Roman Koplatadze

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ON ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

ROMAN KOPLATADZE

ABSTRACT. Sufficient (necessary and sufficient) conditions are given for a functional differential equation to have properties A and B.

Consider the equation

$$u^{(n)}(t) + F(u)(t) = 0\,, \tag{1}$$

where $F\colon C^{n-1}(\mathbb{R}_+;\mathbb{R}) \to L_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R})$ is a continuous operator. Everywhere below we shall assume that a nondecreasing function $\sigma\colon \mathbb{R}_+ \to \mathbb{R}_+$ exists such that $\lim\limits_{t\to+\infty} \sigma(t) = +\infty$ and for any $t \in \mathbb{R}_+$

$$F(x)(t) = F(y)(t) \qquad \text{if} \quad x, y \in C^{n-1}(\mathbb{R}_+;\mathbb{R})$$

and

$$x(s) = y(s) \qquad \text{for} \quad s \geq \sigma(t)\,.$$

For any $t_0 \in \mathbb{R}_+$ let M_{t_0} denote the set of $u \in C^{n-1}(\mathbb{R}_+;\mathbb{R})$ satisfying $u(t) \neq 0$ for $t \geq t^*$, where $t^* = \min\{t_0, \sigma(t_0)\}$. The following assumption will always be fulfilled: either

$$F(u)(t)\,u(t) \geq 0 \qquad \text{for } t \geq t_0\,, \quad \text{for any } t_0 \in \mathbb{R}_+ \text{ and } u \in M_{t_0}\,, \tag{2}$$

or

$$F(u)(t)\,u(t) \leq 0 \qquad \text{for } t \geq t_0\,, \quad \text{for any } t_0 \in \mathbb{R}_+ \text{ and } u \in M_{t_0}\,. \tag{3}$$

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Definition 1. Let $t_0 \in \mathbb{R}_+$. A function $u\colon [t_0,+\infty[\to \mathbb{R}$ is said to be the *proper solution* of the equation (1) if it is locally absolutely continuous up to the order $n-1$ inclusively, there exists a function $\overline{u} \in C^{n-1}(\mathbb{R}_+;\mathbb{R})$ such that $\overline{u}(t) \equiv u(t)$ for $t \geq t_0$, almost everywhere on $[t_0,+\infty[$

$$\overline{u}^{(n)}(t) + F(\overline{u})(t) = 0$$

and

$$\sup\{|u(s)|: s \in [t,+\infty[\} > 0 \quad \text{for any} \quad t \in [t_0,+\infty[.$$

Definition 2. We say that the equation (1) *has the property* A provided any of its proper solutions is oscillatory if n is even and either is oscillatory or satisfies

$$|u^{(i)}(t)| \downarrow 0 \qquad \text{as} \quad t \uparrow +\infty \quad (i = 0,\dots,n-1) \tag{4}$$

if n is odd.

Definition 3. We say that the equation (1) *has the property* B provided any of its proper solutions either is oscillatory or satisfies (4) or

$$|u^{(i)}(t)| \uparrow 0 \qquad \text{as} \quad t \uparrow +\infty \quad (i = 0,\dots,n-1) \tag{5}$$

if n is even and either is oscillatory or satisfies (5) if n is odd.

Conditions for an ordinary differential equation to have the properties A and B are studied well enough (see [1, 2] and references therein). The analogous problems for the equations with deviating arguments are investigated in [3, 4].

Theorem 1. *Let (2) ((3)) hold and let for any $t_0 \in \mathbb{R}_+$*

$$\big|F(u)(t)\big| \geq \sum_{i=1}^{m} \int_{\sigma_i(t)}^{\overline{\sigma}_i(t)} \big|u(s)\big| d_s r_i(t,s) \qquad \text{for } t \geq t_0, \quad u \in M_{t_0}, \tag{6}$$

where the measurable functions $r_i(t,s)$ $(i = 1,\dots,m)$ are nondecreasing in s, $\sigma_i, \overline{\sigma}_i \in C(\mathbb{R}_+;\mathbb{R}_+)$, $\sigma_i(t) \leq \overline{\sigma}_i(t)$ $(i = 1,\dots,m)$ for $t \geq 0$ and

$$\varliminf_{t\to+\infty} \frac{\sigma_i(t)}{t} > 0 \quad (i = 1,\dots,m). \tag{7}$$

Suppose, moreover, that there exists $\varepsilon > 0$ such that for any $l \in \{1,\dots,n-1\}$ and $\lambda \in [l-1,l[$ where $l+n$ is odd (even) the inequality

$$\varliminf_{t\to+\infty} t^{l-\lambda} \int_t^{+\infty} \xi^{n-l-1} \sum_{i=1}^{m} \int_{\sigma_i(\xi)}^{\overline{\sigma}_i(\xi)} s^{\lambda} d_s r_i(\xi,s)\, d\xi \geq \prod_{i=0,i\neq l}^{n-1} |\lambda - i| + \varepsilon$$

holds. Then the equation (1) has the property A (B).

THEOREM 2. *Let* (2), (6), (7) $\big($(3), (6), (7)$\big)$ *hold. Suppose, moreover, that there exists* $\varepsilon > 0$ *such that for any* $\lambda \in \bigcup\limits_{k=0}^{(n-2)/2} [2k, 2k+1[$ $\Big(\lambda \in \bigcup\limits_{k=1}^{(n-2)/2} [2k-1, 2k[\Big)$ *if* n *is even and for any* $\lambda \in \bigcup\limits_{k=1}^{(n-1)/2} [2k-1, 2k[$ $\Big(\lambda \in \bigcup\limits_{k=0}^{(n-3)/2} [2k, 2k+1[\Big)$ *if* n *is odd the inequality*

$$\varliminf_{t\to+\infty} t \int\limits_t^{+\infty} \xi^{n-\lambda-2} \sum_{i=1}^m \int\limits_{\sigma_i(\xi)}^{\overline{\sigma}_i(\xi)} s^\lambda d_s r_i(\xi, s)\, d\xi \geq \prod_{i=0}^{n-1} |\lambda - i| + \varepsilon$$

holds. Then the equation (1) *has the property* A (B).

COROLLARY 1. *Let* (2), (7) $\big($(3), (7)$\big)$ *hold and let for any* $t_0 \in \mathbb{R}_+$

$$\big|F(u)(t)\big| \geq \sum_{i=1}^m p_i(t)\big|u\big(\sigma_i(t)\big)\big| \quad \text{for} \quad t \geq t_0\,, \quad u \in M_{t_0}\,,$$

where $p_i \in L_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}_+)$, $\sigma_i \in C(\mathbb{R}_+; \mathbb{R}_+)$, $\sigma_i(t) \leq t$ $(i = 1, \dots, m)$. *Suppose, moreover, that there exists* $\varepsilon > 0$ *such that for any* $\lambda \in [n-2, n-1[$ *(for any* $\lambda \in [n-3, n-2[\cup [0,1[$ *if* n *is odd and for any* $\lambda \in [n-3, n-2[$ *of* n *is even) the inequality*

$$\varliminf_{t\to+\infty} t \int\limits_t^{+\infty} \xi^{n-\lambda-2} \sum_{i=1}^m p_i(\xi)\sigma_i\sigma_i^\lambda(\xi)\, d\xi \geq \prod_{i=0,}^{n-1} |\lambda - i| + \varepsilon$$

holds. Then the equation (1) *has the property* A (B).

COROLLARY 2. *Let* (2) ((3)) *hold and let for any* $t_0 \in \mathbb{R}_+$,

$$\big|F(u)(t)\big| \geq \frac{c}{t^{n+1}} \int\limits_{\alpha t}^{\overline{\alpha} t} \big|u(s)\big| ds \qquad \text{for } t \geq t_0\,, \quad u \in M_{t_0}\,,$$

where $0 < \alpha < \overline{\alpha}$, *and*

$$c > \max\big\{-(\lambda+1)\lambda(\lambda-1)\cdots(\lambda-n+1)\big(\overline{\alpha}^{\lambda+1} - \alpha^{\lambda+1}\big) : \lambda \in [0, n-1]\big\}$$
$$\big(c > \max\big\{(\lambda+1)\lambda(\lambda-1)\cdots(\lambda-n+1)\big(\overline{\alpha}^{\lambda+1} - \alpha^{\lambda+1}\big) : \lambda \in [0, n-1]\big\}\big) \tag{8}$$

Then the equation (1) *has the property* A (B).

COROLLARY 3. *Let $\alpha > 0$ and $c \in \,]0,+\infty[$ $\;(c \in \,]-\infty, 0\,[\,)$. Then the condition*

$$c > \max\bigl\{-\alpha^{-\lambda}\lambda(\lambda-1)\cdots(\lambda-n+1)\colon \lambda \in [0,n-1]\bigr\}$$
$$\Bigl(c < -\max\bigl\{\alpha^{-\lambda}\lambda(\lambda-1)\cdots(\lambda-n+1) : \lambda \in [0,n-1]\bigr\}\Bigr)$$

is necessary and sufficient for the equation

$$u^{(n)}(t) + \frac{c}{t^n}\,u(\alpha t) = 0$$

to have the property A (B).

COROLLARY 4. *Let $0 < \alpha < \overline{\alpha}$ and $c \in \,]0,+\infty[$ $(c \in \,]-\infty, 0\,[\,)$. Then the condition (8)*

$$\Bigl(c < -\max\bigl\{(\lambda+1)\lambda(\lambda-1)\dots(\lambda-n+1)\bigl(\overline{\alpha}^{\lambda+1} - \alpha^{\lambda+1}\bigr)^{-1} : \lambda \in [0,n-1]\bigr\}\Bigr)$$

is necessary and sufficient for the equation

$$u^{(n)}(t) + \frac{c}{t^{n+1}} \int\limits_{\alpha t}^{\overline{\alpha} t} u(s)\,ds = 0$$

to have the property A (B).

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I. N. Vekua Institute of Applied Mathematics
University str. 2
380043 Tbilisi
REPUBLIC OF GEORGIA
E-mail: kig@imath.kheta.georgia.su