1.1) which contain exactly $e = 9$ hyperbolic equilibria. Up to connection equivalence there are sixteen cases; see Table 2.1. We group these cases according to the dimension of the attractor $\mathcal{A}$:

$$\dim \mathcal{A} = \max_{v \in E} i(v)\,. \tag{3.1}$$

Proposition 1.1 implies

$$1 \leq \dim \mathcal{A} \leq 4\,. \tag{3.2}$$

Indeed, the Morse indices of adjacent equilibria differ by one. Moreover $i(v_k) = 0$ for $k = 1$ and for $k = e = 9$, by dissipativeness. This proves (3.2).

Up to trivial equivalence, there are eighteen permutations $\pi_1, \ldots, \pi_{18}$ such that all Morse indices $i(v_k)$, $k = 1, \ldots, 9$ are nonnegative. Here are these permutations, in lexicographic order:

$$\pi_1 = \{1,2,3,4,5,6,7,8,9\} = (\,)\,;$$
$$\pi_2 = \{1,2,3,4,5,8,7,6,9\} = (6\ 8)\,;$$
$$\pi_3 = \{1,2,3,6,5,4,7,8,9\} = (4\ 6)\,;$$
$$\pi_4 = \{1,2,3,6,7,8,5,4,9\} = (4\ 6\ 8)(5\ 7)\,;$$
$$\pi_5 = \{1,2,3,8,5,6,7,4,9\} = (4\ 8)\,;$$
$$\pi_6 = \{1,2,3,8,7,6,5,4,9\} = (4\ 8)(5\ 7)\,;$$
$$\pi_7 = \{1,4,3,2,5,8,7,6,9\} = (2\ 4)(6\ 8)\,;$$
$$\pi_8 = \{1,4,5,6,7,8,3,2,9\} = (2\ 4\ 6\ 8)(3\ 5\ 7)\,;$$
$$\pi_9 = \{1,4,5,8,7,6,3,2,9\} = (2\ 4\ 8)(3\ 5\ 7)\,;$$
$$\pi_{10} = \{1,6,7,8,3,4,5,2,9\} = (2\ 6\ 4\ 8)(3\ 7\ 5)\,;$$
$$\pi_{11} = \{1,6,7,8,5,2,3,4,9\} = (2\ 6)(3\ 7)(4\ 8)\,;$$
$$\pi_{12} = \{1,6,7,8,5,4,3,2,9\} = (2\ 6\ 4\ 8)(3\ 7)\,;$$
$$\pi_{13} = \{1,8,3,4,5,6,7,2,9\} = (2\ 8)\,;$$
$$\pi_{14} = \{1,8,3,4,7,6,5,2,9\} = (2\ 8)(5\ 7)\,;$$
$$\pi_{15} = \{1,8,3,6,5,4,7,2,9\} = (2\ 8)(4\ 6)\,;$$
$$\pi_{16} = \{1,8,5,6,7,4,3,2,9\} = (2\ 8)(3\ 5\ 7)(4\ 6)\,;$$
$$\pi_{17} = \{1,8,7,4,5,6,3,2,9\} = (2\ 8)(3\ 7)\,;$$
$$\pi_{18} = \{1,8,7,6,5,4,3,2,9\} = (2\ 8)(3\ 7)(4\ 6)\,;$$

TABLE 3.1. Permutations and corresponding cases, sorted by $\dim \mathcal{A}$.

$\dim \mathcal{A}$	$\tau\pi\tau^{-1} = \pi^{-1} = \pi$	$\pi^{-1} = \pi$	$\tau\pi^{-1}\tau^{-1} = \pi$	no symm	cases
1	π_1				1
2	$\pi_3, \pi_{11}, \pi_7, \pi_{17}, \pi_{18}$	π_2, π_6	π_8, π_{12}	π_4, π_9	2.1–2.11
3	π_{13}	π_5, π_{14}	π_{10}, π_{16}		3.1–3.5
4	π_{15}				4

In Table 3.1 we sort these permutations by increasing dimension of $\mathcal{A}$. Within each case, the sorting is by decreasing symmetry; the permutations for which $\tau\pi\tau^{-1} = \pi^{-1} = \pi$ are first, then the permutations for which only $\pi^{-1} = \pi$, or only $\tau\pi^{-1}\tau^{-1} = \pi$, and finally permutations π which do not satisfy any such identity. In terms of nonlinearities $f = f(x, u, p)$ these symmetries were interpreted in (2.2) - (2.4), putting $f_1 = f_2 = f$. Also recall that $f = f(u)$ implies π being an involution: this excludes permutations π_k with $k \in \{4, 8, 9, 10, 12, 16\}$ and the corresponding attractors. Within each symmetry class, finally, examples with fewer equilibria of high Morse index come first. In that sense, we proceed from the simple to the more complicated examples. We note that $\tau\pi\tau^{-1} = \pi$ implies $\pi^{-1} = \pi$ and hence $\tau\pi^{-1}\tau^{-1} = \pi$. Indeed τ reverses the orientation of any cycles of length greater then two. Therefore Table 3.1 does not contain a column $\tau\pi\tau^{-1} = \pi$.

As a final general remark we consider the Morse sequence $i_k = i(v_k)$, $k = 1, \dots, e$, for the permutations $\pi_1, \dots, \pi_{18}$. We note that for $e = 9$ all sequences are realized, which satisfy the following three properties

(i) $i_1 = i_e = 0$,

(ii) $i_k \geq 0$, for all k,

(iii) $|i_k - i_{k+1}| = 1$ for all $1 \leq k < e$.

There exist different attractors with the same corresponding Morse sequence; see, e.g., π_7, π_9, π_{17}.

Throughout the figures, k represents v_k. Also, the vertical axis of the S–curves is labeled by the counting index of the permutation which is represented.

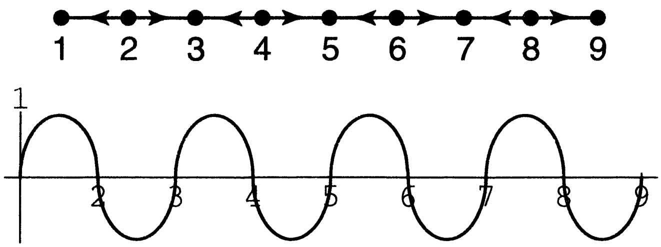

FIGURE 3.1. $\pi = \pi_1 = \text{id}$; see Case 1.

Case 1: $\dim \mathcal{A} = 1$

See Fig. 3.1 for the only corresponding S–curve, $\pi = \pi_1 = \text{id}$, and the attractor. The attractor $\mathcal{A}$ is just an interval with e hyperbolic equilibria.

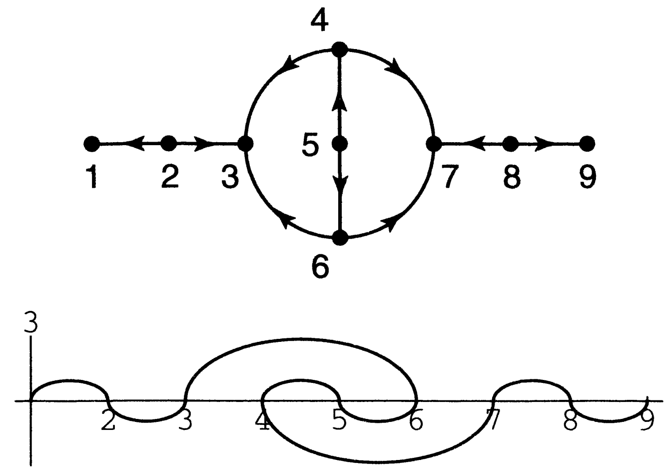

FIGURE 3.2. $\pi = \pi_3$; see Case 2.1.

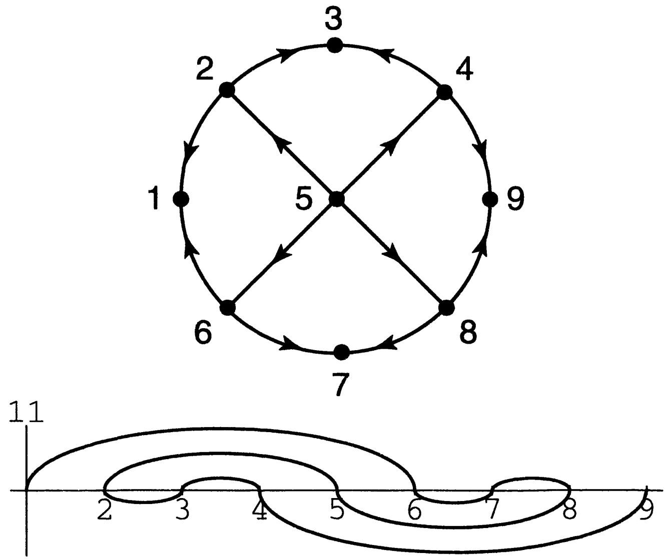

FIGURE 3.3. $\pi = \pi_{11}$; see Cases 2.2 and 2.8.

Case 2: $\dim \mathcal{A} = 2$

There are eleven subcases, nine of which possess distinct connection patterns. In 2.1 – 2.5 we collect the cases with maximal symmetry $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ of the permutations. In two cases, 2.6 and 2.7, $\pi = \pi^{-1}$. Two more cases are concerned with $\pi = \tau\pi^{-1}\tau$, see 2.8, 2.9. The final two cases, 2.10 and 2.11, do not possess any trivial symmetry apparent in π.

Case 2.1: $\pi = \pi_3$

See Fig. 3.2. The attractor is a planar five equilibria Chafee–Infante problem with two three equilibria intervals attached at the local attractors v_3, v_7.

Case 2.2: $\pi = \pi_{11}$

See Fig. 3.3. This attractor arises, e.g., from the same planar Chafee–Infante attractor by subcritical pitchfork bifurcations involving the triples $\{2,3,4\}$ and $\{6,7,8\}$. For a nontrivially connection equivalent match see Case 2.8 below.

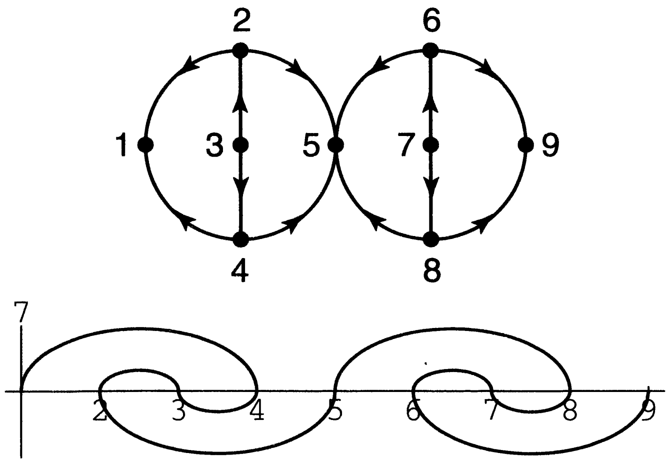

FIGURE 3.4. $\pi = \pi_7$; see Case 2.3.

Case 2.3: $\pi = \pi_7$

See Fig. 3.4. The attractor consists of two planar five equilibrium Chafee–Infante attractors, glued together at the common local attractor v_5.

Case 2.4: $\pi = \pi_{17}$

See Fig. 3.5. This example is intriguing because the S–curve, restricted to the seven equilibria $\{2, 3, \dots, 8\}$, is a forbidden (left-winding) example at that level; see Figure 2.2. By the one-dimensionally unstable "suspension" through equilibria v_1 and v_9, it becomes realizable. This, by the way, is a feasible operation for any (not necessarily Morse type) Jordan permutation: adjoining $m+m$ equilibria, we may raise all previous Morse indices by m.

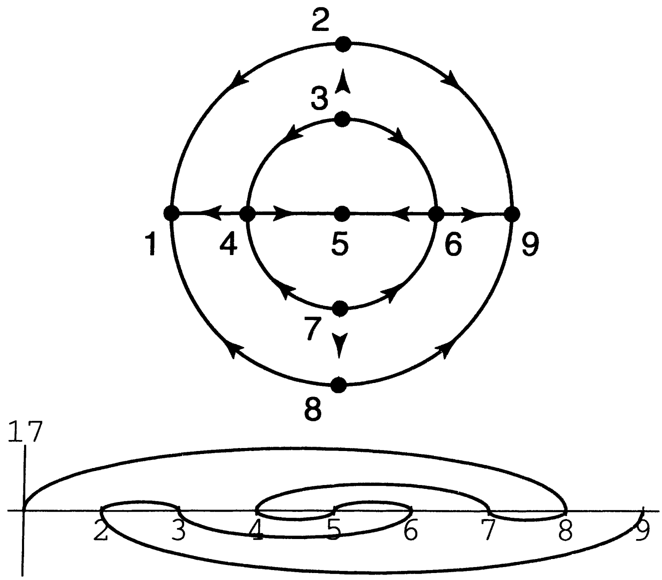

FIGURE 3.5. $\pi = \pi_{17}$; see Case 2.4.

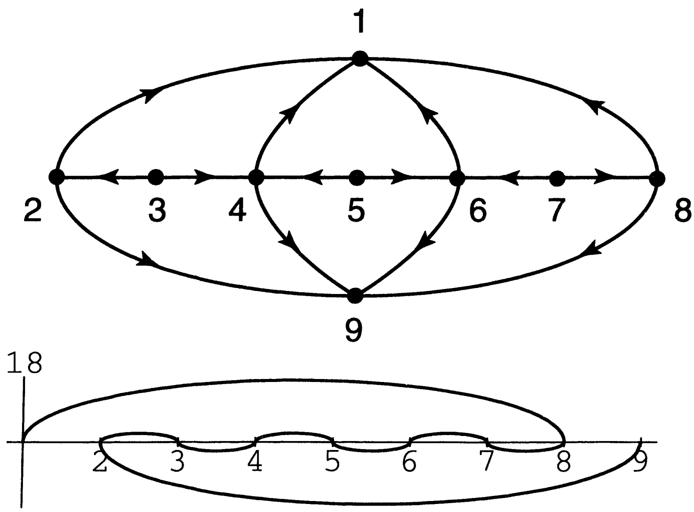

FIGURE 3.6. $\pi = \pi_{18}$; see Case 2.5.

Case 2.5: $\pi = \pi_{18}$

See Fig. 3.6. Again this is a suspension flow: the interval $\{2, 3, \ldots, 8\}$ is one-dimensionally unstably suspended. The resulting flow is a product of that inter-

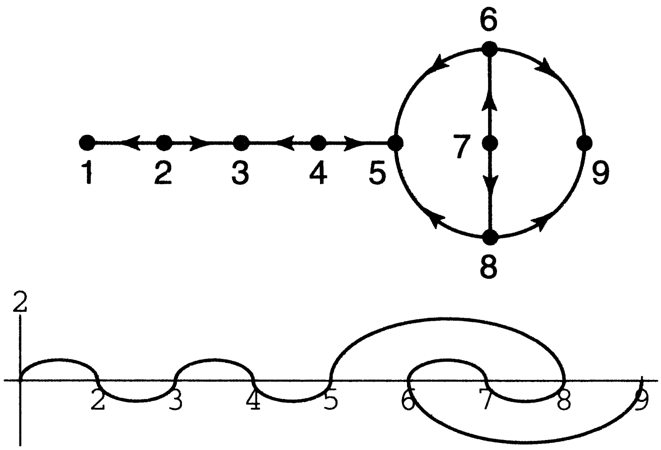

FIGURE 3.7. $\pi = \pi_2$; see Case 2.6.

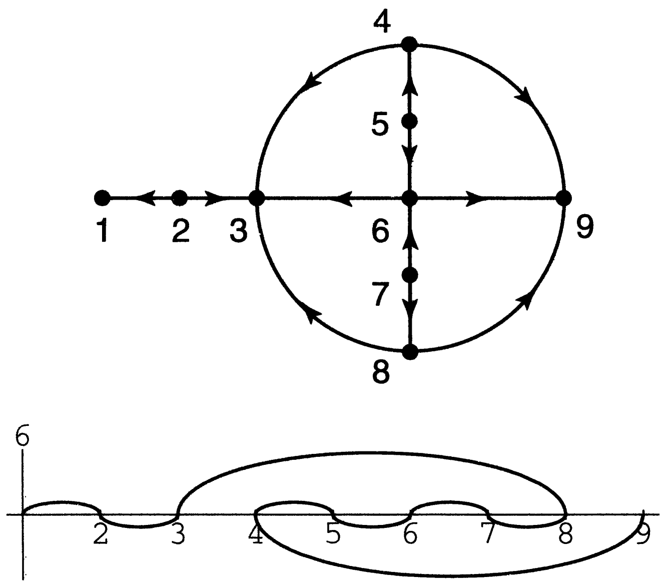

FIGURE 3.8. $\pi = \pi_6$; see Case 2.7.

val with the dissipative three equilibria attractor.

Case 2.6: $\pi = \pi_2$

This is the first of two cases with symmetry given by only $\pi = \pi^{-1}$. For $\pi = \pi_2$ see Fig. 3.7. It consists of a planar five equilibria Chafee–Infante attractor with a five equilibria interval glued onto one of its attracting equilibria.

Case 2.7: $\pi = \pi_6$

See Fig. 3.8. The attractor consists of an interval $\{1, 2, 3\}$ glued to a disk $\{3, 4, \ldots, 9\}$. The disk itself is a one-dimensionally unstable suspension, by $3, 9$, of an interval $\{4, 5, 6, 7, 8\}$.

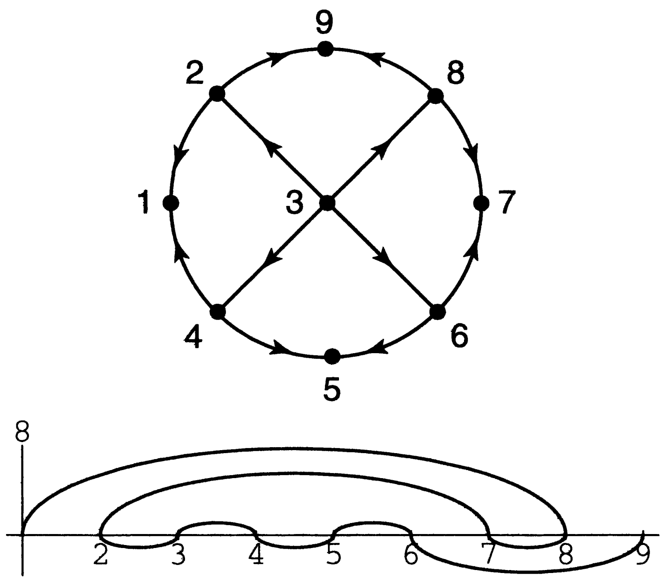

FIGURE 3.9. $\pi = \pi_8$; see Cases 2.2 and 2.8.

Case 2.8: $\pi = \pi_8$

The following two examples possess a τ-symmetry, but contain cycles of length exceeding two. For $\pi = \pi_8$ see Fig. 3.9. Note that the symmetry of the attractor is not apparent in the S–curve. Surprisingly, the attractor is connection equivalent to the fully symmetric case $\pi = \pi_{11}$; see Case 2.2 and Fig. 3.3.

Case 2.9: $\pi = \pi_{12}$

See Fig. 3.10. The attractor consists of a "peace" piece $\{1, 4, 5, 6, 7, 8, 9\}$ and a planar Chafee–Infante piece $\{1, 2, 3, 4, 9\}$. They are glued together along the common boundary interval $\{1, 4, 9\}$.

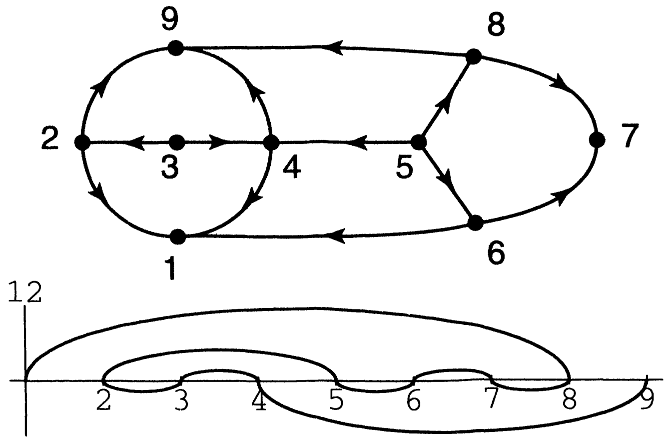

FIGURE 3.10. $\pi = \pi_{12}$; see Cases 2.9 and 2.11.

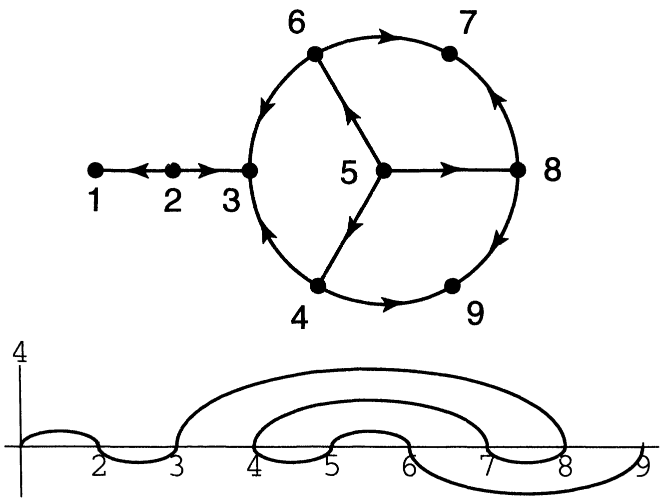

FIGURE 3.11. $\pi = \pi_4$; see Case 2.10.

Case 2.10: $\pi = \pi_4$

This is the first of two final planar examples with no trivial symmetry of π. For $\pi = \pi_4$ see Fig. 3.11. The attractor consists of a "peace" disk with a three equilibrium interval attached at 3. Again, the attractor reveals a symmetry which is not present in π.

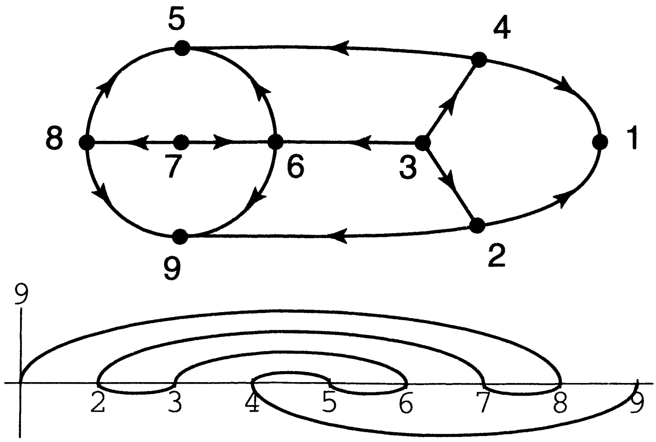

FIGURE 3.12. $\pi = \pi_9$; see Cases 2.11 and 2.9.

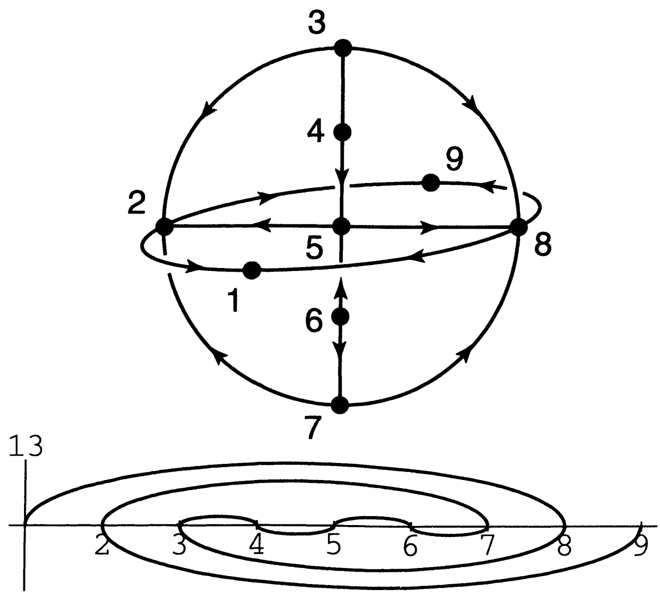

FIGURE 3.13. $\pi = \pi_{13}$; see Case 3.1.

Case 2.11: $\pi = \pi_9$

See Fig. 3.12. As announced above, this attractor is connection equivalent to the τ-symmetric Case 2.9, $\pi = \pi_{12}$. See Fig. 3.10.

Case 3

We now describe the five possible connection patterns of three-dimensional attractors. Case 3.1 describes the example with maximal symmetry. Cases 3.2 and 3.3 consider involutory π, and 3.4, 3.5 are devoted to τ-symmetric permutations. Nonsymmetric cases do not arise here.

Case 3.1: $\pi = \pi_{13}$

See Fig. 3.13. The attractor can be viewed as a two-dimensionally unstable suspension of the interval $\{3,4,5,6,7\}$. The interval replaces, by substitution, the center, say 5, in a Chafee–Infante disk $\{1,2,5,8,9\}$. Alternatively, we could argue that the center, say 5, in a seven equilibria Chafee–Infante ball $\{1,2,3,5,7,8,9\}$ has undergone a subcritical pitchfork, being replaced by the interval $\{4,5,6\}$.

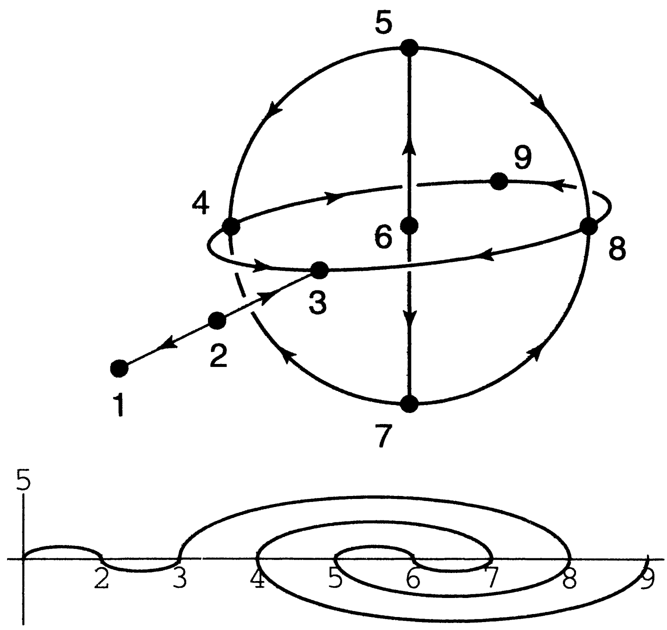

FIGURE 3.14. $\pi = \pi_5$; see Case 3.2.

Case 3.2: $\pi = \pi_5$

This is the first of two involutory cases; see Fig. 3.14 for $\pi = \pi_5$. The attractor consists of a three-dimensional Chafee–Infante ball $\{3,4,\dots,9\}$ with an interval $\{1,2,3\}$ attached at 3.

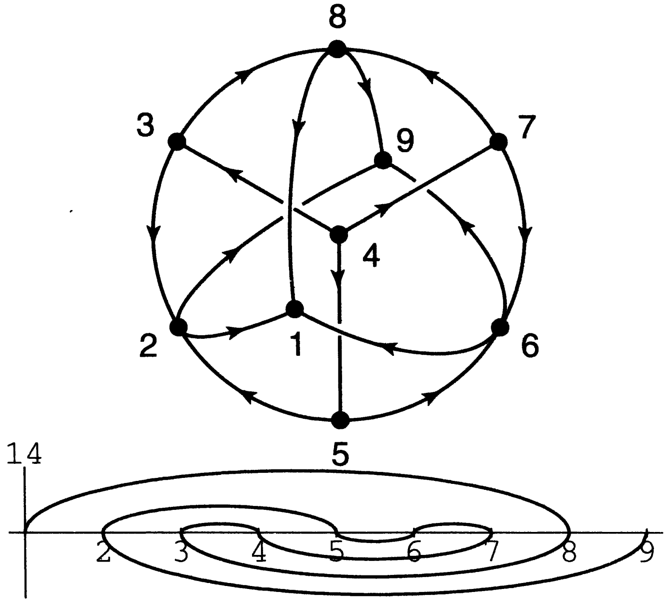

FIGURE 3.15. $\pi = \pi_{14}$; see Case 3.3.

Case 3.3: $\pi = \pi_{14}$

See Fig. 3.15. The attractor is the one-dimensionally unstable suspension of a "peace"-disk $\{2, 3, \ldots, 8\}$.

Case 3.4: $\pi = \pi_{10}$

See Fig. 3.16. The attractor can be obtained by a supercritical pitchfork from a "peace"-disk $\{1, 2, 4, 6, 7, 8, 9\}$, replacing 4 by the two-dimensionally unstable interval $\{3, 4, 5\}$.

Case 3.5: $\pi = \pi_{16}$

See Fig. 3.17. Again, the attractor can be viewed as a one-dimensionally unstable suspension of a planar object described by $\{2, 3, \ldots, 8\}$. That seven equilibria object, in turn, consists of a planar Chafee–Infante disk $\{4, 5, 6, 7, 8\}$ with an interval $\{2, 3, 4\}$ attached at 4. Alternatively, the attractor decomposes into a Chafee–Infante disk $\{1, 2, 3, 4, 9\}$ glued to a Chafee–Infante ball $\{1, 4, 5, 6, 7, 8, 9\}$ along the interval $\{1, 4, 9\}$ in the common boundary.

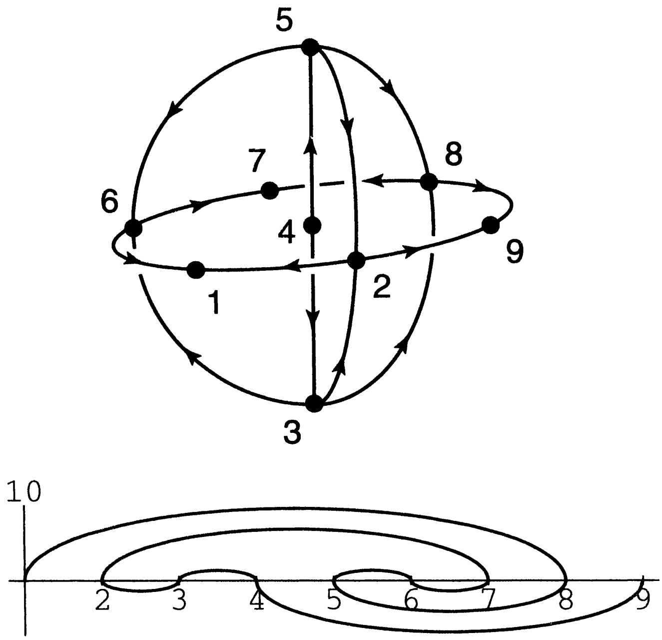

FIGURE 3.16. $\pi = \pi_{10}$; see Case 3.4.

Case 4: dim $\mathcal{A} = 4$

Our final example, $\pi = \pi_{15}$, is a four-dimensional Chafee–Infante ball. The one-dimensional connections are indicated in Fig. 3.18. For any $e = 2n + 1$, the Chafee–Infante example reaches the maximal possible dimension, dim $\mathcal{A} = n$. Also, the corresponding permutation

$$\pi = (2, 2n)(4, 2n-2)\dots(n', 2n+2-n')$$

is uniquely determined; here n' denotes the largest even integer not exceeding n. The corresponding attractors can, for example, be viewed as successive one-dimensionally unstable suspension of $\{2, \dots, e-1\}$ by 1 and e.

For completeness, we note that all attractors with less than nine equilibria are represented in our study by the above permutations $\pi_1, \dots, \pi_6$. Indeed, we just have to remove the spikes corresponding to intervals of π-fixed points like $\{1,2,3\}$ from the attractors.

So much for the sixteen different attractors with $e=9$ equilibria, up to connection equivalence. All of the corresponding permutations π are *pitchforkable* in the sense of Fusco & Rocha; [FR91]. This means that there exists a one

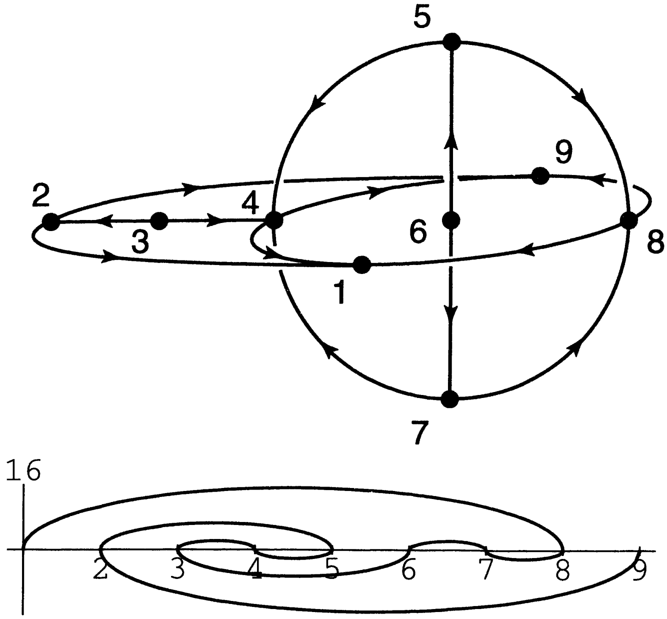

FIGURE 3.17. $\pi = \pi_{16}$; see Case 3.5.

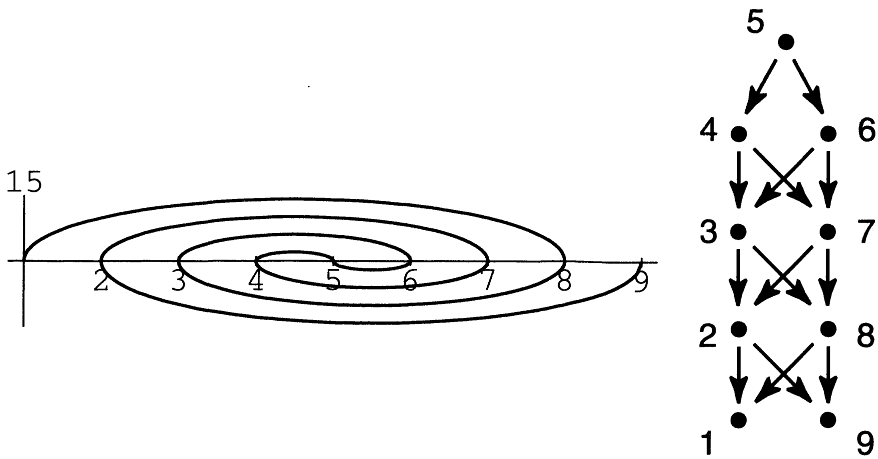

FIGURE 3.18. $\pi = \pi_{15}$; see Case 4.

parameter family of dissipative nonlinearities $f = f_\lambda$ which only exhibits pitchfork bifurcations of the stationary solutions. At $\lambda = 0$, the attractor is trivial $(e=1)$, and at $\lambda = 1$ the permutation π is realized. In particular, all the sixteen connection graphs sketched above actually occur for suitable choices of f.

For $e \geq 11$ non-pitchforkable permutations arise. For example, the two non-pitchforkable permutations for $e = 11$ are given by

$$\pi = \{1,4,5,10,9,6,3,2,7,8,11\},$$
$$\pi = \{1,6,7,10,3,4,9,8,5,2,11\},$$

up to trivially symmetric variants. Note that our derivation of the connection graphs for $e=9$ does not use the pitchforkable nature of the permutations for the derivation of the connection graph. Moreover, the results of Section 1 imply that *any* dissipative f with a pitchforkable associated permutation π is in fact connection equivalent to the particular model constructed by Fusco & Rocha.

We hope that our above study of attractors with nine equilibria can be useful for more than just mindless classification. It is clear, by now, that the crucial bifurcation which could be the key to an understanding of all attractors of (1.1) is the saddle node bifurcation of equilibria — rather than the pitchfork bifurcation. Our study can be viewed as a first step towards understanding the effect of saddle node bifurcations on attractors with less than nine equilibria.

In conclusion, we admit that our classification of attractors is somewhat awkward from a geometric point of view. We start with an elaborate machinery on permutations and — after a long and exhausting detour — arrive at a few pictures which we hope describe the geometry of attractors. A more elegant approach would attempt a classification of the attractors, directly, by understanding the combinatorics of their geometries. Suspensions, products, and substitutions can serve as elements of such a description. Perhaps there is such a shortcut ...

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