Karel Rektorys

Numerical and theoretical treating of evolution problems by the method of discretization in time

In: Jaromír Vosmanský and Miloš Zlámal (eds.): Equadiff 6, Proceedings of the International Conference on Differential Equations and Their Applications held in Brno, Czechoslovakia, Aug. 26 - 30, 1985. J. E. Purkyně University, Department of Mathematics, Brno, 1986. pp. 71--84.

Persistent URL: http://dml.cz/dmlcz/700157

NUMERICAL AND THEORETICAL TREATING OF EVOLUTION PROBLEMS BY THE METHOD OF DISCRETIZATION IN TIME

K. REKTORYS
Technical University Prague
Thákurova 7, 166 29 Prague 6, Czechoslovakia

More than fifty years ago, E. Rothe suggested an approximate method of solution of parabolic problems. He divided the interval $I = [0,T]$ for the variable t into p subintervals I_j of the length $h = T/p$ and at each point $t_j = jh$, $j = 1,\dots,p$, he approximated the function $u(x,t_j)$ by the function $z_j(x)$ and the derivative $\partial u/\partial t$ by the difference quotient $[z_j(x) - z_{j-1}(x)]/h$. Starting with z_0 given by $z_0(x) = u(x,0) = u_0(x)$, he found, successively for $j = 1,\dots,p$, the approximations $z_j(x)$ as solutions of the so arisen ordinary boundary value problems. The problem, solved originally by E. Rothe, was a very simple one. However, his method turned out to be a very useful tool for solution of substantially more complicated evolution problems (at first linear and quasilinear parabolic problems of the second order in n dimensions, later parabolic problems of arbitrary order, nonlinear problems, hyperbolic problems, the Stephan problem, integrodifferential problems, mixed parabolic-hyperbolic problems, etc.). The development of the Rothe method, called also the method of discretization in time, or the horizontal method of lines, is connected with such names as O. A. Ladyženskaja, T. D. Ventcel, A. M. Iljin, A. S. Kalašnikov, O. A. Olejnik, J. I. Ibragimov, P. S. Mosolov, O. A. Liskovec, R. D. Richtmayer, N. N. Janěnko, M. Zlámal, J. Nečas, J. Kačur, A. G. Kartsatos, M. E. Parrot, W. Ziegler, J. W. Jerome, E. Martensen and his school, U. v. Welck, J. Naumann, C. Corduneanu, etc. Theoretical as well as numerical questions have been examined (existence and convergence theorems, regularity questions, numerical aspects, etc.). Many of the obtained results were obtained as well by other methods - method of compactness, theory of semigroups, method of monotone operators, Fourier transform, etc. (A. Friedman, M. Krasnoselskij, P. E. Sobolevskij, F. E. Browder, J. L. Lions, E. Magenes, H. Brézis, V. Barbu, D. Pascali, M. G. Crandal, W. v. Wahl, etc.). As concerns numerical methods, related to the Rothe method, the methods of space - or time-space discretization were applied (V. N. Fadějeva, J. Douglas, T. Dupont, M. Zlámal, R. Glowinski, J. L. Lions,

R. Tremolière, P. A. Raviart, W. Walter, K. Gröger, etc.). Each of the mentioned methods, including the Rothe method, has its preferences and its drawbacks. However, the Rothe method has its significance both as a numerical method and theoretical tool. Existence theorems are proved in a constructive way. Thus no other methods are needed to give preliminary information on existence, or regularity of the solution as required in many other numerical methods when questions on convergence, or order of convergence, etc. are to be answered. The Rothe method is a stable method. To the solution of elliptic problems generated by this method, current methods, especially the variational ones, can be applied. As concerns theoretical results, they are obtained in a relatively simple way, as usual. Moreover, the Rothe method, being a very natural one, makes it possible to get a particularly good insight into the structure of the solutions. Often a brief inspection of the corresponding elliptic problems gives an information what can be expected as concerns properties of the solution. This is why I prefer it.

In 1971, a slightly different technics than that applied currently in this method appeared in my work [2], making it possible to treat corresponding elliptic problems in a particularly simple way. This technics was followed by other authors (in our country J. Nečas, J. Kačur) and became a base for work of my seminar at the Technical University in Prague. Results obtained in this seminar were summarized in my monograph [1] in 1982. I would like to present some of them here, pointing out the very simple way in which they have been obtained.

1. Existence and convergence theorem. Let us start with a relatively simple parabolic problem

$$\frac{\partial u}{\partial t} + Au = f \quad \text{in} \quad G \times (0,T) \ , \tag{1}$$

$$u(x,0) = 0 \ , \tag{2}$$

$$B_i u = 0 \quad \text{on} \quad \Gamma \times (0,T) \ , \quad i = 1,\dots,\mu \ , \tag{3}$$

$$C_i u = 0 \quad \text{on} \quad \Gamma \times (0,T) \ , \quad i = 1,\dots,k-\mu \ . \tag{4}$$

Here, G is a bounded region in E_N with a Lipschitz boundary Γ ,

$$A = \sum_{|i|,|j|\leq k} (-1)^{|i|} D^i\left(a_{ij}(x)D^j\right) \tag{5}$$

with a_{ij} bounded and measurable in G , $f \in L_2(G)$; (3), or (4) are (linear) boundary conditions, stable (thus containing derivatives of orders $\leq k-1$), or unstable with respect to the operator A , respectively. Denote

$$V = \{v;\ v \in W_2^{(k)}(G),\quad B_i v = 0 \text{ on } \Gamma \text{ in the sense of traces}, \quad i = 1,\dots,\mu\}, \tag{6}$$

let $((.,.))$ be the bilinear form, corresponding to the operator A and to the boundary conditions (3), (4), familiar from the theory of variatiinal methods. (Roughly speaking, $((v,u))$ is obtained of (v,Au), applying to every integral $\int_G v\, D^i(a_{ij}\, D^j u)\, dx$ i-times the Green theorem in the usual way, see e.g. [1], or [3]. For example, if $A = -\Delta$ and $u = 0$ on Γ is prescribed, then

$$V = \overset{o}{W}{}_2^{(1)}(G) \quad \text{and} \quad ((v,u)) = \sum_{i=1}^{N} \int_G \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_i}\, d\mathbf{x}\ .)$$

Let the form $((.,.))$ be bounded in $V \times V$ and V-elliptic, i.e. let two positive constants K and α (independent of v and u) exist such that the inequalities

$$|((v,u))| \leqq K||v||_V||u||_V\ , \tag{7}$$

$$((v,v)) \geqq \alpha||v||_V^2 \tag{8}$$

hold for all $v, u \in V$. Let to the solution of the problem (1) - (4) the Rothe method be applied. Denote

$$Z_i(x) = \frac{z_i(x) - z_{i-1}(x)}{h}\ , \quad i = 1,\dots,p\ . \tag{9}$$

(Thus $Z_i(x)$ "corresponds" to the derivative $\partial u/\partial t$ for $t = t_i$.) In the weak formulation, we have to find successively for $j = 1,\dots,p$, the functions $z_j \in V$, satisfying,

$$((v,z_j)) + (v,Z_j) = (v,f) \quad \forall\, v \in V\ , \tag{10}$$

with $z_0(x) = u(x,0) = 0$. Under the assumptions (7), (8), each of these problems has exactly one solution $z_j \in V$. <u>Apriori estimates</u>: Put $v = Z_1 = (z_1 - z_0)h = z_1/h$ into (10) written for $j = 1$. We obtain

$$h((Z_1,Z_1)) + (Z_1,Z_1) = (Z_1,f)\ . \tag{11}$$

Because of (8) and $|(Z_1,f)| \leqq ||Z_1||\ ||f||$, (11) yields

$$||Z_1||^2 \leqq ||Z_1||\ ||f|| \Longrightarrow ||Z_1|| \leqq ||f||\ . \tag{12}$$

Subtracting (10), written for $j - 1$, from (10) gives

$$h((v,Z_j)) + (v,\ Z_j - Z_{j-1}) = 0\ .$$

Putting $v = Z_j$, we obtain, in a similar way as before,

$$||Z_j|| \leqq ||Z_{j-1}||\ , \quad j = 2,\dots,p\ ,$$

what gives, together with (12)

$$||z_j|| \leq ||f|| = c_1 \ . \tag{13}$$

Let us refine our division, considering the divisions d_n with the steps $h_n = h_1/2^{n-1}$, $n = 1,2,\ldots$, $h_1 = h$. Denote the corresponding functions

$$z_j^n \ , \quad Z_j^n = \frac{z_j^n - z_{j-1}^n}{h_n} \ .$$

The estimate (13) having been obtained independently of the lengh of the step h , it remains valid as well for the division d_n ,

$$||Z_j^n|| \leq c_1 \ . \tag{14}$$

Because $z_j^n = (z_j^n - z_{j-1}^n) + \ldots + (z_1^n - z_0^n)$, it follows

$$||z_j^n|| \leq jh_n(||Z_j|| + \ldots + ||Z_1||) \leq Tc_1 = c_2 \ . \tag{15}$$

Putting then $v = z_j^n$ into (10) written for the functions z_j^n and Z_j^n and using (8), we get

$$||z_j^n||_V \leq c_3 \ . \tag{16}$$

(14), (15) and (16) are the basic needed a priori estimates. They have actually been obtained in a very, very simple way. What follows, is a standard procedure, now. Let

$$\begin{aligned} u_n(t) &= z_{j-1}^n + (z_j^n - z_{j-1}^n)\,\frac{t - t_{j-1}^n}{h} \\ &\quad \text{for} \quad t_{j-1}^n \leq t \leq t_j^n \ , \quad j = 1,\ldots,p\cdot 2^{n-1} \end{aligned} \tag{17}$$

($n = 1,2,\ldots$) (the so-called Rothe functions), or

$$U_n(t) = \begin{cases} Z_1^n & \text{for} \quad t = 0 \ , \\ Z_j^n & \text{for} \quad t_{j-1}^n < t \leq t_j^n \ , \quad j = 1,\ldots,p\cdot 2^{n-1} \end{cases} \tag{18}$$

($n = 1,2,\ldots$) be abstract funcions, considered as functions from $I = [0,T]$ into V , or $L_2(G)$, respectively. In consequence of their form and of (16) and (14), they are uniformly bounded (with respect to n) in $L_2(I,V)$, or $L_2(I,L_2(G))$, respectively (even in $C(I,V)$, or $L_\infty(I,L_2(G))$). The space $L_2(I,V)$, and $L_2(I,L_2(G))$ being Hilbert spaces, a subsequence $\{u_{n_k}\}$, or $\{U_{n_k}\}$ can be found such that

$$u_{n_k} \rightharpoonup u \quad \text{in} \quad L_2(I,V) \ , \quad U_{n_k} \rightharpoonup U \quad \text{in} \quad L_2(I,L_2(G)) \ . \tag{19}$$

Now, (17), (18) imply

$$\int_0^t U_n(\tau)d\tau = u_n(t) \ \forall n ,$$

yielding easily

$$\int_0^t U(\tau)d\tau = u(t) ,$$

and thus $U = u'$ in $L_2(I,L_2(G))$. Consequently, $u \in AC(I,L_2(G))$ and $u(0) = 0$ in $C(I,L_2(G))$. So the function u satisfies

$$u \in L_2(I,V) \cap AC(I,L_2(G)), \tag{20}$$

$$u' = U \in L_2(I,L_2(G)), \tag{21}$$

$$u(0) = 0 \text{ in } C(I,L_2(G)). \tag{22}$$

Moreover, on base of integral identities (10) and of (19), one comes to the integral identities

$$\int_0^T ((v,u))\,dt + \int_0^T (v,u')\,dt = \int_0^T (v,f)\,dt. \tag{23}$$

A function with the properties (20) - (23) is called the <u>weak solution</u> of the problem (1) - (4).

<u>Uniqueness</u>: Let u_1, u_2 be two functions satisfying (20) - (23). Then their difference $u = u_2 - u_1$ has the properties (20) - (22) and satisfies

$$\int_0^T ((v,u))\,dt + \int_0^T (v,u')\,dt = 0 \quad \forall\, v \in L_2(I,V).$$

Let $a \in I$ be arbitrary. Choose

$$v(t) = \begin{cases} u(t) & \text{for } 0 \leqq t \leqq a, \\ 0 & \text{for } a < t \leqq T. \end{cases}$$

We have

$$\int_0^T (v,u')\,dt = \int_0^a (u,u')\,dt = \tfrac{1}{2}\|u(a)\|^2 - \tfrac{1}{2}\|u(0)\|^2 = \tfrac{1}{2}\|u(a)\|^2.$$

In consequence of (8) we thus obtain $\|u(a)\|^2 = 0$; the point a having been chosen arbitrarily, $u = 0$ in I.

Uniqueness implies in the familiar way that $u_n \rightharpoonup u$ (not only $u_{n_k} \rightharpoonup u$) in $L_2(I,V)$. Moreover, for every $t \in I$ the sequence $\{u_n(x)\}$ is bounded in V and thus compact in $L_2(G)$. The functions $u_n(t)$ being uniformly

bounded in I, in the metric of the space $L_2(G)$, and equicontinuous on base of (13), the Ascoli theorem can be applied, implying (strong) uniform convergence, in I, of $\{u_n\}$ to u.
Summarising, we thus have:

Theorem 1. Let (7), (8) be satisfied, let $f \in L_2(G)$. Then there exists exactly one weak solution of the problem (1) - (4) and

$$u_n \rightharpoonup u \text{ in } L_2(I,V), \quad u_n \rightrightarrows u \text{ in } C(I,L_2(G)). \tag{24}$$

Remark 1. By a more detailed treatment it can be proved that even $u_n \rightrightarrows u$ in $C(I,V)$, $u' \in L_\infty(I,L_2(G))$. We shall not go into details here. See [1].

2. Error Estimates. As can be excepted, to get an efficient error estimate, some supplementary assumptions are needed: Let the assumptions of Theorem 1 be fulfilled. Let, moreover,

$$f \in V, \quad Af \in L_2(G), \quad ((v,f)) = (v,Af) \ \forall \ v \in V. \tag{25}$$

Then

$$\|u(x,t_j) - z_j(x)\| \leq \frac{Mjh^2}{2}, \quad j = 1,\dots,p, \tag{26}$$

where $M = \|Af\|$. If, moreover, the coefficient C of positive definiteness can be easily found, for which thus

$$((v,v)) \geq C^2 \|v\|^2 \quad \forall \ v \in V$$

holds, then the following (slightly better) estimate can be used:

$$\|u(x,t_j) - z_j(x)\| \leq \frac{Mh}{2C^2}\left(1 - e^{-C^2 jh}\right), \quad j = 1,\dots,p. \tag{27}$$

Proof of (26) (the proof of (27) is similar): Let us investigate the division d_2 (for the notation see the text following (13)) and denote

$$z_{2i}^2 - z_i = q_i^1, \quad i = 0,1,\dots,p$$

(with $z_i^1 = z_i$). The functions z_{2i}^2 satisfy the integral identities

$$((v,z_{2i}^2)) + \frac{1}{h/2}(v,z_{2i}^2 - z_{2i-1}^2) = (v,f) \ \forall \ v \in V.$$

Subtracting the integral identity (10), with i written for j, we obtain

$$((v,z_{2i}^2 - z_i)) + \frac{1}{h}(v,(z_{2i}^2 - z_i) - (z_{2i-2}^2 - z_{i-1})) =$$

$$= - \frac{1}{h}(v,z_{2i}^2 - 2z_{2i-1}^2 + z_{2i-2}^2) \ \forall \ v \in V,$$

or, denoting

$$s_i^2 = \frac{z_i^2 - 2z_{i-1}^2 + z_{i-2}^2}{(h/2)^2} \quad (s_i^n = \frac{z_i^n - 2z_{i-1}^n + z_{i-2}^n}{h_n^2}, \text{ in general}),$$

$$((v,q_i^1)) + \frac{1}{h}(v,q_i^1 - q_{i-1}^1) = -\frac{h}{4}(v,s_{2i}^2) \ \forall\ v \in V,$$

with $q_0^1 = z_0^2 - z_0 = 0$. Putting $v = q_1^1$ for $i = 1$ and taking (8) into account, we get

$$||q_1^1|| \leq \frac{h^2}{4} ||s_2^2|| \leq \frac{h^2 M}{4},$$

because under the assumptions (25)

$$||s_i^n|| \leq M \text{ for all } n \text{ and } i.$$

Similarly, for $i = 2$ we obtain

$$||q_2^1|| \leq ||q_1^1|| + \frac{h^2 M}{4} \leq \frac{2h^2 M}{4},$$

and, finally,

$$||q_i^1|| \leq \frac{ih^2 M}{4}, \quad i = 0,1,\dots,p.$$

In general, we have, for $q_i^n = z_{2i}^{n+1} - z_i^n$,

$$||q_i^n|| \leq \frac{i(h/2^{n-1})^2 M}{4}, \quad i = 0,1,\dots,2^{n-1}p.$$

Now,

$$||u_n(x,t_j) - z_j(x)|| = ||z^n_{2^{n-1}j} - z_j^1|| \leq$$

$$\leq ||q^{n-1}_{2^{n-2}j} + q^{n-2}_{2^{n-3}j} + \dots + q^2_{2j} + q^1_j|| \leq$$

$$\leq \frac{h^2 M}{4}\left(j + \frac{2j}{2^2} + \dots + \frac{2^{n-1}j}{(2^{n-1})^2}\right) < \frac{jh^2 M}{2}.$$

Coming to the limit for $n \to \infty$ (what is allowed because of Theorem 1), we obtain (26).

The estimate (26) is very sharp (and the more is the estimate (27)), as can be seen from the following simple example:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \sin x \text{ in } (0,\pi) \times (0,1), \tag{28}$$

$$u(x,0) = 0, \tag{29}$$

$$u(0,t) = 0, \quad u(\pi,t) = 0. \tag{30}$$

The assumptions (25) are easily established. Further, we have

$$M = ||Af|| = ||-(\sin x)''|| = ||\sin x|| = \sqrt{(\pi/2)} .$$

Choosing, for, example, $h = 0.01$ and $j = 20$, or $j = 40$, the Rothe method yields

$$z_{20} = 0.1805 \sin x ,$$
$$z_{40} = 0.3284 \sin x ,$$

respectively. The exact solution is known (this was the reason why such a simple example has been chosen): $u = (1 - e^{-t}) \sin x$. Thus

$$u(x, 0.20) = 0.1813 \sin x ,$$

$$u(x, 0.40 = 0.3297 \sin x .$$

The actual errors then are

$$\begin{aligned} ||u(x, 0.20) - z_{20}|| &= 0.0008||\sin x|| = 0.00101 , \\ ||u(x, 0.40) - z_{40}|| &= 0.0013||\sin x|| = 0.00163 , \end{aligned} \tag{31}$$

while (26) gives

$$\begin{aligned} ||u(x, 0.20) - z_{20}|| &\leqq \frac{20\cdot 0.01^2}{2} \sqrt{\frac{\pi}{2}} = 0.00125 , \\ ||u(x, 0.40) - z_{40}|| &\leqq \frac{40\cdot 0.01^2}{2} \sqrt{\frac{\pi}{2}} = 0.00251 . \end{aligned} \tag{32}$$

Finding $C = 1$ (see [1], p. 90) and using (27), we get the estimates

$$\begin{aligned} ||u(x, 0.20) - z_{20}|| &\leqq \frac{0.01}{2} (1 - e^{-0.2})\sqrt{\frac{\pi}{2}} = 0.00113 , \\ ||u(x, 0.40) - z_{40}|| &\leqq \frac{0.01}{2} (1 - e^{-0.4})\sqrt{\frac{\pi}{2}} = 0.00206 \end{aligned} \tag{33}$$

which are still better then the estimates (32). The example demonstrates very well the sharpness of the estimates (26), (27) and the fact that they cannot be substantially improved.

3. Nonhomogeneous initial and boundary conditions. Let us first investigate the problem (1) - (4) with homogeneous equation and nonhomogeneous initial condition $u(x,0) = u_0 \in L_2(G)$, i.e. the problem

$$\frac{\partial u}{\partial t} + Au = 0 \quad \text{in} \quad G \times (0,T) , \tag{34}$$

$$u(x,0) = u_0(x) \tag{35}$$

$$B_i u = 0 \quad \text{on} \quad \Gamma \times (0,T) , \quad i = 1,\dots,\mu , \tag{36}$$

$$G_i u = 0 \quad \text{on} \quad \Gamma \times (0,T) , \quad i = 1,\dots,k-\mu . \tag{37}$$

The corresponding integral identities are

$$((v,z_j)) + (v,Z_j) = 0 \quad \forall v \in V , \quad j = 1,\dots,p , \tag{38}$$

with $Z_j = (z_j - z_{j-1})/h$, $z_0 = u_0$. (39)

Similarly as in the case of the problem (1) - (4), we come to the inequality

$$||Z_j|| \leq ||Z_{j-1}|| .$$

However, if only $u_0 \in L_2(G)$ is assumed, it is not possible to put $v = Z_1$ into the first of the integral identities (38) to obtain a simple estimate for $||Z_1||$ as in (11), (12), because we have not $Z_1 \in V$ here, in general. Thus an existence theorem is derived, first for "sufficiently smooth" $u_0 = s \in V$, more precisely for u_0 from the set M of such functions $s \in V$ for which a unique $g \in L_2(G)$ exists satisfying

$$((v,s)) = (v,g) \quad \forall\ v \in V.$$

Putting then

$$z_j = r_j + s$$

into the integral identities

$$((v,z_j)) + \frac{1}{h}(v,z_j - z_{j-1}) = 0 \quad \forall\ v \in V,$$

with $z_0 = s$, we come to the identities

$$((v,r_j)) + \frac{1}{h}(v,r_j - r_{j-1}) = -(v,g) \quad \forall\ v \in V,$$

with $r_0 = 0$, corresponding to the problem (1) - (4) in which u is replaced by r and f by -g. Having obtained its weak solution r(t), the weak solution of (34) - (37) with $u_0 = s \in M$ is defined by $u(t) = r(t) + s$. Moreover, if we put z_j for v into the original integral identities, we obtain, subsequently,

$$||z_1|| \leq ||s||, \quad ||z_2|| \leq ||z_1|| \leq ||s||, \text{ etc.}$$

The function u(t) being the limit, in $C(I,L_2(G))$, of the corresponding Rothe sequence, it follows

$$||u(t)|| \leq ||s|| \quad \text{for all } t \in I.$$

Now, the form ((v,u)), being V-elliptic, the set M is dense in V, thus as well in $L_2(G)$. Let $u_0 \in L_2(G)$ and let $s_i \in M$, $i = 1,2,\ldots$, be (an arbitrary) sequence converging to u_0 in $L_2(G)$. Then the sequence of corresponding weak solutions $u^{(i)}(t)$ is a Cauchy sequence in $C(I,L_2(G))$, because

$$||u^{(j)}(t) - u^{(k)}(t)|| \leq ||s_i - s_k|| \quad \forall\ t \in I.$$

Its limit is then called the <u>very weak solution</u> of the problem (34) - (37). Obviously, this very weak solution is uniquely determined by the initial function $u_0 \in L_2(G)$.
About convergence, in $C(I, L_2(G))$, of the corresponding Rothe sequence to this very weak solution as well about nonhomogeneous boundary conditions see [1].

<u>4. The Ritz-Rothe method.</u> Let us investigate the problem (34) - (37) with $f \in L_2(G)$ on the right-hand side of (34) instead of zero. The solution $u(t)$ of this problem is the sum of solutions of the problems (1) - (4) and (34) - (37). (The problems are linear.) The corresponding integral identities when applying the Rothe method are:

$$((v,z_j)) + \frac{1}{h}(v, z_j - z_{j-1}) = (v,f) \quad \forall v \in V, \quad j = 1,\dots,p, \tag{40}$$

with $z_0 = u_0$. Let us solve each of these problems approximately - to be concrete, by the Ritz method (or by a method with similar properties). So let $v_1,\dots,v_n$ be the first n terms of a base in V and let z_1^* be the Ritz approximation of the function z_1. Put z_1^* instead of z_1 into the second of the identities (40),

$$((v,\tilde{z}_2)) + \frac{1}{h}(v, \tilde{z}_2 - z_1^*) = (v,f) \tag{41}$$

and let z_2^* be the Ritz approximation of the function $\tilde{z}_2$, etc. We thus can construct the function

$$u_1^*(t) = z_{j-1}^* + \frac{z_j^* - z_{j-1}^*}{h}(t - t_{j-1}) \quad \text{for} \quad t_{j-1} \leqq t \leqq t_j, \quad j = 1,\dots,p \tag{42}$$

which is an analogue of the Rothe function $u_1(t)$. (41) announces that using the Ritz method, the errors become cumulated with increasing j. Fortunately, according to a very simple law: Subtract (41) from the second of the identities (40). We obtain

$$((v, z_2 - \tilde{z}_2)) + \frac{1}{h}(v, (z_2 - \tilde{z}_2) - (z_1 - z_1^*)) = 0 \quad \forall v \in V.$$

Putting $v = z_2 - \tilde{z}_2$, we get

$$||z_2 - \tilde{z}_2|| \leqq ||z_1 - z_1^*||.$$

Etc. Using this result, convergence of this "Ritz-Rothe" method is easily proved: Let $\varepsilon > 0$ be given. According to Theorem 1, such a (fine) division of the interval I into p subintervals can be found - let us preserve the notation h for the length of these subintervals - that

$$||u(t) - u_1(t)|| < \frac{\varepsilon}{2} \quad \forall\ t \in I$$

(where $u_1(t)$ is the corresponding Rothe function). Denote $\eta = \varepsilon/2p$. Let the number of terms in the Ritz approximation be sufficiently large so that

$$||z_1 - z_1^*|| < \eta$$

be fulfilled. Then - as just shown -

$$||z_2 - \tilde{z}_2|| < \eta.$$

Let the Ritz approximation z_2^* of $\tilde{z}_2$ be such that

$$||\tilde{z}_2 - z_2^*|| < \eta$$

again. Thus

$$||z_2 - z_2^*|| < 2\eta.$$

In a similar way we come to the estimates

$$||z_j - z_j^*|| < j\eta = \frac{\varepsilon}{2}, \quad j = 3,\dots,p.$$

Because of the form of the Rothe functions $u_1(t)$ and $u_1^*(t)$ (they are piecewise linear in t), we have

$$||u(t) - u_1^*(t)|| < \varepsilon \quad \forall\ t \in I.$$

5. Regularity of the solution. a) Regularity with respect to t, smoothing effect. In [1], Chap. 12 and 13, regularity properties of the weak, or very weak solutions with respect to t are examined. We shall not go into details and show the very simple idea of these investigations on the example of the problem (34) - (37). Let the form $((\cdot,\cdot))$ satisfy (7) and (8) (boundedness and V-ellipticity) and let it be, moreover, symmetric in V, i.e. let

$$((v,u)) = ((u,v)) \quad \forall\ v,\ u \in V \tag{43}$$

be fulfilled. Thus $((\cdot,\cdot))$ has the properties of a scalar product. Let h be sufficiently small (in order that the points t^0, $2t^0$, etc., investigated below, lie in the interval $[0,T]$) and choose an arbitrary $t^0 \in (0,T)$ such that $t^0 = jh$ (j being a positive integer). Take the first of the integral identities (38) and put $v = z_1$. We obtain

$$h((z_1,z_1)) + (z_1,\ z_1 - u_0) = 0\ .$$

Writing $(z_1,\ z_1 - u_0)$ in the form $\frac{1}{2}\left(||z_1||^2 + ||z_1 - u_0||^2 - ||u_0||^2\right)$, we get

$$h((z_1,z_1)) + \frac{1}{2}\ ||z_1||^2 \leqq \frac{1}{2}\ ||u_0||^2\ .$$

Similarly,

$$h((z_2,z_2)) + \frac{1}{2}\,||z_2||^2 \leqq \frac{1}{2}\,||z_1||^2 \,,$$

$$\cdots\cdots\cdots$$

$$h((z_j,z_j)) + \frac{1}{2}\,||z_j||^2 \leqq \frac{1}{2}\,||z_{j-1}||^2 \,.$$

Making the sum, we obtain

$$h\sum_{i=1}^{j}((z_i,z_i)) \leqq \frac{1}{2}\,||u_0||^2 \,. \tag{44}$$

Putting, in the second of the identities (38), $v = z_2 - z_1$, we get, similarly,

$$((z_2 - z_1,\, z_2)) + \frac{1}{h}\,(z_2 - z_1,\, z_2 - z_1) = 0 \,,$$

$$\frac{1}{2}\,[((z_2,z_2)) + ((z_2 - z_1,\, z_2 - z_1)) - ((z_1,z_1))] \leqq 0 \,,$$

$$((z_2,z_2)) \leqq ((z_1,z_1)) \,.$$

Going on in this way, we obtain

$$((z_j,z_j)) \leqq ((z_{j-1},z_{j-1})) \leqq \dots \leqq ((z_2,z_2)) \leqq ((z_1,z_1)) \,. \tag{45}$$

Thus replacing in (44) all the summands by $((z_j,z_j))$ and taking into account that $jh = t^0$, we have

$$((z_j,z_j)) \leqq \frac{1}{2t^0}\,||u_0||^2 \,, \tag{46}$$

and, because of the V-ellipticity of the form $((.,.))$ (see (8)),

$$||z_j||_V \leqq \frac{1}{\sqrt{(2\alpha t^0)}}\,||u_0|| \,. \tag{47}$$

In consequence of (45), this result holds for all larger indices, too, and also for all divisions d_n with $n \geqq 1$,

$$||z_i^n||_V \leqq \frac{1}{\sqrt{(2\alpha t^0)}}\,||u_0|| \quad \forall\, t_i^n \geqq t^0 \,. \tag{48}$$

Using this result, interchanging the role of z_i and Z_i and assuming $2t^0 \in (0,T)$, we get, similarly,

$$||Z_i^n|| \leqq \frac{1}{2t^0}\,||u_0|| \quad \forall\, t_i^n \geqq 2t^0 \,. \tag{49}$$

A simple consideration leads then to the conclusion that for the restrictions $\tilde{u}(t)$ and $\tilde{u}'(t)$ of the functions $u(t)$ and $u'(t)$ on the interval $[2t^0,T]$, (48), (49) imply

$$\tilde{u} \in L_2([2t^0,T],V) \,, \quad \tilde{u}' \in L_2([2t^0,T],L_2(G)) \,.$$

Going on in the same way, we prove similarly (assuming $4t^0 \in (0,T)$)

$$\tilde{u}' \in L_2([4t^0,T],V) \,, \quad \tilde{u}'' \in L_2([4t^0,T],L_2(G)) \,,$$

etc. Let $\eta \in (0,T)$ be arbitrary, $q > 0$ an arbitrary integer. Chosing $t^0 \leqq \eta/(2q + 4)$, we come, in this way, to the result that

$$\tilde{u}^{(i+1)} \in L_2([\eta,T],V) \ , \quad \tilde{u}^{(i+2)} \in L_2([\eta,T],L_2(G)) \ , \quad i = 0,\dots,q \ ,$$

what implies, among others,

$$\tilde{u}^{(i)} \in AC([\eta,T],V) \ , \quad \tilde{u}^{(i+1)} \in AC([\eta,T],L_2(G)) \ , \quad i = 0,\dots,q \ .$$

The numbers η and q having been chosen arbitrarily, we have come, in this very simple way, to the following

Theorem 2. Let (7), (8), (43) be fulfilled, $u_0 \in L_2(G)$. Then the very weak solution $u(t)$ of the problem (34) - (37), considered as an abstract function $[0,T] \to V$, has on the interval $(0,T]$ continuous derivatives of all orders.

Let us remark that applying the just shown idea in a properly modified way, J. Kačur obtained rather strong regularity results for the equation $\partial u/\partial t + A(t)u = f(t)$. See [4].

b) Regularity with respect to x. While the basic method how to examine regularity with respect to t has been shown in [1], the idea how to obtain regularity results with respect to x belongs to J. Kačur. Regularity results known for elliptic problems are utilized. For details see [4].

6. Other parabolic problems. Using the same technics as above, linear parabolic equations of the form $\partial u/\partial t + A(t)u = f(t)$, nonlinear equations, integrodifferential equations as well as some nontraditional problems (problem with an integral condition, for example) can be examined. For details see [1].

7. Hyperbolic problems. Also hyperbolic problems can be treated in the same way. Under the assumption of boundedness in $V \times V$, V-ellipticity and V-symmetry of the form $((.,.))$, an existence and convergence theorem has been derived, in [1], and convergence of the "Ritz-Rothe method" proved. Moreover, for $f \in V$, the following error estimate has been found, in a similar way as in the case (26):

$$||u(x,t_j) - z_j(x)|| \leqq Mh^3\, j(j + 1) \quad \text{with} \quad M = \sqrt{\tfrac{3}{2}\,((f,f))}. \tag{50}$$

For some regularity results see [5]. For generalization to the case of quasilinear hyperbolic equations see [4].

R e f e r e n c e s

[1] REKTORYS, K.: *The Method of Discretization in Time and Partial Differential Equations*. Dordrecht-Boston-London, D. Reidel 1982.

[2] REKTORYS, K.: *On Application of Direct Variational Methods to the Solution of Parabolic Boundary Value Problems of Arbitrary Order.* Czech. Math. J. 21 (1971), 318-339.

[3] REKTORYS, K.: *Variational Methods in Mathematics, Science and Engineering,* 2nd Ed. Dordrecht-Boston-London, D. Reidel 1979.

[4] KAČUR, J.: *Method of Rothe in Evolution Equations.* Leipzig, Teubner. To appear.

[5] PULTAR, M.: *Solution of Abstract Hyperbolic Equations by Rothe Method.* Aplikace matematiky 29, (1984), 23-39.