# STABILITY AND BIFURCATION PROBLEMS FOR REACTION-DIFFUSION SYSTEMS WITH UNILATERAL CONDITIONS

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Let us consider a reaction-diffusion system of the type

$$\begin{aligned} u_t &= d\Delta u + f(u,v) \\ v_t &= \Delta v + g(u,v) \end{aligned} \quad \text{on} \quad \langle 0,\infty) \times \Omega \tag{RD}$$

where $f$, $g$ are real smooth functions on $R^2$, $d$ is a nonnegative parameter (diffusion coefficient), $\Omega$ is a bounded domain in $R^n$. Suppose that $\bar{u}, \bar{v} > 0$ is an isolated solution of $f(\bar{u},\bar{v}) = g(\bar{u},\bar{v}) = 0$. Thus, $\bar{u}$, $\bar{v}$ is a stationary spatially homogeneous solution of (RD) with Neumann boundary conditions and also with the boundary conditions

$$u = \bar{u} \quad \text{on} \quad (0,\infty) \times \Gamma_D, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad (0,\infty) \times \Gamma_N, \tag{BC$_1$}$$

$$v = \bar{v} \quad \text{on} \quad (0,\infty) \times \Gamma_D, \quad \frac{\partial v}{\partial n} = 0 \quad \text{on} \quad (0,\infty) \times \Gamma_N, \tag{BC$_2$}$$

where $\Gamma_D \cup \Gamma_N = \partial\Omega$ (the boundary of $\Omega$). Set $b_{11} = \frac{\partial f}{\partial u}(\bar{u},\bar{v})$, $b_{12} = \frac{\partial f}{\partial v}(\bar{u},\bar{v})$, $b_{21} = \frac{\partial g}{\partial u}(\bar{u},\bar{v})$, $b_{22} = \frac{\partial g}{\partial v}(\bar{u},\bar{v})$, $B = \begin{pmatrix} b_{11}, b_{12} \\ b_{21}, b_{22} \end{pmatrix}$ and suppose

$$b_{11} > 0,\ b_{21} > 0,\ b_{12} < 0,\ b_{22} < 0,\ b_{11} + b_{22} < 0,\ \det B > 0. \tag{B}$$

Then there exists the greatest bifurcation point $d_0$ of (RD), (BC) at which spatially nonhomogeneous stationary solutions bifurcate from the branch of trivial solutions $\{[d,\bar{u},\bar{v}];\ d \in R\}$; the solution $\bar{u}$, $\bar{v}$ is stable for any $d > d_0$ and unstable for any $0 < d < d_0$. (All eigenvalues of the corresponding linearized problem have negative real parts for $d > d_0$ and there exists a positive eigenvalue for $d < d_0$.) For the case of Neumann conditions and $n = 1$ (i.e. $\Omega = (0,1)$) see e.g. [8], for the general case see [2]. Notice that (B) is fulfilled in models connected with population dynamics, chemistry, morphogenesis etc. In these cases $u$ represents a prey or an activator, $v$ represents a predator or an inhibitor. The existence of stationary spatially nonhomogeneous solutions explains the occurence of the so called striking patterns.

The aim of this lecture is to present some results obtained by the author together with P. Drábek, M. Míková and J. Neustupa, showing how this situation can change by introducing unilateral conditions given by a cone in a suitable Hilbert space. One of the simplest examples are unilateral boundary conditions

$$\begin{aligned} &v = \bar{v} \quad \text{on} \quad (0,\infty) \times \Gamma_D \, , \quad v \geqq \bar{v} \, , \quad \frac{\partial v}{\partial n} \geqq 0 \, , \\ &(v - \bar{v})\frac{\partial v}{\partial n} = 0 \quad \text{on} \quad (0,\infty) \times \Gamma_0 \, , \quad \frac{\partial v}{\partial n} = 0 \quad \text{on} \quad (0,\infty) \times (\Gamma_N \setminus \Gamma_0) \end{aligned} \tag{1}$$

where $\Gamma_0$ is a given subset of $\Gamma_N$. Roughly speaking, the spatially homogeneous solution $\bar{u}$, $\bar{v}$ becomes unstable even for some $d > d_0$ and the greatest bifurcation point shifts to the right of $d_0$ if classical conditions are replaced by unilateral ones for $v$; the greatest bifurcation point shifts to the left if unilateral conditions are introduced for $u$.

In what follows, we shall suppose $\bar{u} = \bar{v} = 0$ without loss of generality.

## 1. Abstract formulation.

Let $K$ be an arbitrary closed convex cone in the space $V = \{u \in W_2^1(\Omega)\,;\ u = 0 \ \text{on} \ \Gamma_D\}$ with its vertex at the origin. Consider the problem

$$\begin{aligned} &\int_\Omega [u_t(t,x)\phi(x) + d\nabla u(t,x)\nabla\phi(x) - f(u(t,x),v(t,x))\phi(x)] = 0 \, , \\ &\quad v(t,.) \in K \, , \\ &\int_\Omega \{v_t(t,x)[\psi(x) - v(t,x)] + \nabla v(t,x)\nabla[\psi(x) - v(t,x)] - \\ &- g(u(t,x),v(t,x))[\psi(x) - v(t,x)]\}\, dx \geqq 0 \\ &\quad \text{for all} \quad \phi \in V \, , \quad \psi \in K \, , \ \text{a.a.} \quad t \geqq 0 \, , \end{aligned} \tag{2}$$

where $\nabla u \cdot \nabla\phi = \sum_{i=1}^{n} u_{x_i}\phi_{x_i}$. By a solution we can understand a couple $u, v \in L_2(0,T;V)$ such that $u_t, v_t \in L_2(0,T;V)$.

Notice that the choice $K = \{v \in V;\ v \geqq 0 \ \text{on} \ \Gamma_0\}$ corresponds to the problem (RD), ($BC_1$), (1). ((2) is obtained by multiplying the equations in (RD) by test functions, integrating by parts and using ($BC_1$), (1).)

In general, we can define a solution of (RD) with the boundary conditions ($BC_1$) and unilateral conditions for $v$ given by $K$ as a couple $u$, $v$ satisfying (2). The corresponding linearization reads

$$\int_\Omega [u_t(t,x)\phi(x) + d\nabla u(t,x)\nabla\phi(x) - \big(b_{11}u(t,x) + b_{12}v(t,x)\big)\phi(x)]\,dx$$
$$= 0 , \quad v(t,.) \in K ,$$
$$\int_\Omega \{v_t(t,x)\big(\psi(x) - v(t,x)\big) + \nabla v(t,x)\cdot\nabla[\psi(x) - v(t,x)] \tag{3}$$
$$- \big(b_{21}u(t,x) + b_{22}v(t,x)\big)\big(\psi(x) - v(t,x)\big)\}\,dx \geqq 0$$
$$\text{for all } \phi \in V , \quad \psi \in K , \quad \text{a.a. } t \geqq 0 .$$

Analogously, we can consider (RD) with ($BC_2$) and unilateral conditions for $u$ , i.e.

$$u \in K ,$$
$$\int_\Omega [u_t(\phi - u) + d\nabla u\cdot\nabla(\phi - u) - f(u,v)(\phi - u)]\,dx \geqq 0 \tag{4'}$$
$$\text{for all } \phi \in K , \text{ a.a. } t \geqq 0 ,$$
$$\int_\Omega [v_t\psi + \nabla v\cdot\nabla\psi - g(u,v)\psi]\,dx = 0 \quad \text{for all } \psi \in V , \text{ a.a. } t \geqq 0 .$$

## 2. Destabilization.

EXAMPLE 1. Consider $\Gamma_D = \emptyset$ (i.e. $V = W_2^1$ ) , $K = \{v \in V;\ v \geqq 0$ on $\Omega\}$ . Then (2) corresponds to a free-boundary problem

$$u_t = d\Delta u + f(u,v) \quad \text{on } \langle 0,\infty) \times \Omega ,$$
$$v_t = \Delta v + g(u,v) \quad \text{on } Q_+ ,$$
$$v = 0, \ - g(u,0) \geqq 0 \quad \text{on } \langle 0,\infty) \times \Omega \setminus Q_+ , \tag{5}$$
$$u_{x_i} , v_{x_i} \text{ are continuous, } \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } (0,\infty) \times \partial\Omega ,$$

where the domain $Q_+ = \{[t,x] \in \langle 0,\infty) \times \Omega;\ v(t,x) > 0\}$ is unknown. The couple $u(t,x) = \exp(b_{11}t)\cdot\xi$ , $v(t,x) = 0$ satisfies the linearization of (5) (i.e. also (3)) classically for any $\xi < 0$ . It follows that the trivial solution of (3) is unstable for any $d$ , even for $d > d_0$ , and even with respect to spatially homogeneous perturbations. Notice that spatially homogeneous solutions of (3) (in our special case) are solutions of the inequality

$$U(t) \in K_c$$
$$\langle U_t(t) - BU(t), \Psi - U(t) \rangle \geqq 0 \quad \text{for all } \Psi \in K_c , \text{ a.a. } t \geqq 0 , \tag{6}$$

where $U = [u,v]$ , $K_c = \{[\phi,\psi] \in R^2;\ \psi \geqq 0\}$ , $\langle .,. \rangle$ is the scalar product in $R^2$. It is not difficult to describe all trajectories of (6) and characterize also some spatially homogeneous solution of (2) under the assumption (B) . As a consequence it is possible to prove also the instability for (2) for any $d > 0$ (see [7]).

Notice that the eigenvalue problem determining the stability of the trivial solution of (RD), (BC) can be written in the vector form

$$D(d)\Delta U + BU = \lambda U \tag{7}$$

(with the boundary conditions (BC)), where $U = [u,v]$, $D(d) = \binom{d,0}{0,1}$, $\Delta U = [\Delta u, \Delta v]$. The eigenvalue problem with unilateral conditions corresponding to (3) reads

$$\begin{aligned} &\int_\Omega [d\nabla u\cdot\nabla\phi - (b_{11}u + b_{12}v - \lambda u)\phi]\, dx = 0 \quad \text{for all} \quad \phi \in V\,, \\ &v \in K\,, \\ &\int_\Omega [\nabla v\cdot\nabla(\psi - v) - (b_{21}u + b_{22}v - \lambda v)(\psi - v)]\, dx \geqq 0 \quad \text{for all} \quad \psi \in K. \end{aligned} \tag{8}$$

Denote by $E(d,\lambda)$ the set of all solution of (7), (BC) (for given $d$, $\lambda \in R$). Notice that $E(d_0,0) \neq \{0\}$ because $d_0$ is a bifurcation point of (RD), (BC). Further, we shall suppose that

there exists a completely continuous operator $\beta : V \to V$ (a penalty operator) satisfying $< \beta u - \beta v, u - v > \geqq 0$, $\beta(tu) = t\beta u$ for all $u, v \in V$, $t \geqq 0$, $\beta u = 0$ for all $u \in K$, $< \beta v, v > > 0$ for all $v \notin K$, $< \beta v, u > < 0$ for all $v \notin K$, $u \in K^o$, (P)

where $<.,.>$ is the inner product in $V$, $K^o$ and $\partial K$ is the interior and the boundary of $K$. This assumption is fulfilled in examples. For the cone $K$ mentioned in Section 1 we can consider the penalty operator defined by

$$<\beta v,\psi> = - \int_{\Gamma_0} v^- \psi \, dx \quad \text{for all} \quad v, \psi \in V\,,$$

where $v^-$ denotes the negative part of $v$.

<u>THEOREM 1.</u> *Let* (P), (B) *hold and* $E(d_0,0) \cap V \times K^o \neq \emptyset$. *Then for any* $d \in (d_0,d_1)$ (*with some* $d_1 > d_0$) *there exists a solution of* (3) *of the type* $U(t) = \exp(\lambda_d^I t)\cdot U_d^I$ *with* $\lambda_d^I > 0$, $U_d^I \in \partial K$.

<u>THEOREM 2.</u> *Let* (B) *hold and* $E(d_0,0) \cap V \times K^o \neq \emptyset$, $\dim E(d_0,0) = 1$, $\operatorname{meas} \Gamma_D > 0$. *Then there exists a bifurcation point* $d_I > d_0$ *of* (2) *at which spatially nonhomogeneous stationary solutions bifurcate from* $\{[d,0,0];\ d \in R\}$.

<u>P r o o f of Theorem 1</u> is based on a modification of the method developed in [5]. We shall explain main ideas only (more precisely see [3]

cf. also [2]). It is sufficient to show that for any $d \in (d_0,d_1)$ there exists a positive eigenvalue $\lambda_d^I$ of (8) with the corresponding eigenvector $U_d^I = [u_d^I, v_d^I] \in V \times \partial K$. Suppose that $\dim E(d_0,0)=1$. (The general case can be reduced to this situation - see [3].) Choose a fixed $d > d_0$ and consider the system with the penalty

$$\int_\Omega [d\nabla u\cdot\nabla\phi - (b_{11}u + b_{12}v - \lambda u)\phi]\, dx = 0 \quad \text{for all} \quad \phi \in V\,,$$
$$\int_\Omega [\nabla v\cdot\nabla\psi - (b_{21}u + b_{22}v - \lambda v)\psi]\, dx + \varepsilon\langle\beta v,\psi\rangle = 0 \text{ for all } \psi \in V\,. \tag{9}$$

It is equivalent to (7), (BC) for $\varepsilon = 0$ and its eigenvalues and eigenvectors approximate those of (8) for $\varepsilon \to +\infty$. (The last assertion can be proved by standard penalty method technique.) We shall consider only solutions of (9) satisfying the norm condition

$$||U||^2 \ ( = ||u||^2 + ||v||^2 \ ) = \frac{\varepsilon}{1+\varepsilon}\,. \tag{10}$$

Set $C_d = \overline{\{[\lambda,U,\varepsilon] \in R \times V \times V \times R;\ ||U|| \neq 0\ ,(9),\ (10) \text{ is fulfilled}\}}$ (the closure in $R \times V \times V \times R$). The main idea is to show that the greatest eigenvalue $\lambda_d$ of (7), (BC) can be joined with an eigenvalue $\lambda_d^I$ of (8) by a connected (in $R \times V \times V \times R$) subset $C_d^+$ of $C_d$ and to prove $\lambda_d^I > 0$ on the basis of the properties of this branch $C_d^+$ (for any $d \in (d_0,d_1)$ with some $d_1 > d_0$). The existence of a global continuum $C_d^+ \subset C_d$ of solution of (9), (10) starting at $[\lambda_d,0,0]$ in the direction $U_d = [u_d,v_d] \in E(d,\lambda_d) \cap V \times (-K^o)$ follows from a slight generalization of a Dancer's bifurcation result [1]. (Setting $x = [U,\varepsilon]$, (9), (10) can be written as the usual bifurcation equation $x - L(\mu)x + N(\mu,x) = 0$ in the space $X = V \times V \times R$ with compact linear operators $L(\mu)$ depending continuously on $\mu \in R$ and a small compact perturbation $N$; "starting in the direction $U_d$" means that for any $\delta > 0$ there is $[\lambda,U,\varepsilon] \in C_d^+$ with $||U/||U|| - U_d|| < \delta$ in any neighbourhood of $[\lambda_d,0,0]$.) An elementary investigation of solutions of (9), (10) yields that in a small neighbourhood of $[\lambda_d,0,0]$ can be only solutions $[\lambda,U,\varepsilon]$ of (9), (10) satisfying $\lambda > \lambda_d$ and that for all solutions of (9), (10) different from $[\lambda_d,0,0]$ the following implications hold: $\lambda > \lambda_d \Longrightarrow v \notin \partial K$; $v \notin K \Longrightarrow \lambda \neq \lambda_d$. This together with the fact that $C_d^+$ starts in the direction $U_d \notin V \times K$ and with the connectedness of $C_d^+$ implies $\lambda > \lambda_d$, $v \notin K$ for any $[\lambda,U,\varepsilon] \in C_d^+$, $U = [u,v]$. It follows that $C_d^+$ cannot intersect an analogous branch $C_d^-$ of solution of (9), (10) starting in the direction $-U_d \in V \times K^o$. Dancer's result (see [1], Theorem 2) states that in this case $C_d^+$ is unbounded. It follows that there exists a sequence $\{[\lambda_n, U_n, \varepsilon_n]\} \subset C_d^+$ with $\varepsilon_n \to +\infty$. The penalty method technique gives $\lambda_n \longrightarrow \lambda_d^I$, $U_n \to U_d^I$, where $\lambda_d^I$ and $U_d^I$ is an eigenvalue and the corresponding eigenvector of (8), $U_d^I \in V \times \partial K$. If $\lambda_d^I > 0$ was not true for all $d \in (d_0, d_1)$ with some $d_1 > d_0$ then we would obtain $\lambda_{d_n}^I \to 0$ for some $d_n \to d_0+$ because $\lambda_{d_n} \to 0-$, $\lambda_d^I \geqq \lambda_d$. We could suppose $U_{d_n}^I \to U \in \partial K$ and this would contradict the assumptions $\dim E(d_0,0) = 1$, $E(d_0,0) \cap V \times K^o \neq \emptyset$. (Any solution $U$ of (8) with $d = d_0$, $\lambda = 0$ lies in $E(d_0,0)$ under the last assumption.)

<u>P r o o f of Theorem 2</u> is based on the same method as that of Theorem 1. The greatest bifurcation point $d_0$ of the stationary problem corresponding to (RD), (BC) can be joined (raughly speaking) with a bifurcation point $d_I > d_0$ of (2) by a branch of solutions of the corresponding penalty equation (with the variable $d$ instead of $\lambda$). See [4].

<u>REMARK 1.</u> If meas $\Gamma_D = 0$ in Theorem 2 (the case of Neumann conditions) then the bifurcation point $d_I$ can coincide with infinity in a certain sense (see [4]).

<u>REMARK 2.</u> If $K = \{v \in V;\ v \geqq 0 \text{ on } \Gamma_0\}$ then $E(d_0,0) \cap V \times K^o \neq \emptyset$ holds if there exists $U = [u,v] \in E(d_0,0)$ with $v \geqq \delta$ on $\Gamma_0$ ( $\delta > 0$ ).

## <u>3. Stabilization.</u>

<u>EXAMPLE 2.</u> Consider the cone $K = \{u \in V;\ u \geqq 0 \text{ on } \Omega\}$ with $V = W_2^1$ again. Then spatially constant solutions of the linearization of (4) are solutions of (6) with $K_c = \{\psi = [\phi,\psi] \in R^2;\ \phi \geqq 0\}$. If $B$ has a pair of complex eigenvalues and $b_{12}, b_{22} < 0$ then any solution of (6) (coinciding with the solution of $U_t = BU$ as long as it is in $K^o \times V$ ) touches the line $\{[0,v];\ v \geqq 0\}$ after some time $t_0$ and then coincides with the solution of the type $u(t) = 0$, $v(t) = \exp(b_{22}(t - t_0))\cdot\xi$. It follows that the trivial solution of the linearization of (4) is stable with respect to spatially homogeneous perturbations even if the trivial solution of $U_t = BU$ is unstable (more precisely see [7]). Of course, in this way we cannot obtain any information about the stability with respect to nonhonogeneous perturbations.

<u>THEOREM 3.</u> *Let* (B) *hold and let* $E(d_0,0) \cap K \times V = \{0\}$, meas $\Gamma_D > 0$.

*Then there is no bifurcation point of* (4) *at which stationary spatially nonhomogeneous solutions bifurcate from* $\{[d,0,0];\ d \in R\}$ *in* $(d_0 - \delta, +\infty)$ *(with some* $\delta > 0$ *).*

<u>P r o o f</u> . Introduce the inner product $\langle .,. \rangle$ and the operator $A$ in $V$ by

$$\langle u,\phi \rangle = \int_\Omega \nabla u \nabla \phi \, dx \ , \quad \langle Au,\phi \rangle = \int_\Omega u\, \phi \, dx \quad \text{for all} \quad u,\ \phi \in V \ .$$

The linearization of the stationary problem corresponding to (4) can be written as

$$\begin{aligned} & u \in K \ , \\ & \langle du - b_{11}Au - b_{12}Av,\ \phi - u \rangle \geqq 0 \quad \text{for all} \quad \phi \in K \ , \\ & v - b_{21}Au - b_{22}Av = 0 \ . \end{aligned} \tag{11}$$

Calculating $v$ from the second equation in (11) and substituting to the first inequality we obtain the inequality of the type

$$\begin{aligned} & u \in K \ , \\ & \langle du - Tu,\ \phi - u \rangle \geqq 0 \quad \text{for all} \quad \phi \in K \end{aligned} \tag{12}$$

with a compact linear symmetric operator $T$ in $V$ . It follows that any bifurcation point $d_I$ of our unilateral stationary problem is simultaneously an eigenvalue of the inequality (12) and therefore $d_I \leqq \max\limits_{||u||=1,\ u \in K} \langle Tu,u \rangle$ . Further, the greatest bifurcation point $d_0$ of (RD), (BC) is simultaneously the greatest eigenvalue of $T$ , i.e. $d_0 = \max\limits_{||u||=1,\ u \in V} \langle Tu,u \rangle$ and any $u \in V$ realizing this maximum is an eigenvector of $T$ corresponding to $d_0$ . This together with the assumption $E(d_0) \cap K \times V = \{0\}$ implies $\max\limits_{||u||=1,\ u \in K} \langle Tu,u \rangle < d_0$ . For the details see [6] where a more general case is considered.

<u>REMARK 3.</u> Let $K = \{u \in V;\ u \geqq 0 \text{ on } \Gamma_0\}$ . Then $E(d_0,0) \cap K \times V = \{0\}$ is fulfilled if there exists $U = [u,v] \in E(d_0,0)$ such that $u$ changes its sign on $\Gamma_0$ .

## <u>4. Final remarks.</u>

It is possible to consider also more general inequalities

$$\int_\Omega u_t(\phi - u) + d\nabla u\cdot\nabla(\phi - u) - f(u,v)(\phi - u)\,dx$$
$$+ \Phi_1(\phi) - \Phi_1(u) \geqq 0\ ,$$
$$\int_\Omega v_t(\psi - v) + \nabla v\cdot\nabla(\psi - v) - g(u,v)(\psi - v)\,dx \tag{13}$$
$$+ \Phi_2(\psi) - \Phi_2(v) \geqq 0 \quad \text{for all} \quad \phi,\ \psi \in V\ ,\ \text{a.a.} \quad t \geqq 0\ ,$$

where $\Phi_1$ , $\Phi_2$ are convex proper functionals on $V$ . More general unilateral conditions are included in this formulation. An analogy of Theorem 3 for (13) with $\Phi_2 = 0$ is contained in [6], a destabilizing effect of such unilateral conditions (for $\Phi_1 = 0$) will be the subject of a forthcomming paper.

## R e f e r e n c e s


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