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NON-EXPANSIVE MAPPINGS IN CONVEX LINEAR TOPOLOGICAL SPACES

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A self-mapping f of a metric space (X, d) is said to be non-expansive iff for all $x, y \in X$, $d[f(x), f(y)] \leq d(x, y)$. In [1] and [2], F. BROWDER proved that if X is a closed bounded convex subset of a uniformly convex Banach space B on which the metric d is that induced from the norm on B, then every non-expansive self-mapping of X has a fixed-point. Browder's result is included in a similar result by KIRK [3] who showed that, if X is a closed bounded convex subset of a reflexive Banach space possessing "normal structure", then the self-mapping f has a fixed-point in X. A convex subset K of a Banach space has normal structure iff, for each non-trivial bounded convex subset C of K, there is an $x \in C$ such that

$$\operatorname{diam}(C) > \sup_{y \in C} ||x - y||.$$

It is a simple matter to see that if B is a uniformly convex Banach space then every non-trivial convex set K in B has normal structure.

In proving these theorems the authors rely heavily upon the special properties of reflexive Banach spaces. We observe, however, that these results, among others, may be proved directly from a rather simple, but general principle about semi-continuous mappings on locally convex spaces. In the sequel V will denote a locally convex linear topological space (over the reals or complexes) and ϱ_K a lower semi-continuous non-negative convex function defined on a convex subset K of V. ϱ_K is said to be normal if in addition ϱ_K is non-constant on each non-trivial closed convex subset C of K. We then have the following.

Proposition. *Let K be a weakly compact convex subset of the locally convex space V and let ϱ_K be a normal function on K. If f is a self-mapping of K such that*

$$\varrho_K(f(x)) \leq \varrho_K(x), \qquad x \in K, \tag{1}$$

then f has a fixed-point in K.

Proof: ϱ_K is lower semi-continuous if and only if the sets, defined for $t \geq 0$,

$$A_t = \{x \in K | \varrho_K(x) \leq t\}$$

are closed. In view of the convexity of ϱ_K, the A_t are closed *convex* sets and, consequently, are weakly closed. Hence ϱ_K is weakly lower semi-continuous. Because of the weak compactnes of K,

$$M = \{x \in K | \varrho_K(x) = \inf \varrho_K(y)\}$$

is then a non-void closed convex subset of K. Indeed, since ϱ_K is normal M consists of exactly one point x_0. On the other hand, $f(x_0) \in K$ so that $\varrho_K(x_0) \leq \leq \varrho_K(f(x_0))$. But assumption (1) yields $\varrho(x_0) = \varrho(f(x_0))$, thus $f(x_0) \in M$ and hence $f(x_0) = x_0$.

A simple generalization of Kirk's result now follows immediately. Let p be a continuous semi-norm on V. We shall call a convex subset K of V p-normal iff for each non-trivial weakly compact convex subset C of K it is true that $\sup_{y \in C} p(x - y)$ is non-constant on C.

Theorem 1. *Let V be a locally convex linear topological space, K a weakly compact convex subset of V, p a continuous semi-norm on V. If K is p-normal and f is a self mapping of V such that, for all $x, y \in K$*

$$p(f(x) - f(y)) \leq p(x - y), \tag{2}$$

then f has a fixed-point in K.

Proof: p is a continuous semi-norm on V, in particular p is convex. Hence p is weakly lower semi-continuous on V and, consequently, $\{x | p(x) \leq 1\}$ is a barrel relative to the weak topology on V. Thus, by [4] (Lemma 1, page 66), $\varrho_0(x) = \sup_{y \in K_0} p(x - y)$ is a finite-valued non-negative convex function on K_0 which is weakly lower semi-continuous. Here K_0 is any weakly compact convex subset of V.

We now determine $K_0 \subset K$ such that $f(K_0) \subset K_0$ on which ϱ_0 satisfies (1). To this end, let $\mathscr{K}$ be the collection of all closed convex subsets C of K such that $f(C) \subset C$. These sets are, therefore, weakly closed and, thus, weakly compact, so that K possesses the finite intersection property. Since $K \in \mathscr{K}$, Zorn's lemma is applicable and, hence, there is a minimal weakly compact $K_0 \in \mathscr{K}$ such that $f(K_0) \subset K_0$. Since $\overline{C_0} f(K_0)$, the closed convex hull of $f(K_0)$, belongs to $\mathscr{K}$, and since $\overline{C_0} f(K_0) \subset K_0$, the minimality of K_0 allows us to conclude that $K_0 = \overline{C_0} f(K_0)$. This is the convex set K_0 that we use to apply the proposition.

To this end, for $x \in K_0$ we set

$$\varrho_0(x) = \sup_{y \in K_0} p(x - y).$$

Since $K_0 = \overline{C_0} f(K_0)$ and since p is both continuous and convex,

$$\varrho_0(f(x)) = \sup_{y \in K_0} p(f(x) - f(y)), \qquad x \in K_0,$$

so that $\varrho_0(f(x)) \leq \varrho_0(x)$ for $x \in K_0$. Moreover, ϱ_0 is lower-semi continuous non-negative convex on K_0. That ϱ_0 is, indeed, normal follows from the minimality of K_0; for if $C \subset K_0$ is closed convex and non-trivial such that $\varrho_0(x) = k = \text{constant}$ on C, then actually $C = \{x \in K_0 | \varrho(x) \leq k\}$ and is a closed convex subset of K_0 on which $\varrho_0(f(z)) \leq \varrho_0(z)$. Thus $f(C) \subset C$ and, hence $C = K_0$, which is impossible by the P-normality of K. Hence the proposition is applicable and yields a fixed-point for f in K.

We may now state the KIRK and BROWDER results as immediate corollaries of Theorem 1.

Corollary 1. (KIRK). *Let B be a reflexive Banach space and K a closed bounded convex subset of B which possesses normal structure. If f is a non-expansive self-mapping of K, f has a fixed-point in K.*

Corollary 2. (F. BROWDER). *Let B be a uniformly convex Banach space and K a closed bounded convex subset of B. A non-expansive self-mapping of K has a fixed-point in K.*

The proof of Corollary 1 follows from the following facts: K is weakly compact because of the reflexivity of B; the norm is continuous, and K is norm-normal. As noted earlier Corollary 2 is a consequence of Corollary 1, since a uniformly convex Banach space is reflexive and has the property that any convex set has normal structure.

For the sake of completeness, we state two other results which follow from Theorem 1, or rather Corollary 1. The first is stated and proved in [3]. The second is due to BROWDER [2] under the more restrictive condition that B is uniformly convex. It is a partial generalization of the MARKOV—KAKUTANI result [5], [6] on commuting families of linear contractions.

Corollary 3. *Let B be a reflexive Banach space and K a closed convex subset of B possessing normal structure. A non-expansive self-mapping f of K has a fixed-point if and only if there exists an $x_0 \in K$ such that the sequence of iterates $\{f^{(n)}(x_0)\}$ is bounded.*

Corollary 4. *Let B be a reflexive, strictly convex Banach space and K a bounded closed convex subset of B possessing normal structure. If $\{f_\lambda\}$, $\lambda \in \Lambda$ is a commuting set of self-mappings of K, then there is an $x_0 \in K$ such that $f_\lambda(x_0) = x_0$ for all $\lambda \in \Lambda$.*

The proof of Corollary 3 is obtained from Corollary 1 as follows: Let $r = \sup ||x_0 - f^{(n)}(x_0)||$ and let K_n be the intersection of the closed ball of radius r about $f^{(n)}(x_0)$ with K. The K_n are closed, bounded and convex.

Because of the non-expansiveness of f, it follows that $f^{(m)}(x_0) \in K_n$ for all $m \geq n$. Thus the collection of weakly compact convex sets $\{K_n\}$ have the finite intersection property and, hence, $C = \bigcup_{s=1}^{\infty} \bigcap_{n=s}^{\infty} K_n$ is non-void convex and weakly compact. Moreover $f(C) \subset C$. Taking the closure $\overline{C}$ of C, we observe that $\overline{C}$ is a closed, bounded, convex subset of K on which f is a self-mapping. Since K has normal structure so does $\overline{C}$ and hence Corollary 1 is applicable to yield the desired result.

With respect to Corollary 4, we first note that, in view of the strict convexity of B, the fixed-point set of a non-expansive self-mapping of a convex subset of B is itself convex. If $C_\lambda = \{x \in K | f_\lambda(x) = x\}$, $\lambda \in \Lambda$, then these sets are closed convex subsets of K which by Corollary 1 are non-void. Taking account of the commutivity of the f_λ it is readily seen that the $\{C_\lambda\}$ have the finite intersection property. Since B is reflexive, $\bigcap_{\lambda \in \Lambda} C_\lambda \neq \emptyset$.

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