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SOME BOUNDARY PROBLEMS FOR THE EQUATIONS WITH STRONG DEGENERATION

S. Nikolskij, Moskva

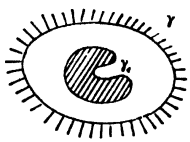
Let Ω be a bounded open set of the n-dimentional space R of the points $x = (x_1, \ldots, x_n)$ with enough smooth boundary Γ.

We shall consider two cases:

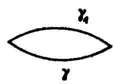
1) $\varrho = \varrho(x)$ is the distance from x to Γ,

2) $\varrho = \varrho(x)$ is the distance from x to γ, where γ is a part of Γ, $\gamma + \gamma_1 = \Gamma$, $\gamma\gamma_1 = 0$.

Here are two characteristic plots for the case 2)



Plot 1.



Plot 2.

But I must warn that in different problems under consideration it's necessary to propose some conditions on disposition γ to γ_1, as it will be seen below.

We shall use the following notation

$$||f||_{L_p(\varepsilon)} = (\int_\varepsilon |f(x)|^p \,dx)^{1/p} \qquad (1 \leq p \leq \infty, \ \varepsilon \subset R).$$

By definition $W^{\tau}_{p,\alpha}(\Omega)$ is the class of the functions defined on Ω, which have finite norm

$$||f||_{W^{\tau}_{p,\alpha}(\Omega)} = ||f||_{L_p(\Omega)} + \sum_{|K|=\tau} \left\| \frac{f^{(K)}}{\varrho^\alpha} \right\|_{L_p(\Omega)}.$$

Here $\sum$ extends on all derivatives of the order τ. At first we shall mean that

$$\tau + \alpha - \frac{1}{p} > 0$$

and

$$s - 1 = \left[\tau + \alpha - \frac{1}{p}\right]$$

is its entier. Thus s is an integer depending on τ, α, p and satisfying inequalities

$$1 \leq s \leq \tau.$$

It is well known (see [1] theorem 38) that every function $f \in W^{\tau}_{p,\alpha}(\Omega)$ has traces on Γ.

They are in the case 1)

$$\left.\frac{\partial^K f}{\partial h^K}\right|_{\Gamma} = \varphi_K \qquad (K = 0, 1, \ldots, s-1)$$

and in the case 2)

$$\left.\frac{\partial^K f}{\partial h^K}\right|_{\gamma} = \varphi_K \qquad (K = 0, 1, \ldots, s-1),$$

$$\left.\frac{\partial^K f}{\partial h^K}\right|_{\gamma_1} = \varphi_K \qquad (K = 0, 1, \ldots, \tau-1).$$

If

$$\tau + \alpha - \frac{1}{p} \leq 0$$

well let $s = 0$. It is natural because in this case function f of the class $W^{\tau}_{p,\alpha}(\Omega)$ generally speaking has no traces on

1) Γ or 2) γ.

But in the case 2) f has still traces on γ_1, corresponding $K = 0, 1, \ldots, \tau - 1$.

If $s = \tau$ we shall say that the weak degeneration takes place and if $s < \tau$ — — the strong one.

With the class $W^{\tau}_{2,\alpha}(\Omega)$ $(p = 2)$ we relate a differential equation

$$(1) \qquad Lu = \sum_{|K|,|l| \leq \tau} (-1)^{|l|} \mathscr{D}^{(l)}(Q_{Kl} U^{(K)}) = f \qquad x \in \Omega,\ Q_{Kl} = Q_{lK}(x)$$

with conditions

$$\sum_{|K|,|l| \leq \tau} Q_{Kl}\, \xi_K\, \xi_l > \frac{\varkappa}{\varrho^{2\alpha}} \sum_{|K| = \tau} \xi_K^2,$$

$$|Q_{Kl}(x)| < \frac{M}{\varrho^{2\alpha_{Kl}}}, \qquad \alpha_{Kl} = \tau + \alpha - \max(|K|, |l|),$$

$$\alpha_{Kl} < \tfrac{1}{2}.$$

Here Q_{Kl} are functions of x and vector parameters $\boldsymbol{K}$, τ; ξ_K are variables related with considered vectors $\boldsymbol{K}$ and $\boldsymbol{\varkappa}$, M don't depend on x, ξ_K, ξ_l.

As asualy to consider questions of the smootniss of the classical solution it is necessary to propose in addition the usual conditions on differentiability of Q_{Kl}.

Such restrietions on the coefficients are necessary in our considerations too, because we consider not only generalised solutions but classical ones (belonging to $W^{\tau}_{2,\alpha}(\Omega)$).

We consider here the problem:

To find solution of the equality (1) belonging to the class $W^{\tau}_{2,\alpha}(\Omega)$ with boundary conditions 1) or 2).

This problem includes at $\alpha = 0$ the usual. Dirichlet problem for differential equation of the elliptic type.

If $s = \tau$ we shall call our problem "the weak problem" and if $s < \tau$ — "the strong one".

The series of investigations has been devoted to different problems with degeneration; see [16] §6, [18] where are given the lists of literature and also [1—15], [19—23].

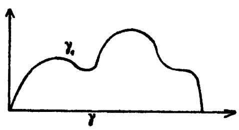
Now we are interested only in the mentioned formulated strong (boundary!) problem ($s < \tau$).

First investigations on this problem referred to the case 2) for the equation of the second order when therefore boundary values are given only on a part γ of Γ, (because $s = 0$ in this case).

M. B. Келдыш [10] has considered (in metric C) such a problem for the equations, which includes in particular the following one

$$Lu = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} y^m + cu = 0 \tag{2}$$

on a two dimentional domain of such a kind (Plot 3). M. B. Келдыш has discovered in particular that for $m > 1$



Plot 3.

the Dirichlet problem for the equation (1) is not correct, but it is perfectly correct if to give boundary values only on γ_1.

It is possible to show that in the case 2) of the strong problem for the equations of the high order there exists the unique solution.

But the case 1) is quite different. In general in this case uniquess breaks for the strong problem.

For instance uniquess breaks for the equation (2), where $c = 0$, because every constant then satisfies corresponding homogeneous equation.

Recently (1964 [18]) П. И. Лизоркин and I have proved that the strong problem in case 1) has always the unique solution if $2s \geq \tau$, and in case $2s < \tau$ it is not right, generally speaking.

From the point of view of the variational method these questions may be explained as follows.

Uniquess of the generalised solution of the boundary problem depends essentialy on answer to the question: does the inequality of the Poincare type (for $p = q$):

$$(3) \qquad ||f||_{L_p(\Omega)} \leq c \left(\sum ||\varphi_K||_{\Gamma} + \sum_{|K|=\tau} \left\| \frac{f^{(K)}}{\varrho^{\alpha}} \right\|_{L_p(\Omega)} \right)$$

hold or not?

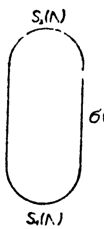
Here the norms $||\varphi_K||_{\Gamma}$ are taken in corresponding metric $W_2(\Gamma)$.

As for the uniquess of the classical solution this question in strong problem, to compare it with the weak one, has not principal differences.

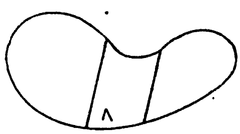
Let Λ be an open set, which is cut out from a circular cilinder by two not intersecting
smooth surfaces $S_1 = S_1(\Lambda)$, $S_2 = S_2(\Lambda)$. It is important that every line belonging to our closed cilinder intersects S_1, as well as S_2 only at one point being not tangent to S_1 (S_2) at this point. Such a domain Λ we shall name a regular one. We propose also that $\sigma = \sigma(\Lambda)$ is the side surface of Λ without points belonging to S_1, S_2 and name the $a \times e$ of the considered cilinder the $a \times e$ of our regular domain.

Let now as above Ω be an open bounded set with smooth boundary Γ.

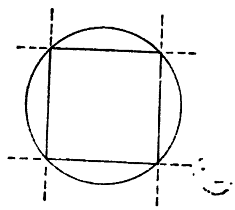
By definition Λ is a regular "bridge" of Ω if it satisfies the following conditions:



Plot 4.



Plot 5.


Plot 6.

1) Λ is a regular domain belonging to Ω,
2) $S_1(\Lambda)$, $S_2(\Lambda) \subset \Gamma$,
3) $\sigma(\Lambda) \not\subset \Gamma$.

We proved [17] the following

Lemma. *It is possible to cover Ω by a finite number of the regular bridges Λ.*

For istance, a two dimentional circle we can cover by two regular bridges, as on this plot

The proof of the Poincare inequality in the case 1) for $2s \geq \tau$ may be obtained by following steps.

At first we prove (П. И. Лизоркин and С. М. Никольский [18]) this inequality for the functions given on a one-dimentional segment $[a, b]$:

$$(4) \qquad ||f||_{L_p(a,b)} < C_l \left\{ \sum_0^{S-1} (|f_{(a)}^{(K)}| + |f_{(b)}^{(K)}|) + \left\| \frac{f^{(k)}}{\varrho^\alpha} \right\|_{L_p(a,b)} \right\}.$$

It is right for $2s \geq \tau$, but it's not right for $2s < \tau$. Here C_l is a constant continuosly depending on $l = b - a > 0$.

The next step is the generalising of this inequality for regular bridges Λ:

$$(5) \qquad ||f||_{L_p(\Lambda)} < c \left\{ \sum_0^{S-1} (||\varphi_K||_{L_p(S_1)} + ||\varphi_K||_{L_p(S_2)}) + \left\| \frac{\frac{\partial^2 f}{\partial \xi^2}}{\varrho_\xi^\alpha} \right\|_{L_p(\Lambda)} \right\},$$

$$\varphi_K = \left. \frac{\partial^K f}{\partial \xi^K} \right|_{S_1, S_2}.$$

Here ϱ_ξ is the distance from x to Γ in direction of the axe ξ of the bridge. To obtain (5) we introduce the new coordinates $(\xi, \eta) = (\xi, \eta_1, \ldots, \eta_{n+1})$. The coordinate axe ξ is directed as the axe of the considered bridge and the other coordinate axes $\eta_1, \ldots, \eta_{n-1}$, are for instance ortogonal to it.

First we use the inequality (4) for $f = f(\xi, \eta)$, when η is fixed, then take (4) in power p and integrate on η. It leads to (5) if to take in account that the constant in (4) is bounded for $0 < l_1 \leq l \leq l_2$.

Lastly we substitute $\varrho(x)$ instead of $\varrho_\xi(x)$ in (5). It is possible because ϱ_ξ and ϱ have the same order $(c_1\varrho(x) < \varrho_\xi(x) < c_2\varrho(x))$ for all x belonging to a regular bridge Λ.

It is also possible to substitute normes $||\varphi_K||_\Gamma$ instead of the normes $||\varphi_K||_{L_p(S_i)}$ $(i = 1, 2)$, where already $\varphi_K = \left. \frac{\partial^K h}{\partial h^K} \right|_\Gamma$ and $|| \ \ ||_\Gamma$ are understood in the corresponding metric $W_p^l(\Gamma)$ (instead $L_p(\Gamma)$). Finaly using the mentioned cover lemma we obtain the Poincare inequality (3).

To prove the inequality (3) in the case 2) one can begin from the following one dimentional inequality

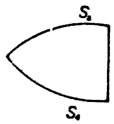
$$(6) \qquad ||f||_{L_p(a,b)} < c_e \left\{ \sum_0^{r-1} |f_{(b)}^{(K)}| + \left\| \frac{f^{(K)}}{(x-a)^\alpha} \right\|_{L_p(a,b)} \right\}.$$

Here degeneration takes place only at one boondary point of $[a, b]$, namely at a. But there is no at all degeneration at other boundary point b.

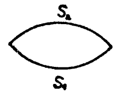
Let's now consider the same domain as on the plot 1. We propose also that our domain may be covered by bridges which connect either γ_1 with γ, or

γ_1 with γ_1. For the bridges of the first kind we use the inequality which generalises (6) and for the ones of the second kind the inequality (5) for $\alpha = 0$.

Pay attention that (6) differs from (4). In (4) C_l; is continuous only for $l > 0$, and in (6) for $l \geq 0$. The last gives possibility to generalise (6) for domains, more general than on the plot 4.



Plot 7.



Plot 8.

Now surfaces S_1 and S_2 can have common points.

Some remarks.

1) The mentioned method of the covering Ω by regular bridges may be used for transfer many other inequalities from one dimentional segment to the domains with enough smooth boundary, for instance, inequalities in the approximation theory by polinomials.

2) It is possible to extend the method on the domains with Lipshitz boundary.

3) Ю. Салманов received some development of the results. Namely he has obtained the corresponding inequality in the case of the strong degeneration on a domain with pricked out a point.

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