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In: Pavol Brunovský and Milan Medveď (eds.): Equadiff 8, Czech - Slovak Conference on Differential Equations and Their Applications. Bratislava, August 24-28, 1993. Mathematical Institute, Slovak Academy of Sciences, Bratislava, 1994. Tatra Mountains Mathematical Publications, 4. pp. 117--124.

Persistent URL: http://dml.cz/dmlcz/700102

Tatra Mountains Math. Publ. **4** (1994), 117–124

**TATRA MOUNTAINS**
*Mathematical Publications*

# REGULARITY FOR DOUBLY NONLINEAR PARABOLIC EQUATIONS

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ABSTRACT. Hölder estimates and existence of regular solutions of Cauchy–Dirichlet problem for doubly nonlinear parabolic equations are established. Similar equations arise in the study of turbulent filtration of a gas or a fluid through porous media.

Let $\Omega$ is a bounded open set in $\mathbb{R}^n$, $n \geq 1$, $Q_T = \Omega \times (0,T]$, $S_T = \partial\Omega \times (0,T]$, $\Gamma_T = S_T \times \big(\Omega \times \{t=0\}\big)$ ($\Gamma_T$ is a parabolic boundary of $Q_T$). Consider in $Q_T$ equation of the type

$$\partial u/\partial t - \operatorname{div} \vec{a}(x,t,u,\nabla u) + a_0(x,t,u,\nabla u) = 0\,, \tag{1.1}$$

where $\vec{a} = (a^1,\dots,a^n)$, $\nabla u = \left(\dfrac{\partial u}{\partial x_1},\dots,\dfrac{\partial u}{\partial x_n}\right)$. Assume that $a^i(x,t,u,p)$, $a_0(x,t,u,p)$ satisfy the Caratheodory condition and let for a.e. $(x,t) \in Q_T$ and any $u \in \mathbb{R}\backslash\{0\}$, $p \in \mathbb{R}^n$ inequalities

$$\begin{aligned} &\vec{a}(x,t,u,p)\cdot p \geq \nu_0 |u|^l\,|p|^m - \Phi_0(x,t,u), \quad \nu_0 > 0\,;\\ &\big|\vec{a}(x,t,u,p)\big| \leq \mu_1 |u|^l\,|p|^{m-1} + \Phi_1(x,t,u)\,, \end{aligned} \tag{1.2}$$

hold with some $m > 1$, $l > 1-m$ and $\Phi_i(x,t,u) \geq 0$.

The prototypes of equations of the type (1.1), (1.2) are

$$\partial u/\partial t - \operatorname{div}\big\{|u|^l\,|\nabla u|^{m-2}\nabla u\big\} = 0\,, \quad m>1,\ l>1-m \tag{1.3}$$

and

$$\partial u/\partial t - \sum_{i=1}^n \frac{d}{dx_i}\big\{|u|^l\,|\partial u/\partial x_i|^{m-2}\partial u/\partial x_i\big\} = 0, \quad m>1,\ l>1-m\,. \tag{1.3$'$}$$

Equations (1.3), (1.3′) and close equations arise in the study of turbulent filtration of a gas or a liquid through porous media and non–Newtonian fluids.

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AMS Subject Classification (1991): 35K55, 35K65.
Key words: regularity, turbulent flow, porus media.

For example in the case $m > 1$, $l > 0$ equation (1.3) is called in the survey [1] as the equation of non–Newtonian polythropic filtration. We shall use this title for (1.3) further only in the case $m > 2$, $l > 0$.

On the other hand in the case $n = 1$, $m \in [3/2, 2)$, $l \geq m - 1$ equation (1.3) arises in the study of turbulent flow of a gas in one-dimensional porous media. This phenomenon was described at first by L e i b e n s o n in [2]. Further we shall call (1.3) as the Leibenson equation for any $n \geq 1$, $m \in (1,2)$, $l > 0$, $m + l \geq 2$.

At last in the study of flow through three-dimensional porous media in turbulent regimes (i.e., in the case when in view of large velocities the classical Darcy's law fails to be true) several authors (see [3]) consider equation of the type

$$\partial u/\partial t - \operatorname{div}\left\{|u|^l \left|\nabla u - c_0|u|^k \vec{Z}\right|^{m-2}(\nabla u - c_0|u|^k \vec{Z})\right\} = 0 \tag{1.4}$$

with $m \in (1,2)$, $l = (s-1)(m-1)$, $k = r - s + 1$, $\vec{Z} = (0,0,1)$, from the physical point of view it is natural to assume that $1 < s \leq r$. Thus we have in (1.4) $l > 0$, $k \geq 1$. Furthermore we shall call on equation (1.4) briefly as the equation of turbulent filtration.

In the case of equation (1.3) we have

$$a^i = |u|^l\,|p|^{m-2}p_i\,, \quad \frac{\partial a^i}{\partial p_j}\,\xi_i\xi_j \geq \min(1, m-1)|u|^l\,|p|^{m-2}|\xi|^2. \tag{1.5}$$

In view of (1.5) it is natural to use the following classification for equations (1.1), (1.2):

| $m > 2,\ l > 0$ | $1 < m < 2,\ l > 0$ | $m > 2,\ l < 0$ | $1 < m < 2,\ l < 0$ |
|---|---|---|---|
| doubly degenerate par. eq. | singular-degenerate par. eq. | degenerate-singular par. eq. | doubly singular par. eq. |

On the other hand rewrite (1.3′) as

$$\partial u/\partial t - D(u, \nabla u)\Delta u + \mathcal{E}(u, \nabla u) = 0\,,$$

where $D(u,p) = (m-1)|u|^l \sum\limits_{i=1}^{n} |p_i|^{m-2}$, $\mathcal{E}(u,p) = l\,|u|^{l-2}\,u \sum\limits_{i=1}^{n} |p_i|^m$. It is evident that

$$\lim_{\mu\downarrow 0} D(\mu u, \mu p) = \begin{cases} 0, & \text{if } m + l > 2\,, \\ D(u,p)\,, & \text{if } m + l = 2\,, \\ +\infty\,, & \text{if } m + l > 2\,. \end{cases} \tag{1.6}$$

In view of (1.6) we shall use also the following classification for general equations (1.1), (1.2):

| $m + l > 2$ | $m + l = 2$ | $m + l < 2$ |
|---|---|---|
| equations of the type of slow diffusion | equations of the type of normal diffusion | equations of the type of fast diffusion |

So the equation of non–Newtonian polythropic filtration is a doubly degenerate parabolic equation and at the same time is an equation of the type of slow diffusion, while the Leibenson equation is a singular-degenerate parabolic equation and an equation of the type of slow or normal diffusion. On the other hand the equation of turbulent filtration is a singular-degenerate parabolic equation and an equation of the type of fast diffusion if $s < \frac{1}{m-1}$, of normal diffusion if $s = \frac{1}{m-1}$ and of slow diffusion if $s > \frac{1}{m-1}$.

Remark now that equations (1.1), (1.2) can be rewritten as equations of the type

$$\frac{\partial b(v)}{\partial t} - \operatorname{div} \vec{A}(x,t,v,\nabla v) + A_0(x,t,v,\nabla v) = 0 \tag{1.7}$$

where

$$v = |u|^\sigma u, \quad \sigma = \frac{l}{m-1}, \quad b(v) = |v|^{\beta-1}v, \quad \beta = \frac{1}{\sigma+1} = \frac{m-1}{l+m-1}$$

and coefficients $A^i(x,t,v,q)$ satisfy the following inequalities

$$\begin{aligned} &\vec{A}(x,t,v,q)\cdot q \geq \hat{\nu}_0|q|^m - \hat{\Phi}_0(x,t,v)\,, \qquad \hat{\nu}_0 > 0\,; \\ &|\vec{A}(x,t,v,q)| \leq \hat{\mu}_1|q|^{m-1} + \hat{\Phi}_1(x,t,v)\,, \end{aligned} \tag{1.8}$$

with $m > 1,\ \beta > 0,\ \hat{\Phi}_i(x,t,v) \geq 0$. It is evident that equations of the type (1.7), (1.8) are only another form of equations of the type (1.1), (1.2).

Equations (1.7), (1.8) (and hence (1.1), (1.2)) are known as doubly nonlinear parabolic equations. Existence results for equations of this type were obtained in the pioneering papers [4, 5] by Raviart and J.–L. Lions and then in [6–13, 3] etc. The purpose of this talk is to discuss the regularity problem for equations (1.1), (1.2). Up to recent time there were no regularity results for weak solutions of doubly nonlinear parabolic equations. Under regular solution we mean in this paper Hölder continuous weak solution. The simple modification of Barenblatt explicit self-similar solution (see [14]) lets to show that holderness is the best possible smoothness of weak solutions (1.1), (1.2) in the case $m > 1,\ l > 1$.

More precisely this paper devotes mainly to discussion of two problems for equations (1.1), (1.2):

1. regularity (Hölder continuity) of every bounded weak solution;
2. existence of regular solution of Cauchy–Dirichlet problem.

For discussion of the first of these problems it is necessary to specify the growth conditions (1.2) for equation (1.1). Suppose that for a.e. $(x,t)$ and any $u \in \mathbb{R},\ p \in \mathbb{R}^n$

$$\begin{aligned} &\vec{a}(x,t,u,p)\cdot p \geq \nu_0|u|^l\,|p|^m - \varphi_0\,, && \nu_0 > 0\,; \\ &\big|\vec{a}(x,t,u,p)\big| \leq \mu_1|u|^l\,|p|^{m-1} + \varphi_1|u|^\alpha, && \alpha = l/m\,; \\ &\big|a_0(x,t,u,p)\big| \leq \mu_2|u|^{lm'}|p|^m + \varphi_2, && 1/m + 1/m' = 1\,, \end{aligned} \tag{1.9}$$

where $m > 1$, $l > 1 - m$, $\varphi_i = \varphi_i(x,t) \geq 0$, $\varphi_0 \equiv \varphi_1 \equiv 0$ if $l < 0$; $\varphi_0$, $\varphi_1^{m'}$, $\varphi_2 \in L_{q,q_0}(Q_T)$ with appropriate exponents $q, q_0$. For the sake of simplicity we shall assume however that $\varphi_0, \varphi_1$ and $\varphi_2$ are some given constants.

Recall that the pioneering results and methods of establishing of Hölder estimates for weak solutions of linear and quasilinear uniformly elliptic and parabolic equations in divergence form appeared in the classical papers by De Giorgi, Nash, Moser and Ladyzhenskaya–Uraltseva.

Hölder continuity of weak solutions of quasilinear uniformly parabolic equations of the type (1.1), (1.9) (i.e., in the case $m = 2$, $l = 0$) were established with the aid of different methods in sixtieth by Ladyzhenskaya–Uraltseva, Aronson–Serrin, author and others (see in particular [15–19]).

Continuity or Hölder continuity of weak solutions of different subclasses of (1.1), (1.9) in the case $m = 2$, $l \neq 0$ were proved only in eightieth in [20–29].

In 1986 the important advancement was made by Di Benedetto who was able to establish Hölder estimates for (1.1), (1.9) in the case $m > 2$, $l = 0$ ([30]). In particular an interesting development of approaches by De Giorgi and Ladyzhenskaya–Uraltseva of establishing of Hölder estimates is given in [30]. Later in [31] Hölder estimates in the singular case $1 < m < 2$, $l = 0$ were obtained under additional assumptions concerning the structure of equation and properties of weak solution.

Hölder estimates for multidimensional ($n > 1$) weak solutions of quasilinear doubly degenerate parabolic equations (i.e., for (1.1), (1.9) in the case $m > 2$, $l > 0$) were obtained in 1989 by myself with the aid of appropriate development of Di Benedetto approaches (see [32–34, 37, 38]). These estimates were used in [35, 36] for proving of existence of nonnegative Hölder continuous weak solutions of Cauchy–Dirichlet problem for (1.1), (1.9) in the case $m > 2$, $l > 0$. In all my papers I considered only nonnegative weak solutions namely because they have physical sense. More precisely I established Hölder estimates for the following (natural) class of weak solutions.

**DEFINITION 1.1.** Any nonnegative bounded in $Q_T$ function $u$ is a *weak solution* [*supersolution, subsolution*] if

a) $u \in C([0,T]; L_1(\Omega))$, $\quad \partial u^{\sigma+1}/\partial x_i \in L_m(Q_T)$, $\quad \sigma = \frac{l}{m-1}$, $\quad i = 1, \ldots, n$;

b) for any $\Phi \in \overset{\circ}{C}{}^1(Q_T)$ and any $t_1, t_2 \in [0,T]$

$$\int_\Omega u\Phi \, dx \,\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega \{-u\Phi_t + \vec{a}(x,t,u,u_x)\cdot\nabla\Phi + a_0(x,t,u,u_x)\Phi\}\, dxdt = 0 \tag{1.10}$$

$$[\geq 0, \ \leq 0]$$

where $u_x = (u_{x_1}, \ldots, u_{x_n})$ and $u_{x_i}$ are defined by

$$u_{x_i} = \begin{cases} (1+\sigma)^{-1} u^{-\sigma} \partial \hat{u} / \partial x_i & \text{in } \{Q_T : u > 0\} \\ 0 & \text{in } \{Q_T : u = 0\} \end{cases}, \quad \hat{u} = u^{\sigma+1} \in W_m^{1,0}(Q_T). \tag{1.11}$$

Remark that Definition 1.1 is available in the general case $m > 1$, $l > 1 - m$. It should be to mention that in the case $n = 1$, $m > 1$, $l > 2 - m$ Hölder continuity in $\mathbb{R}^1 \times [\varepsilon, T]$ for any $\varepsilon > 0$ with the best possible Hölder exponent was proved by Esteban–Vazquez for weak solutions of Cauchy problem (see [39]).

Recently some new investigations of regularity for doubly nonlinear parabolic equations appeared. In preprint [40] in the case $m > 2$, $l > 0$ Hölder estimates for weak solution belonging to $W_m^{1,0}(Q_T)$ are obtained without assumption that weak solution is nonnegative. On the other hand using approaches of paper [31] Vespri in [42] established Hölder estimates in the singular case $1 < m < 2$, $l \neq 0$ under (roughly speaking) the following additional assumptions:

(i) $\vec{a} = \vec{a}(x, u, p)$, $\left[\vec{a}(x, u, p) - \vec{a}(x, u, q)\right] \cdot (p - q) \geq 0$;

(j) functions $u \to a^i(x, u, p)$ are Lipschitz $(i = 1, \ldots, n)$;

(k) $u \in W_m^{1,0}(Q_T)$;

(l) $\partial u / \partial t \in L_2(Q_T)$.

It should be to say that Hölder estimates established in [31, 42] are independent of Lipshitz constants in (j) and norms $\|\nabla u\|_{L_m(Q_T)}$, $\|\partial u/\partial t\|_{L_2(Q_T)}$. Moreover authors of [31, 42] suppose that weak solution under consideration is a weak $W_m^{1,0}(Q_T)$ — limit of solutions $u_\varepsilon$ of some regularized equations for which conditions (1.9) and $(i)$–$(l)$ are fulfilled. But in any case condition (l) is non-pleasant with the point of view of proving of existence of regular solution of Cauchy–Dirichlet problem (because it is difficult to find a regularized problems for which all conditions (1.2) and $(i)$–$(l)$ would be fulfilled). As far as we know up to present nobody has proved the existence of a Hölder continuous weak solution of the Cauchy–Dirichlet problem in the singular case $1 < m < 2$.

Recently I established some new results concerning Hölderness and existence of regular solutions for doubly nonlinear parabolic equations of the type (1.1), (1.2) which I consider here either in the case

$$\text{(SN)} \qquad m > 1, \quad l \geq 0, \quad m + l \geq 2$$

or in the case

$$\text{(F)} \qquad m > 1, \quad 1 < m + l < 2.$$

So in the case (SN) I deal with equations of the type of slow or normal diffusion under additional assumption $l \geq 0$, while in the case (F) I consider the full class of equations of the type of fast diffusion.

At first I establish Hölder estimates. If in the former case the establishment of these estimates requires only the structure conditions (1.9), in the latter case I have to assume beside (1.9) additional conditions (i), (j), (k) (but not (l)). Thus in particular as against of Vespri result [41] I eliminate in the singular case $1 < m < 2$ all additional conditions if $m + l \geq 2$, $l \geq 0$ and condition (l) if $m + l < 2$. These results will be published in [43–45].

We applied these Hölder estimates for proving of existence of Hölder continuous solutions of Cauchy–Dirichlet problem for equations of the type (1.1), (1.2) with parameters satisfying conditions

$$m > \max\bigl(2n/(n+2), 1\bigr), \qquad l \geq 0 .$$

Moreover we proved that regular solution of Cauchy–Dirichlet problem is unique. In particular existence and uniqueness of regular solution of Cauchy–Dirichlet problem is established for nonhomogeneous equations with the principal parts like in (1.3) and (1.4) (see [43], [46]).

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Received November 29, 1993

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