Aristide Halanay

Invariant manifolds for discrete systems

In: Valter Šeda (ed.): Differential Equations and Their Applications, Proceedings of the Conference held in Bratislava in September 1966. Slovenské pedagogické nakladateľstvo, Bratislava, 1967. Acta Facultatis Rerum Naturalium Universitatis Comenianae. Mathematica, XVII. pp. 173--191.

Persistent URL: http://dml.cz/dmlcz/700213

ACTA FACULTATIS RERUM NATURALIUM UNIVERSITATIS COMENIANAE
MATHEMATICA XVII — 1967

INVARIANT MANIFOLDS FOR DISCRETE SYSTEMS

A. HALANAY, Bucharest

In this paper a general theorem on the existence of invariant manifolds for discrete systems is obtained using the general method of J. KURZWEIL [1]. As this theorem is true for discrete systems in a Banach space it may be applied to prove the existence of invariant manifolds for some systems with time lag.

I. General theory

Consider a discrete system $x_{n+1} = f_n(x_n)$; $f_n : G_n \subset X \to X$, n is an integer, X is a Banach space and G_n is a domain in X. If $\tilde{x} \in G_{\tilde{n}}$ we may define the solution $x_n(\tilde{n}, \tilde{x})$ for $n \geq \tilde{n}$ such that $x_{\tilde{n}}(\tilde{n}, \tilde{x}) = \tilde{x}$; if $f_n(x_n(\tilde{n}, \tilde{x})) \in G_{n+1}$ this solution is defined for all $n \geq \tilde{n}$. Suppose this is the case; then obviously $x_n(n_1, x_{n_1}(\tilde{n}, \tilde{x})) = x_n(\tilde{n}, \tilde{x})$ for all $n \geq n_1 \geq \tilde{n}$.

The general theorem we shall prove concerns discrete systems in a product space $\mathfrak{X} = C \times \mathfrak{C}$; these systems will be described by two functions $c_n(\tilde{n}, \tilde{c}, \tilde{\gamma})$ and $\gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma})$ defined for $n \geq \tilde{n}$, $\tilde{c} \in C$, $\tilde{\gamma} \in \mathfrak{C}$, $c_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) \in C$, $\gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) \in \mathfrak{C}$, and such that $c_n(n_1, c_{n_1}(\tilde{n}, \tilde{c}, \tilde{\gamma}), \gamma_{n_1}(\tilde{n}, \tilde{c}, \tilde{\gamma})) = c_n(\tilde{n}, \tilde{c}, \tilde{\gamma})$, $\gamma_n(n_1, c_{n_1}(\tilde{n}, \tilde{c}, \tilde{\gamma}), \gamma_{n_1}(\tilde{n}, \tilde{c}, \tilde{\gamma})) = \gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma})$.

Theorem 1. *Consider a discrete system in the product space $C \times \mathfrak{C}$. Suppose there exist positive constants l, L, N, α_1, α_2, $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1$, k_1, k_2 such that:* 1° *$||\tilde{c}|| \leq l$ imply that $c_n(\tilde{n}, \tilde{c}, \tilde{\gamma})$ is defined for all $n \geq \tilde{n}$ and $||c_n(\tilde{n}, \tilde{c}, \tilde{\gamma})|| \leq l$ for $n \geq \tilde{n} + N$.*

2°. *$||\tilde{c}_1|| \leq l$, $||\tilde{c}_2|| \leq l$, $\tilde{n} + N \leq n \leq \tilde{n} + 2N$ imply*
$||c_n(\tilde{n}, \tilde{c}_1, \tilde{\gamma}) - c_n(\tilde{n}, \tilde{c}_2, \tilde{\gamma})|| + L\,||\gamma_n(\tilde{n}, \tilde{c}_1, \tilde{\gamma}) - \gamma_n(\tilde{n}, \tilde{c}_2, \tilde{\gamma})|| \leq \alpha_1\,||\tilde{c}_1 - \tilde{c}_2||$.

3°. *$||\tilde{c}_1|| \leq l$, $||\tilde{c}_2|| \leq l$, $||\tilde{c}_1 - \tilde{c}_2|| \leq L\,||\tilde{\gamma}_1 - \tilde{\gamma}_2||$ imply*

a) $||\gamma_n(\tilde{n}, \tilde{c}_1, \tilde{\gamma}_1) - \gamma_n(\tilde{n}, \tilde{c}_2, \tilde{\gamma}_2) - \tilde{\gamma}_1 + \tilde{\gamma}_2|| \leq \alpha_2\,||\tilde{\gamma}_1 - \tilde{\gamma}_2||$
for $\tilde{n} \leq n \leq \tilde{n} + 2N$

b) $||c_n(\tilde{n}, \tilde{c}_1, \tilde{\gamma}_1) - c_n(\tilde{n}, \tilde{c}_2, \tilde{\gamma}_2)|| \leq (1 - \alpha_2) L ||\tilde{\gamma}_1 - \tilde{\gamma}_2||$
for $\tilde{n} + N \leq n \leq \tilde{n} + 2N$

4°. $||c_n(\tilde{n}, \tilde{c}_1, \tilde{\gamma}_1) - c_n(\tilde{n}, \tilde{c}_2, \tilde{\gamma}_2)|| + ||\gamma_n(\tilde{n}, \tilde{c}_1, \tilde{\gamma}_1) - \gamma_n(\tilde{n}, \tilde{c}_2, \tilde{\gamma}_2)|| \leq$ $\leq k_1 k^{n-\tilde{n}}(||\tilde{c}_1 - \tilde{c}_2|| + ||\tilde{\gamma}_1 - \tilde{\gamma}_2||)$ *for all* $n \geq \tilde{n}$ *for which the functions are defined. Then for each integer* n *there exist a function* p_n: $\mathfrak{C} \to C$ *and positive constants* K, $0 < \alpha < 1$ *such that*

a) $||p_n(\gamma)|| \leq l$;
b) $||p_n(\gamma_1) - p_n(\gamma_2)|| \leq L ||\gamma_1 - \gamma_2||$;
c) $||\tilde{c}|| \leq l$ *implies* $||c_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) - p_n(\gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma})) \leq K\alpha^{n-\tilde{n}} ||\tilde{c} - p_{\tilde{n}}(\tilde{\gamma})||$;
d) $\tilde{c} = p_{\tilde{n}}(\tilde{\gamma})$ *implies* $c_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) = p_n(\gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma}))$ *for all* n;
e) p_n *is uniquely determined by the above properties*;
f) 1°. *If* $c_{n+\nu}(\tilde{n} + \nu, \tilde{c}, \tilde{\gamma}) \equiv c_n(\tilde{n}, \tilde{c}, \tilde{\gamma})$, $\gamma_{n+\nu}(\tilde{n} + \nu, \tilde{c}, \tilde{\gamma}) \equiv \gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma})$ *for all* n, $\tilde{n}$, $\tilde{c}$, $\tilde{\gamma}$ *for which the functions are defined, then* $p_{n+\nu}(\gamma) \equiv p_n(\gamma)$.
2°. *If* $c_n(\tilde{n}, \tilde{c}, \tilde{\gamma} + \omega) = c_n(\tilde{n}, \tilde{c}, \tilde{\gamma})$, $\gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma} + \omega) \equiv \gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) + \omega$ *for all* n, $\tilde{n}$, $\tilde{c}$, $\tilde{\gamma}$ *for which the functions are defined, then* $p_n(\gamma + \omega) \equiv p_n(\gamma)$.
g) *If each sequence* $n_k \to \infty$ *contains a subsequence* n_{k_l} *such that* $c_{n+n_{k_l}}(\tilde{n} + n_{k_l}, \tilde{c}, \tilde{\gamma})$, $\gamma_{n+n_{k_l}}(\tilde{n} + n_{k_l}, \tilde{c}, \tilde{\gamma})$ *are convergent for* $l \to \infty$, *uniformly on each finite set of values* $n \geq \tilde{n}$ *and uniformly with respect to* $\tilde{n}$, $\tilde{c}$, $\tilde{\gamma}$, *then the sequence* p_n *is almost periodic uniformly with respect to* γ.

Proof. A. Denote by $Q(l, L)$ the set of functions q: $\mathfrak{C} \to C$ such that $||q(\gamma_1) - q(\gamma_2)|| \leq L ||\gamma_1 - \gamma_2||$, $||q(\gamma)|| \leq l$. Let $\vartheta^q_{n,\tilde{n}} : \mathfrak{C} \to \mathfrak{C}$ be defined by $\vartheta^q_{n,\tilde{n}}(\tilde{\gamma}) = \gamma_n(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma})$. From condition 3° a) follows for $\tilde{n} \leq n \leq \tilde{n} + 2N$ that $||\vartheta^q_{n,\tilde{n}}(\tilde{\gamma}_1) - \vartheta^q_{n,\tilde{n}}(\tilde{\gamma}_2) - \tilde{\gamma}_1 + \tilde{\gamma}_2|| \leq \alpha_2 ||\tilde{\gamma}_1 - \tilde{\gamma}_2||$ hence $(1 - \alpha_2) ||\tilde{\gamma}_1 - \tilde{\gamma}_2|| \leq ||\vartheta^q_{n,\tilde{n}}(\tilde{\gamma}_1) - \vartheta^q_{n,\tilde{n}}(\tilde{\gamma}_2)|| \leq (1 + \alpha_2) ||\tilde{\gamma}_1 - \tilde{\gamma}_2||$.

It is proved then by a lemma of Kurzweil that for $\tilde{n} \leq n \leq \tilde{n} + 2N$ $\vartheta^q_{n,\tilde{n}}$ is a one-to-one mapping of $\mathfrak{C}$ onto $\mathfrak{C}$; let $\sigma^q_{n,\tilde{n}}$: $\mathfrak{C} \to \mathfrak{C}$ be the inverse mapping.

B. Define the mapping $P_{n,\tilde{n}}q$: $\mathfrak{C} \to C$ by

$$[P_{n,\tilde{n}}q] (\tilde{\gamma}) = c_n(\tilde{n}, q[\sigma^q_{n,\tilde{n}}(\tilde{\gamma})], \sigma^q_{n,\tilde{n}}(\tilde{\gamma})), \qquad \tilde{n} \leq n \leq \tilde{n} + 2N.$$

For $\tilde{n} + N \leq n \leq \tilde{n} + 2N$ we have $||[P_{n,\tilde{n}}q] (\tilde{\gamma})|| \leq l$ from condition 1° and $||[P_{n,\tilde{n}}q](\tilde{\gamma}_1) - [P_{n,\tilde{n}}q](\tilde{\gamma}_2)|| \leq (1 - \alpha_2) L ||\sigma^q_{n,\tilde{n}}(\tilde{\gamma}_1) - \sigma^q_{n,\tilde{n}}(\tilde{\gamma}_2) || \leq L ||\tilde{\gamma}_1 - \tilde{\gamma}_2||$ from conditions 3° a) and b).

It follows that for $\tilde{n} + N \leq n \leq \tilde{n} + 2N$ we have $P_{n,\tilde{n}}q \in Q(l, L)$, hence $P_{n,\tilde{n}} : Q(l, L) \to Q(l, L)$.

C. Let $\tilde{n} + N \leq \tilde{n}_1 \leq \tilde{n} + 2N$, $\tilde{n}_1 + N \leq \tilde{n}_2 \leq \tilde{n}_1 + 2N$, $q_1 = P_{\tilde{n}_1,\tilde{n}}q$. We have $\vartheta^{q_1}_{\tilde{n}_2,\tilde{n}_1}(\tilde{\gamma}) = \gamma_{\tilde{n}_2}(\tilde{n}_1, c_{\tilde{n}_1}(\tilde{n}, q[\sigma^q_{\tilde{n}_1,\tilde{n}}(\tilde{\gamma})], \sigma^q_{\tilde{n}_1,\tilde{n}}(\tilde{\gamma})), \tilde{\gamma}) =$

$$= \gamma_{\tilde{n}_2}(\tilde{n}_1, c_{\tilde{n}_1} (\tilde{n}, q[\sigma^q_{\tilde{n}_1,\tilde{n}}(\tilde{\gamma})], \sigma^q_{\tilde{n}_1,\tilde{n}}(\tilde{\gamma})), \vartheta^q_{\tilde{n}_1,\tilde{n}}[\sigma^q_{\tilde{n}_1,\tilde{n}}(\tilde{\gamma})]) =$$
$$= \gamma_{\tilde{n}_2}(\tilde{n}, q[\sigma^q_{\tilde{n}_1,\tilde{n}}(\tilde{\gamma})], \sigma^q_{\tilde{n}_1,\tilde{n}}(\tilde{\gamma})) = \vartheta^q_{\tilde{n}_2,\tilde{n}}[\sigma^q_{\tilde{n}_1,\tilde{n}}(\tilde{\gamma})].$$

From here we deduce $\vartheta^{q_1}_{\tilde{n}_2,\tilde{n}_1}[\vartheta^q_{\tilde{n}_1,\tilde{n}}(\tilde{\gamma})] = \vartheta^q_{\tilde{n}_2,\tilde{n}}(\gamma)$. The mapping $\vartheta^q_{\tilde{n}_2,\tilde{n}}$ is the product of two mappings which are one-to-one and onto hence $\vartheta^q_{\tilde{n}_2,\tilde{n}}$ has an inverse $\sigma^q_{\tilde{n}_2,\tilde{n}}$ defined on $\mathfrak{C}$. It follows that $\vartheta^q_{\tilde{n}_1,\tilde{n}}$ has an inverse defined on $\mathfrak{C}$ for all $\tilde{n} \leq n \leq \tilde{n} + 4N$; the reasoning may be repeated and we deduce that $\vartheta^q_{n,\tilde{n}}$ has for all $n \geq \tilde{n}$ an inverse defined on $\mathfrak{C}$. In our proof we used the fact that $q_1 = P_{\tilde{n}_1,\tilde{n}}q$ belongs to $Q(l, L)$; hence we must prove that $P_{n,\tilde{n}}q \in Q(l, L)$ for all $n \geq \tilde{n} + N$.

We have $[P_{\tilde{n}_2,\tilde{n}}q]\,(\tilde{\gamma}) = c_{\tilde{n}_2}(\tilde{n}, q[\sigma^q_{\tilde{n}_2,\tilde{n}}(\tilde{\gamma})], \sigma^q_{\tilde{n}_2,\tilde{n}}(\tilde{\gamma})) =$
$= c_{\tilde{n}_2}(\tilde{n}_1, c_{\tilde{n}_1}(\tilde{n}, q[\sigma^q_{\tilde{n}_2,\tilde{n}}(\tilde{\gamma})], \sigma^q_{\tilde{n}_2,\tilde{n}}(\tilde{\gamma})), \gamma_{\tilde{n}_1}(\tilde{n}, q[\sigma^q_{\tilde{n}_2,\tilde{n}}(\tilde{\gamma})], \sigma^q_{\tilde{n}_2,\tilde{n}}(\tilde{\gamma})))$.
But $\sigma^q_{\tilde{n}_2,\tilde{n}} = \sigma^q_{\tilde{n}_1,\tilde{n}}(\sigma^{q_1}_{\tilde{n}_2,n_1})$, hence $c_{\tilde{n}_1}(\tilde{n}, q[\sigma^q_{\tilde{n}_2,\tilde{n}}(\tilde{\gamma})], \sigma^q_{\tilde{n}_2,\tilde{n}}(\tilde{\gamma})) =$
$= c_{\tilde{n}_1}(\tilde{n}, q[\sigma^q_{\tilde{n}_1,\tilde{n}}(\sigma^{q_1}_{\tilde{n}_2,\tilde{n}_1}(\tilde{\gamma}))], \sigma^q_{\tilde{n}_1,\tilde{n}}(\sigma^{q_1}_{\tilde{n}_2,\tilde{n}_1}(\tilde{\gamma}))) =$
$= [P_{\tilde{n}_1,\tilde{n}}q]\,(\sigma^{q_1}_{\tilde{n}_2,\tilde{n}_1}(\tilde{\gamma})) = q_1[\sigma^{q_1}_{\tilde{n}_2,\tilde{n}_1}(\tilde{\gamma})]$ and
$\gamma_{\tilde{n}_1}(\tilde{n}, q[\sigma^q_{\tilde{n}_2,\tilde{n}}(\tilde{\gamma})], \sigma^q_{\tilde{n}_2,\tilde{n}}(\tilde{\gamma})) = \vartheta^q_{\tilde{n}_1,\tilde{n}}(\sigma^q_{\tilde{n}_2,\tilde{n}}(\tilde{\gamma})) = \sigma^{q_1}_{\tilde{n}_2,\tilde{n}_1}(\tilde{\gamma})$.

It follows that

$[P_{\tilde{n}_2,\tilde{n}}q]\,(\tilde{\gamma}) = c_{\tilde{n}_2}(\tilde{n}_1, q_1[\sigma^{q_1}_{\tilde{n}_2,\tilde{n}_1}(\tilde{\gamma})], \sigma^{q_1}_{\tilde{n}_2,\tilde{n}_1}(\tilde{\gamma})) = [P_{\tilde{n}_2,\tilde{n}_1}q_1]\,(\tilde{\gamma})$
hence $P_{\tilde{n}_2,\tilde{n}}q \in Q(l, L)$ and $P_{\tilde{n}_2,\tilde{n}}\,q = P_{\tilde{n}_2,\tilde{n}_1}P_{\tilde{n}_1,\tilde{n}}q$.

The reasoning may be repeated and we deduce that $P_{n,\tilde{n}}q \in Q(l, L)$ for all $n \geq \tilde{n} + N$ and that $P_{\tilde{n}_2,\tilde{n}} = P_{\tilde{n}_2,\tilde{n}_1}P_{\tilde{n}_1,\tilde{n}}$ for all $\tilde{n}_2 \geq \tilde{n}_1 \geq \tilde{n} + N$.

Let us remark the most important relation
$[P_{n,\tilde{n}}q]\,(\gamma_n(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma})) = [P_{n,\tilde{n}}q]\,(\vartheta^q_{n,\tilde{n}}(\tilde{\gamma})) =$
$= c_n(\tilde{n}, q[\sigma^q_{n,\tilde{n}}\vartheta^q_{n,\tilde{n}}(\tilde{\gamma})], \sigma^q_{n,\tilde{n}}\vartheta^q_{n,\tilde{n}}(\tilde{\gamma})) = c_n(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma})$ for all $n \geq \tilde{n}$.

D. We have $\|c_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) - [P_{n,\tilde{n}}q]\,(\gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma}))\| \leq$
$\leq \|c_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) - c_n(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma})\| +$
$+ \|[P_{n,\tilde{n}}q]\,(\gamma_n\,(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma})) - [P_{n,\tilde{n}}q]\,(\gamma_n\,(\tilde{n}, \tilde{c}, \tilde{\gamma}))\| \leq$
$\leq \|c_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) - c_n(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma})\| + L\,\|\gamma_n(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}) - \gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma})\| \leq$
$\leq \alpha_1\|q(\tilde{\gamma}) - \tilde{c}\|$ for $\|\tilde{c}\| \leq l$, $\tilde{n} + N \leq n \leq \tilde{n} + 2N$.

From here follows that
$\|[P_{n,\tilde{n}}q_2]\,(\gamma_n(\tilde{n}, q_2(\tilde{\gamma}), \tilde{\gamma})) - [P_{n,\tilde{n}}q_1]\,(\gamma_n(\tilde{n}, q_2(\tilde{\gamma}), \tilde{\gamma}))\| \leq \alpha_1\,\|q_2(\tilde{\gamma}) - q_1(\tilde{\gamma})\|$
hence
$\|[P_{n,\tilde{n}}q_2]\,(\tilde{\gamma}) - [P_{n,\tilde{n}}q_1]\,(\tilde{\gamma})\| \leq \alpha_1\|q_2[\sigma^{q_1}_{n,\tilde{n}}(\tilde{\gamma})] - q_1[\sigma^{q_2}_{n,\tilde{n}}(\tilde{\gamma})]\|$,
$\tilde{n} + N \leq n \leq \tilde{n} + 2N$.

Let now $q_i \in Q(l, L)$, $\lim_{i\to\infty} q_i(\tilde{\gamma}) = q(\tilde{\gamma})$ uniformly with respect to $\tilde{\gamma} \in \mathfrak{C}$. Let $\tilde{\gamma} \in \mathfrak{C}$, $\tilde{\gamma}_i = \sigma^{q_i}_{n,\tilde{n}}(\tilde{\gamma})$, hence $\tilde{\gamma} = \gamma_n(\tilde{n}, q_i(\tilde{\gamma}_i), \tilde{\gamma}_i)$. We have $\|q_i(\tilde{\gamma}_i) - q_i(\tilde{\gamma}_j)\| \leq \leq L\,\|\tilde{\gamma}_i - \tilde{\gamma}_j\|$ hence from condition 3° a) we deduce
$(1 - \alpha_2)\,\|\tilde{\gamma}_i - \tilde{\gamma}_j\| \leq \|\gamma_n(\tilde{n}, q_i(\tilde{\gamma}_i), \tilde{\gamma}_i) - \gamma_n(\tilde{n}, q_i(\tilde{\gamma}_j), \tilde{\gamma}_j)\| \leq$
$\leq \|\gamma_n(\tilde{n}, q_i(\tilde{\gamma}_i), \tilde{\gamma}_i) - \gamma_n(\tilde{n}, q_j(\tilde{\gamma}_j), \tilde{\gamma}_j)\| +$
$+ \|\gamma_n(\tilde{n}, q_j(\tilde{\gamma}_j), \tilde{\gamma}_j) - \gamma_n(\tilde{n}, q_i(\tilde{\gamma}_j), \tilde{\gamma}_j)\| =$
$= \|\gamma_n(\tilde{n}, q_j(\tilde{\gamma}_j), \tilde{\gamma}_j) - \gamma_n(\tilde{n}, q_i(\tilde{\gamma}_j), \tilde{\gamma}_j)\| \leq k_1k_2^{n-\tilde{n}}\|q_j(\tilde{\gamma}_j) - q_i(\tilde{\gamma}_j)\|$.

For $\varepsilon > 0$ let $N_\varepsilon > 0$ be such that $n \geq N_\varepsilon$ implies $\|q_{n+p}(\gamma) - q_n(\gamma)\| \leq \varepsilon$ for all $\gamma \in \mathfrak{C}$; then $\|q_{n+p}(\tilde{\gamma}_n) - q_n(\tilde{\gamma}_n)\| \leq \varepsilon$ for $n \geq N_\varepsilon$ and $\|\tilde{\gamma}_{j+p} - \tilde{\gamma}_j\| \leq$

$\leq \frac{1}{1-\alpha_2} k_1 k_2^{n-\tilde{n}} ||q_j(\tilde{\gamma}_j) - q_{j+p}(\tilde{\gamma}_j)|| \leq \frac{\varepsilon}{1-\alpha_2} k_1 k_2^{n-\tilde{n}}$ hence $\tilde{\gamma}_j$ is a Cauchy sequence. Let $\tilde{\gamma}_0 = \lim_{j\to\infty} \tilde{\gamma}_j$. We have $\lim_{j\to\infty} q_j(\tilde{\gamma}_j) = q(\tilde{\gamma}_0)$, $\lim_{j\to\infty} c_n(\tilde{n}, q_j(\tilde{\gamma}_j), \tilde{\gamma}_j) =$ $= c_n(\tilde{n}, q(\tilde{\gamma}_0), \tilde{\gamma}_0)$, $\lim_{j\to\infty} \gamma_n(\tilde{n}, q_j(\tilde{\gamma}_j), \tilde{\gamma}_j) = \gamma_n(\tilde{n}, q(\tilde{\gamma}_0), \tilde{\gamma}_0)$, hence

$$\gamma_n(\tilde{n}, q(\tilde{\gamma}_0), \tilde{\gamma}_0) = \tilde{\gamma}, \quad \lim_{j\to\infty} [P_{n,\tilde{n}} q_j] (\tilde{\gamma}) = \lim_{j\to\infty} [P_{n,\tilde{n}} q_j] (\gamma_n(\tilde{n}, q_j(\tilde{\gamma}_j), \tilde{\gamma}_j)) =$$
$$= \lim_{j\to\infty} c_n(\tilde{n}, q_j(\tilde{\gamma}_j), \tilde{\gamma}_j) = c_n(\tilde{n}, q(\tilde{\gamma}_0), \tilde{\gamma}_0) = [P_{n,\tilde{n}} q] (\gamma_n(\tilde{n}, q(\tilde{\gamma}_0), \tilde{\gamma}_0)) =$$
$$= [P_{n,\tilde{n}} q] (\tilde{\gamma}),$$

the convergence being uniform with respect to $\tilde{\gamma} \in \mathfrak{C}$.

We have thus proved that for all $n \geq \tilde{n}$ from $q_i \overset{u}{\to} q$ follows that $P_{n,\tilde{n}} q_i \overset{u}{\to}$ $\overset{u}{\to} P_{n,n} q$.

E. We have $\lim_{\tilde{n}\to-\infty} P_{n,\tilde{n}} q = p_n$, $p_n \in Q(l, L)$, $P_{n_2,\tilde{n}_1} p_{n_1} = p_{n_2}$ for $n_2 \geq n_1$. Let $\tilde{n}_1 = n$, $\tilde{n}_i - 2N \leq \tilde{n}_{i+1} \leq \tilde{n}_i - N$, $j > i > 1$; we have $P_{n,\tilde{n}_j} q =$ $= P_{n,\tilde{n}_i}(P_{\tilde{n}_i,\tilde{n}_j} q)$ and

$$||[P_{n,\tilde{n}_i} q] (\tilde{\gamma}) - [P_{n,\tilde{n}_j} q] (\tilde{\gamma})|| =$$
$$= ||[P_{\tilde{n}_1,\tilde{n}_2}, \ldots, P_{\tilde{n}_{i-1},\tilde{n}_i} q] (\tilde{\gamma}) - [P_{\tilde{n}_1,\tilde{n}_2}, \ldots, P_{\tilde{n}_{i-1},\tilde{n}_j}(P_{\tilde{n}_{i-1},\tilde{n}_j} q)] (\tilde{\gamma})|| \leq$$
$$\leq \alpha_1^{i-1} \sup_{\gamma} ||[P_{\tilde{n}_i,\tilde{n}_i} q] (\tilde{\gamma}) - q(\tilde{\gamma})|| \leq \alpha_1^{i-1} . 2l, \quad \text{hence}$$

$\lim_{i\to\infty} [P_{n,\tilde{n}_i} q] (\tilde{\gamma})$ exists, uniformly with respect to $\tilde{\gamma} \in \mathfrak{C}$.

Moreover

$||[P_{n,\tilde{n}_i} q_2] (\tilde{\gamma}) - [P_{n,\tilde{n}_i} q_1] (\tilde{\gamma})|| \leq \alpha_1^i \sup ||q_1(\tilde{\gamma}) - q_2(\tilde{\gamma})||$, hence $\lim_{i\to\infty} [P_{n,\tilde{n}_i} q_2] (\tilde{\gamma}) = \lim_{i\to\infty} [P_{n,\tilde{n}} q_1] (\tilde{\gamma})$ and p_n does not depend on q. From $P_{n_2,n_1} P_{n_1,\tilde{n}} q = P_{n_2,\tilde{n}} q$ it follows for $\tilde{n} \to -\infty$ that $P_{n_2,n_1} p_{n_1} = p_{n_2}$ $(n_2 \geq n_1)$. Let indeed $\tilde{n}_i \to -\infty$; then $P_{n_1,\tilde{n}_i} q \overset{u}{\to} p_{n_1}$ hence $P_{n_2,n_1} P_{n_1,\tilde{n}_i} q \overset{u}{\to} P_{n_2,n_1} p_{n_1}$ and $P_{n_2,\tilde{n}_i} q \overset{u}{\to} p_{n_2}$.

F. The functions p_n have all properties stated in theorem 1. It is obvious that $p_n \in Q(l, L)$ hence a), b) are verified. We have further

$[P_{n,\tilde{n}} q] (\gamma_n(\tilde{n}, q(\tilde{\gamma},) \tilde{\gamma})) = c_n(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma})$ and $P_{n,\tilde{n}} p_{\tilde{n}} = p_n$ for all $n \geq \tilde{n}$. Let in the first relation $q = p_{\tilde{n}}$; we get

$$c_n(\tilde{n}, p_{\tilde{n}}(\tilde{\gamma}), \tilde{\gamma}) = [P_{n,\tilde{n}} p_{\tilde{n}}] (\gamma_n(\tilde{n}, p_{\tilde{n}}(\tilde{\gamma}), \tilde{\gamma})) = p_n(\gamma_n(\tilde{n}, p_{\tilde{n}}(\tilde{\gamma}), \tilde{\gamma}))$$

for $n \geq \tilde{n}$. We shall prove that the relation holds for all n. We have

$$p_{\tilde{n}}(\tilde{\gamma}) = [P_{\tilde{n},\tilde{n}-i} p_{\tilde{n}-i}] (\tilde{\gamma}) = c_{\tilde{n}}(\tilde{n} - i, p_{\tilde{n}-i}[\sigma^{p\tilde{n}-i}_{\tilde{n},\tilde{n}-i}(\tilde{\gamma})], \sigma^{p\tilde{n}-i}_{\tilde{n},\tilde{n}-i}(\tilde{\gamma}))$$
$$\gamma_{\tilde{n}}(\tilde{n} - i, p_{\tilde{n}-i}[\sigma^{p\tilde{n}-i}_{\tilde{n},\tilde{n}-i}(\tilde{\gamma})], \sigma^{p\tilde{n}-i}_{\tilde{n},\tilde{n}-i}(\tilde{\gamma})) = \vartheta^{p\tilde{n}-i}_{\tilde{n},\tilde{n}-i} \sigma^{p\tilde{n}-i}_{\tilde{n},\tilde{n}-i}(\tilde{\gamma}) = \tilde{\gamma}$$

hence

$$c_n(\tilde{n}, p_{\tilde{n}}(\tilde{\gamma}), \tilde{\gamma}) = c_n(\tilde{n} - i, p_{\tilde{n}-i}[\sigma^{p\tilde{n}-i}_{\tilde{n},\tilde{n}-i}(\tilde{\gamma})], \sigma^{p\tilde{n}-i}_{\tilde{n},\tilde{n}-i}(\tilde{\gamma})) =$$
$$= p_n(\gamma_n(\tilde{n} - i, p_{\tilde{n}-i}[\sigma^{p\tilde{n}-i}_{\tilde{n},\tilde{n}-i}(\tilde{\gamma})], \sigma^{p\tilde{n}-i}_{\tilde{n},\tilde{n}-i}(\tilde{\gamma})))$$

and this relation is true for $n \geq \tilde{n} - i$. We get from here

$c_n(\tilde{n} - i, p_{\tilde{n}-i}(\tilde{\gamma}), \tilde{\gamma}) = p_n(\gamma_n(\tilde{n} - i, p_{\tilde{n}-i}(\tilde{\gamma}), \tilde{\gamma}))$

and relation d) is proved for all n.

To establish c) we start from $||c_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) - [P_{n,\tilde{n}}q]\, \gamma_n\, (\tilde{n}, \tilde{c}, \tilde{\gamma}))|| \leq \leq \alpha_1 ||q(\tilde{\gamma}) - \tilde{c}||$ for $\tilde{n} + N \leq n \leq \tilde{n} + 2N$, $||\tilde{c}|| \leq l$; let in this relation $q = p_{\tilde{n}}$. We get $||c_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) - [P_{n,\tilde{n}}p_{\tilde{n}}]\,(\gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma}))|| \leq \alpha_1 ||p_{\tilde{n}}(\tilde{\gamma}) - \tilde{c}||$, hence $||c_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) - p_n(\gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma}))|| \leq \alpha_1 ||p_n(\tilde{\gamma}) - \tilde{c}||$ for $\tilde{n} + N \leq n \leq \tilde{n} + 2N$. By induction it is then proved that

$||c_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) - p_n(\gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma}))|| \leq \alpha_1^k ||p_{\tilde{n}}(\tilde{\gamma}) - \tilde{c}||$ for $\tilde{n} + kN \leq n \leq \tilde{n} + + (k + 1)\, N$.

For $\tilde{n} \leq n \leq \tilde{n} + N$ we have

$||c_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) - c_n(\tilde{n}, p_{\tilde{n}}(\tilde{\gamma}), \tilde{\gamma})|| + ||\gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) - \gamma_n(\tilde{n}, p_{\tilde{n}}(\tilde{\gamma}), \tilde{\gamma})|| \leq$
$\leq k_1 k_2^N ||\tilde{c} - p_{\tilde{n}}(\tilde{\gamma})||$

$||p_n(\gamma_n(\tilde{n}, p_{\tilde{n}}(\tilde{\gamma}), \tilde{\gamma})) - p_n(\gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma}))|| \leq L\, ||\gamma_n(\tilde{n}, p_{\tilde{n}}(\tilde{\gamma}), \tilde{\gamma}) - \gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma})|| \leq$
$\leq L k_1 k_2^N ||\tilde{c} - p_{\tilde{n}}(\tilde{\gamma})||$, hence

$||c_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) - p_n(\gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma}))|| \leq (1 + L)\, k_1 k_2^N ||\tilde{c} - p_{\tilde{n}}(\tilde{\gamma})||$.

Let $K = (1 + L)\, k_1 \left(\frac{k_2}{\alpha}\right)^N$, $\alpha = \alpha_1^{\frac{1}{N}}$; we have

$||c_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) - p_n(\gamma_n(\tilde{n}, \tilde{c}, \tilde{\gamma}))|| \leq K\alpha^N ||\tilde{c} - p_{\tilde{n}}(\tilde{\gamma})|| \leq K\alpha^{n-\tilde{n}} ||\tilde{c} - p_{\tilde{n}}(\tilde{\gamma})||$,
$\tilde{n} \leq n \leq \tilde{n} + N$

$||c_n(\tilde{n}, \tilde{c}, \tilde{\gamma}) - p_n(\gamma_n\, (\tilde{n}, \tilde{c}, \tilde{\gamma}))|| \leq \alpha^{kN} ||\tilde{c} - p_{\tilde{n}}(\tilde{\gamma})|| \leq \frac{1}{\alpha^N}\, \alpha^{n-\tilde{n}} ||\tilde{c} - p_{\tilde{n}}(\tilde{\gamma})|| \leq$

$\leq K\alpha^{n-\tilde{n}} ||\tilde{c} - p_{\tilde{n}}(\tilde{\gamma})||$

for $\tilde{n} + kN \leq n \leq \tilde{n} + (k + 1)\, N$, hence for all $n \geq \tilde{n}$, and property c) is established. Let us prove property e). Let p'_n with properties a), b), c), d), $\tilde{\gamma} \in \mathfrak{C}$, $\tilde{n}' = \tilde{n} - N$, $\tilde{\gamma}' = \sigma^{p'_{\tilde{n}}}_{\tilde{n},\tilde{n}'}(\tilde{\gamma})$; we have $p'_{\tilde{n}}(\tilde{\gamma}) = p'_{\tilde{n}}(\gamma_{\tilde{n}}(\tilde{n}', p'_{\tilde{n}'}(\tilde{\gamma}'), \tilde{\gamma}')) = = c_{\tilde{n}}(\tilde{n}', p'_{\tilde{n}'}(\tilde{\gamma}'), \tilde{\gamma}')$ (by d)) and $||c_{\tilde{n}}(\tilde{n}', p'_{\tilde{n}'}(\tilde{\gamma}'), \tilde{\gamma}') - p_{\tilde{n}}(\gamma_{\tilde{n}}(\tilde{n}', p'_{\tilde{n}'}(\tilde{\gamma}'), \tilde{\gamma}')|| \leq \leq K\alpha^{\tilde{n}-\tilde{n}'} ||p'_{\tilde{n}'}(\tilde{\gamma}') - p_{\tilde{n}'}(\tilde{\gamma}')||$ (by c)). It follows that $||p'_{\tilde{n}}(\tilde{\gamma}) - p_{\tilde{n}}(\tilde{\gamma})|| \leq \leq K\alpha^N ||p'_{\tilde{n}'}(\tilde{\gamma}') - p_{\tilde{n}'}(\tilde{\gamma}')||$ and by induction $||p'_{\tilde{n}}(\tilde{\gamma}) - p_{\tilde{n}}(\tilde{\gamma})|| \leq \leq K\alpha^{jN} ||p'_{\tilde{n}-jN}(\tilde{\gamma}^{(j)}) - p_{\tilde{n}-jN}(\tilde{\gamma}^{(j)})|| \leq 2lK\alpha^{jN}$ and for $j \to \infty$ we get $p'_{\tilde{n}}(\tilde{\gamma}) = = p_{\tilde{n}}(\tilde{\gamma})$.

G. To obtain property f) 1° we remark that

$[P_{n,\tilde{n}}q]\,(\gamma_n(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma})) = c_n(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}) = c_{n+\nu}(\tilde{n} + \nu, q(\tilde{\gamma}), \tilde{\gamma}) =$
$= [P_{n+\nu,\tilde{n}+\nu}q]\,(\gamma_{n+\nu}(\tilde{n} + \nu, q(\tilde{\gamma}), \tilde{\gamma})) = [P_{n+\nu,\tilde{n}+\nu}q]\,(\gamma_n(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}))$

hence $P_{n,\tilde{n}}q = P_{n+\nu,\tilde{n}+\nu}q$ and for $\tilde{n} \to -\infty$ we get $p_n = p_{n+\nu}$.

Let then in the conditions of f) 2° q be periodic of period ω; we have $[P_{n,\tilde{n}}q]\,(\gamma_n(\tilde{n}, q(\tilde{\gamma} + \omega), \tilde{\gamma} + \omega)) = c_n(\tilde{n}, q(\tilde{\gamma} + \omega), \tilde{\gamma} + \omega) =$
$= c_n(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}) = [P_{n,\tilde{n}}q]\,(\gamma_n(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}))$

hence $[P_{n,\tilde{n}}q]\,(\vartheta^q_{n,\tilde{n}}(\tilde{\gamma}) + \omega) = [P_{n,\tilde{n}}q]\,(\vartheta^q_{n,\tilde{n}}(\tilde{\gamma}))$ and for $\tilde{\gamma} = \sigma^q_{n,\tilde{n}}(\gamma)$ we get

$[P_{n,\tilde{n}}q]\,(\gamma + \omega) = [P_{n,\tilde{n}}q]\,(\gamma)$. For $\tilde{n} \to -\infty$ we get $p_n(\gamma + \omega) = p_n(\gamma)$. We shall now prove g).

Let $\lim\limits_{i\to\infty} c_n^i(\tilde{n}, \tilde{c}, \tilde{\gamma}) = c_n^*(\tilde{n}, \tilde{c}, \tilde{\gamma})$, $\lim\limits_{i\to\infty} \gamma_n^i(\tilde{n}, \tilde{c}, \tilde{\gamma}) = \gamma_n^*(\tilde{n}, \tilde{c}, \tilde{\gamma})$ the convergence being for $n = \tilde{n} + N$ uniform with respect to $\tilde{n}$, $\tilde{c}$, $\tilde{\gamma}$. We have
$c_{\tilde{n}+N}(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}) = [P^{(i)}_{\tilde{n}+N,\tilde{n}}q]\,(\gamma^{(i)}_{\tilde{n}+N}(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}))$
$c^*_{\tilde{n}+N}(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}) = [P^*_{\tilde{n}+N,\tilde{n}}q]\,(\gamma^*_{\tilde{n}+N}(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}))$
since if systems $c_n^i(\tilde{n}, \tilde{c}, \tilde{\gamma})$, $\gamma_n^i(\tilde{n}, \tilde{c}, \tilde{\gamma})$ have all the properties 1⁰, 2⁰, 3⁰, 4⁰, the same is true for the limit system $c_n^*(\tilde{n}, \tilde{c}, \tilde{\gamma})$, $\gamma_n^*(\tilde{n}, \tilde{c}, \tilde{\gamma})$.

We deduce

$$\begin{aligned}
&||[P^{(i)}_{\tilde{n}+N,\tilde{n}}q]\,(\gamma^*_{\tilde{n}+N}(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma})) - [P^*_{\tilde{n}+N,\tilde{n}}q]\,(\gamma^*_{\tilde{n}+N}(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}))|| \le \\
&\quad \le ||c^i_{\tilde{n}+N}(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}) - [P^{(i)}_{\tilde{n}+N,\tilde{n}}q]\,(\gamma^*_{\tilde{n}+N}(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}))|| + \\
&\quad + ||c^i_{\tilde{n}+N}(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}) - c^*_{\tilde{n}+N}(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma})|| \le \\
&\quad \le L\,||\gamma^i_{\tilde{n}+N}(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}) - \gamma^*_{\tilde{n}+N}(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma})|| + \\
&\quad + ||c^i_{\tilde{n}+N}(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma}) - c^*_{\tilde{n}+N}(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma})|| \le k'\varepsilon_i, \quad \lim_{i\to\infty} \varepsilon_i = 0,
\end{aligned}$$

$||[P^{(i)}_{\tilde{n}+N,\tilde{n}}q]\,(\gamma) - [P^*_{\tilde{n}+N,\tilde{n}}q]\,(\gamma)|| \le k'\varepsilon_i$. We have further

$$\begin{aligned}
&||[P^{(i)}_{\tilde{n}+N,\tilde{n}}q_1]\,(\tilde{\gamma}) - [P^*_{\tilde{n}+N,\tilde{n}}q_2]\,(\tilde{\gamma})|| \le ||[P^{(i)}_{\tilde{n}+N,\tilde{n}}q_1]\,(\tilde{\gamma}) - [P^*_{\tilde{n}+N,\tilde{n}}q_1]\,(\tilde{\gamma})|| + \\
&\quad + ||[P^*_{\tilde{n}+N,\tilde{n}}q_1]\,(\tilde{\gamma}) - [P^*_{\tilde{n}+N,\tilde{n}}q_2]\,(\tilde{\gamma})|| \le k'\varepsilon_i + \alpha_1 \sup_{\tilde{\gamma}} ||q_1(\tilde{\gamma}) - q_2(\tilde{\gamma})||.
\end{aligned}$$

It follows that

$$\begin{aligned}
&||[P^{(i)}_{n,n-jN}q]\,(\tilde{\gamma}) - [P^*_{n,n-jN}q]\,(\tilde{\gamma})|| = \\
&\quad = ||[P^{(i)}_{n,n-N}P^{(i)}_{n-N,n-2N}\cdots P^{(i)}_{n-(j-1)N,n-jN}q]\,(\tilde{\gamma}) - \\
&\quad - [P^*_{n,n-N}\cdots P^*_{n-(j-1)N,n-jN}q]\,(\tilde{\gamma})|| \le k'\varepsilon_i(1 + \alpha_1 + \alpha_1^2 + \ldots + \alpha_1^j) \le \\
&\quad \le \frac{k'\varepsilon_i}{1 - \alpha_1}
\end{aligned}$$

and for $j \to \infty$ we get

$$||p_n^{(i)}(\tilde{\gamma}) - p_n^*(\tilde{\gamma})|| \le \frac{k'\varepsilon_i}{1 - \alpha_1}$$

hence $\lim\limits_{i\to\infty} p_n^i(\tilde{\gamma}) = p_n^*(\tilde{\gamma})$ uniformly with respect to n and $\tilde{\gamma} \in \mathfrak{C}$.

Let now $n_k \to \infty$, n_{k_l} the subsequence from the statement of g); denote $c_n^l(\tilde{n}, \tilde{c}, \tilde{\gamma}) = c_{n+n_{k_l}}(n + n_{k_l}, \tilde{c}, \tilde{\gamma})$, $\gamma_n^l(\tilde{n}, \tilde{c}, \tilde{\gamma}) = \gamma_{n+n_{k_l}}(\tilde{n} + n_{k_l}, \tilde{c}, \tilde{\gamma})$. The systems $c_n^l(\tilde{n}, \tilde{c}, \tilde{\gamma})$, $\gamma_n^l(\tilde{n}, \tilde{c}, \tilde{\gamma})$ have all properties 1⁰, 2⁰, 3⁰, 4⁰ from the statement since these properties depend uniquely on the difference $n - \tilde{n}$. Hence $\lim\limits_{l\to\infty} p_n^{(l)}(\tilde{\gamma}) = p_n^*(\tilde{\gamma})$, the convergence being uniform with respect to n and $\tilde{\gamma}$. But $P^{(l)}_{n,\tilde{n}} = P_{n+n_{k_l},\tilde{n}+n_{k_l}}$ hence for $\tilde{n} \to -\infty$ we get $p_n^{(l)} = p_{n+n_{k_l}}$ and $p_{n+n_{k_l}}$ converges to p_n^* uniformly with respect to n and $\tilde{\gamma}$. The almost periodicity of p_n is thus proved.

Remarks. 1⁰. If the system has the property of periodicity from f) 1⁰ we can get p_n by proving that the mapping $P_{\nu,0}$: $Q(l, L) \to Q(l, L)$ has a unique fixed-point. We may organize $Q(l, L)$ as a metric space in the usual way with the distance $\varrho(q_1, q_2) = \sup_\gamma ||q_1(\gamma) - q_2(\gamma)||$. Let h be such that $N \le h\nu \le \le 2N$. We have $||[P_{h\nu,0}q_1](\gamma) - [P_{h\nu,0}q_2](\gamma)|| \le \alpha_1 \sup_\gamma ||q_1(\tilde{\gamma}) - q_2(\tilde{\gamma})|| = = \alpha_1\varrho(q_1, q_2)$ hence $\varrho(P_{h\nu,0}q_1, P_{h\nu,0}q_2) \le \alpha_1\varrho(q_1, q_2)$ and $P_{h\nu,0}$ is a contraction in $Q(l, L)$. It follows that $P_{h\nu,0}$ admits a unique fixed point q_0. But $P_{h\nu,0} = = P_{h\nu,(h-1)\nu}P_{(h-1)\nu,0} = P_{\nu,0}P_{(h-1)\nu,0}$ and by induction $P_{h\nu,0} = (P_{\nu,0})^h$ which shows that q_0 is a fixed point for $P_{\nu,0}$.

For this proof to be complete we must show that $P_{n_3,n_2}P_{n_2,n_1} = P_{n_3,n_1}$ holds for all $n_1 \le n_2 \le n_3$ (and not only for $n_2 \ge n_1 + N$).

From the fact that the fundamental relation $[P_{n,\tilde{n}}q](\gamma_n(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma})) = = c_n(\tilde{n}, q(\tilde{\gamma}), \tilde{\gamma})$ holds for all $n \ge \tilde{n}$ we deduce

$[P_{n_3,n_1}q](\gamma_{n_3}(n_1, q(\tilde{\gamma}), \tilde{\gamma})) = c_{n_3}(n_1, q(\tilde{\gamma}), \tilde{\gamma}) =$
$= c_{n_3}(n_2, c_{n_2}(n_1, q(\tilde{\gamma}), \tilde{\gamma}), \gamma_{n_2}(n_1, q(\tilde{\gamma}), \tilde{\gamma})) =$
$= c_{n_3}(n_2, [P_{n_2,n_1}q](\gamma_{n_2}(n_1, q(\tilde{\gamma}), \tilde{\gamma})), \gamma_{n_2}(n_1, q(\tilde{\gamma}), \tilde{\gamma})) =$
$= [P_{n_3,n_2}P_{n_2,n_1}q][\gamma_{n_3}(n_2, c_{n_2}(n_1, q(\tilde{\gamma}), \tilde{\gamma}), \gamma_{n_2}(n_1, q(\tilde{\gamma}), \tilde{\gamma}))] =$
$= [P_{n_3,n_2}P_{n_2,n_1}q][\gamma_{n_3}(n_1, q(\tilde{\gamma}), \tilde{\gamma})]$; if we set in this relation $\tilde{\gamma} = \sigma^q_{n_3,n_1}(\gamma)$ we get $[P_{n_3,n_1}q](\gamma) = [P_{n_3,n_2}P_{n_2,n_1}q](\gamma)$.

Let then q_0 the fixed point of $P_{\nu,0}$ and $p_n = P_{n,0}q_0$. We have $P_{n,\tilde{n}}p_{\tilde{n}} = = P_{n,\tilde{n}}P_{\tilde{n},0}q_0 = P_{n,0}q_0 = p_n$. Observe that $p_n \in Q(l, L)$; indeed $P_{n,0}q_0 = = P_{n+h\nu,h\nu}q_0 = P_{n+h\nu,h\nu}P_{h\nu,0}q_0 = P_{n+h\nu,0}q_0 \in Q(l, L)$ since $n + h\nu \ge N$. Properties a), b), c), d), e) are eastly verified since in proving them we used only $p_n \in Q(l, L)$ and $P_{n,\tilde{n}}p_{\tilde{n}} = p_n$. We have then $p_{n+\nu} = P_{n+\nu,0}q_0 = P_{n+\nu,\nu}P_{\nu,0}q_0 = = P_{n+\nu,0}q_0 = P_{n,0}q_0 = p_n$ and if we observe that $P_{h\nu,0}$ maps the set of periodic functions of period ω from $Q(l, L)$ in itself when condition f) 2° is verified, it is seen that q_0 is periodic and $p_n(\gamma + \omega) = [P_{n,0}q_0](\gamma + \omega) = [P_{n,0}q_0](\gamma) = = p_n(\gamma)$.

2°. We can use the above method for discrete systems of the form $c_n(\tilde{n}, \tilde{c})$ and obtain conclusions about the existence of an exponentially stable bounded solution which is periodic in the case of periodic systems and almost-periodic in the case of almost-periodic systems. The proof for this case is much simpler.

We state the following *proposition.*

Let a discrete system have the properties:

1°. $||\tilde{c}|| \le l$, $\tilde{n} + N \le n \le \tilde{n} + 2N$ implies $||c_n(\tilde{n}, \tilde{c})|| \le l$.

2°. $||\tilde{c}_1|| \le l$, $||\tilde{c}_2|| \le l$, $\tilde{n} + N \le n \le \tilde{n} + 2N$ imply
$||c_n(\tilde{n}, \tilde{c}_1) - c_n(\tilde{n}, \tilde{c}_2)|| \le \alpha_1||\tilde{c}_1 - \tilde{c}_2||$.

3°. $||c_n(\tilde{n}, \tilde{c}_1) - c_n(\tilde{n}, \tilde{c}_2)|| \le k_1k_2^{n-\tilde{n}}||\tilde{c}_1 - \tilde{c}_2||$ for all $n \ge n$, $||\tilde{c}_i|| \le H$.

Then there exists a sequence $p_n \in C$ such that

a) $||p_n|| \le l$,

b) $p_n = c_n(n_1, p_{n_1})$ hence p_n is a solution,

c) $||c_n(\tilde{n}, \tilde{c}) - p_n|| \le k\alpha^{n-\tilde{n}}||\tilde{c} - p_{\tilde{n}}||$ for $||\tilde{c}|| \le l$, $n \ge \tilde{n}$,

d) if $c_{n+\nu}(\tilde{n}+\nu, \tilde{c}) = c_n(\tilde{n}, \tilde{c})$ then $p_{n+\nu} = p_n$,

e) for almost periodic systems p_n is almost periodic.

We prove this proposition in the same way as we proved the theorem. Let $||\tilde{c}|| \leq l$, $P_{n,\tilde{n}}\tilde{c} = c_n(\tilde{n}, \tilde{c})$; $P_{\tilde{n}_2,\tilde{n}_1}\tilde{c} = P_{\tilde{n}_2,\tilde{n}_1}P_{\tilde{n}_1,\tilde{n}}\,\tilde{c}$ is obvious. Let $n = \tilde{n}_1$, $\tilde{n}_i - 2N \leq \tilde{n}_{i+1} \leq \tilde{n}_i - N$, $j > i > 1$; we have

$||c_n(\tilde{n}_i, \tilde{c}) - c_n(\tilde{n}_j, \tilde{c})|| = ||c_{\tilde{n}_1}(\tilde{n}_2, c_{\tilde{n}_2}(\tilde{n}_i, \tilde{c})) - c_{\tilde{n}_1}(\tilde{n}_2, c_{\tilde{n}_2}(\tilde{n}_i, \tilde{c}'))||$

where $\tilde{c}' = c_{\tilde{n}_i}(\tilde{n}_j, \tilde{c})$. We get $||c_n(\tilde{n}_i, \tilde{c}) - c_n(\tilde{n}_j, c)|| \leq \alpha_1^{i-1}||\tilde{c} - \tilde{c}'|| \leq 2l\alpha_1^{i-1}$ hence $\lim\limits_{\tilde{n}\to-\infty} c_n(\tilde{n}, \tilde{c})$ exists for $||\tilde{c}|| \leq l$. We define $p_n = \lim\limits_{\tilde{n}\to-\infty} c_n(\tilde{n}, \tilde{c})$ and the proof of properties b), c), d), e) is as in the general case.

II. The theorem on continuous dependence on parameters and the stability theorem.

In order to get a system for which the conditions from the general theorem are verified we have to prove a theorem on the continuous dependence on parameters and a stability theorem.

Theorem 2. *Consider the discrete systems* $x_{n+1} = f_n(x_n)$, $x_{n+1} = f_n^\circ(x_n)$ *and suppose that* $||f_n(x) - f_n^\circ(x)|| \leq \xi$, $\left|\left|\frac{\partial f_n}{\partial x}(x) - \frac{\partial f_n^\circ}{\partial x}(x)\right|\right| \leq \xi$ *for all* n *and for all* $x \in G_n$, $\left|\left|\frac{\partial f_n}{\partial x}\right|\right| \leq K_1$, $\left|\left|\frac{\partial f_n^0}{\partial x}\right|\right| \leq K_1$.

Suppose that

$\left|\left|\frac{\partial f_n}{\partial x}(x_1) - \frac{\partial f_n}{\partial x}(x_2)\right|\right| \leq \omega(||x_1 - x_2||)$, $\left|\left|\frac{\partial f_n^\circ}{\partial x}(x_1) - \frac{\partial f_n^\circ}{\partial x}(x_2)\right|\right| \leq \omega(||x_1 - x_2||)$

$\lim\limits_{\varrho\to 0} \omega(\varrho) = 0$, ω *increasing.*

Then $||x_n(\tilde{n}, \tilde{x}) - x_n^\circ(\tilde{n}, \tilde{x})|| \leq \dfrac{K_1^N - 1}{K_1 - 1}\xi$

$||x_n(\tilde{n}, \tilde{x}_2) - x_n(\tilde{n}, \tilde{x}_1) - x_n^\circ(\tilde{n}, \tilde{x}_2) + x_n^\circ(\tilde{n}, \tilde{x}_1)|| \leq \alpha_N(\xi)\,||\tilde{x}_2 - \tilde{x}_1||$

for $\tilde{n} \leq n \leq \tilde{n} + N$, $\lim\limits_{\xi\to\infty} \alpha_N(\xi) = 0$.

Proof. We have $||x_{\tilde{n}+1}(\tilde{n}, \tilde{x}) - x_{\tilde{n}+1}^\circ(\tilde{n}, \tilde{x})|| = ||f_{\tilde{n}}(\tilde{x}) - f_{\tilde{n}}^\circ(\tilde{x})|| < \xi$.

Suppose $||x_{\tilde{n}+p}(\tilde{n}, \tilde{x}) - x_{\tilde{n}+p}^\circ(\tilde{n}, \tilde{x})|| \leq (1 + K_1 + \ldots + K_1^{p-1})\,\xi$. Then

$||x_{\tilde{n}+p+1}(\tilde{n}, \tilde{x}) - x_{\tilde{n}+p+1}^\circ(\tilde{n}, \tilde{x})|| = ||f_{\tilde{n}+p}(x_{\tilde{n}+p}(\tilde{n}, \tilde{x})) - f_{\tilde{n}+p}^\circ(x_{\tilde{n}+p}^\circ(\tilde{n}, \tilde{x}))|| \leq$

$\leq f_{\tilde{n}+p}(x_{\tilde{n}+p}(\tilde{n}, \tilde{x})) - f_{\tilde{n}+p}(x_{\tilde{n}+p}^\circ(\tilde{n}, \tilde{x}))|| +$

$+ ||f_{\tilde{n}+p}(x_{\tilde{n}+p}^\circ(\tilde{n}, \tilde{x})) - f_{\tilde{n}+p}^\circ(x_{\tilde{n}+p}^\circ(\tilde{n}, \tilde{x}))|| \leq$

$\leq K_1||x_{\tilde{n}+p}(\tilde{n}, \tilde{x}) - x_{\tilde{n}+p}^\circ(\tilde{n}, \tilde{x})|| + \xi \leq (1 + K_1 + \ldots + K_1^p)\,\xi$

and the first assertion is proved. Let us remark that from this assertion it follows that if the solution of system $x_{n+1} = f_n^\circ(x_n)$ is defined for $\tilde{n} \leq n \leq \leq \tilde{n} + N$ then if ξ is small enough the solution of the system $x_{n+1} = f_n(x_n)$ will be also defined for such n.

To prove the second assertion we start from

$$x_{\tilde{n}+1}(\tilde{n}, \tilde{x}_2) - x_{\tilde{n}+1}(\tilde{n}, \tilde{x}_1) - x^\circ_{\tilde{n}+1}(\tilde{n}, \tilde{x}^\circ_{\tilde{n}2}) + x_{\tilde{n}+1}(\tilde{n}, \tilde{x}_1) =$$
$$= f_{\tilde{n}}(\tilde{x}_2) - f_{\tilde{n}}(\tilde{x}_1) - f^\circ_{\tilde{n}}(\tilde{x}_2) + f^\circ_{\tilde{n}}(\tilde{x}_1) =$$
$$= \int_0^1 \left[\frac{\partial f_{\tilde{n}}}{\partial x}(\tilde{x}_1 + \lambda(\tilde{x}_2 - \tilde{x}_1))(\tilde{x}_2 - \tilde{x}_1) - \frac{\partial f^\circ_{\tilde{n}}}{\partial x}(\tilde{x}_1 + \lambda(\tilde{x}_2 - \tilde{x}_1))(\tilde{x}_2 - \tilde{x}_1)\right] d\lambda.$$

We get

$$||x_{\tilde{n}+1}(\tilde{n}, \tilde{x}_2) - x_{\tilde{n}+1}(\tilde{n}, \tilde{x}_1) - x^\circ_{\tilde{n}+1}(\tilde{n}, \tilde{x}_2) + x^\circ_{\tilde{n}+1}(\tilde{n}, \tilde{x}_1)|| \leq \xi ||\tilde{x}_2 - \tilde{x}_1||.$$

We have then

$$x_{\tilde{n}+p+1}(\tilde{n}, \tilde{x}_2) - x_{\tilde{n}+p+1}(\tilde{n}, \tilde{x}_1) - x^\circ_{\tilde{n}+p+1}(\tilde{n}, \tilde{x}_2) + x^\circ_{\tilde{n}+p+1}(\tilde{n}, \tilde{x}_1) =$$
$$= f_{\tilde{n}+p}(x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2)) - f_{\tilde{n}+p}(x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1)) - f^\circ_{\tilde{n}+p}(x^\circ_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2)) +$$
$$+ f^\circ_{\tilde{n}+p}(x^\circ_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1) = f^\circ_{\tilde{n}+p}(x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2)) -$$
$$- f_{\tilde{n}+p}[x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1) + x^\circ_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) - x^\circ_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1)] +$$
$$+ f_{\tilde{n}+p}[x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1) + x^\circ_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) - x^\circ_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1)] -$$
$$- f_{\tilde{n}+p}(x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2)) + f_{\tilde{n}+p}(x^\circ_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1)) -$$
$$- f_{\tilde{n}+p}(x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1)) + f_{\tilde{n}+p}(x^\circ_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2)) -$$
$$- f_{\tilde{n}+p}(x^\circ_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1)) - f^\circ_{\tilde{n}+p}(x^\circ_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2)) + f^\circ_{\tilde{n}+p}(x^\circ_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1)) =$$
$$= \int_0^1 \frac{\partial f_{\tilde{n}+p}}{\partial x}[x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) +$$
$$+ \lambda(x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) - x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1) - x^\circ_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) + x^\circ_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1))] \, d\lambda (x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) -$$
$$- x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1) - x^\circ_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) + x^\circ_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1)) -$$
$$- \int_0^1 \frac{\partial f_{\tilde{n}+p}}{\partial x}[x^\circ_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1) +$$
$$+ \lambda(x^\circ_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1) - x^\circ_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2))] \, d\lambda (x^\circ_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1) - x^\circ_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2)) +$$
$$+ \int_0^1 \frac{\partial f_{\tilde{n}+p}}{\partial x}[x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1) +$$
$$+ \lambda(x^\circ_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1) - x^\circ_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2))] \, d\lambda (x^\circ_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1) - x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2)) +$$
$$+ \int_0^1 \frac{\partial f_{\tilde{n}+p}}{\partial x}[x^\circ_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) +$$
$$+ \lambda(x^\circ_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) - x^\circ_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1))] \, d\lambda (x^\circ_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) - x^\circ_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1)) -$$

$$- \int_0^1 \frac{\partial f^{\circ}_{\tilde{n}+p}}{\partial x} [x^{\circ}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) +$$
$$+ \lambda(x^{\circ}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) - x^{\circ}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1))] \, d\lambda (x^{\circ}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) - x^{\circ}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1)).$$

It follows that

$$v_{p+1} = ||x_{\tilde{n}+p+1}(\tilde{n}, \tilde{x}_2) - x_{\tilde{n}+p+1}(\tilde{n}, \tilde{x}_1) - x^{\circ}_{\tilde{n}+p+1}(\tilde{n}, \tilde{x}_2) + x^{\circ}_{\tilde{n}+p+1}(\tilde{n}, \tilde{x}_1)|| \leq$$
$$\leq K_1 ||x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) - x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1) - x^{\circ}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) + x^{\circ}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1)|| +$$
$$+ \omega(||x_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1) - x^{\circ}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1)||) \, ||x^{\circ}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1) - x^{\circ}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2)|| +$$
$$+ \xi ||x^{\circ}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_2) - x^{\circ}_{\tilde{n}+p}(\tilde{n}, \tilde{x}_1)|| \leq$$
$$\leq K_1 v_p + \omega \left(\frac{K_1^N - 1}{K_1 - 1} \xi \right) K_1^p ||\tilde{x}_1 - \tilde{x}_2|| + \xi K_1^p ||\tilde{x}_1 - \tilde{x}_2||$$

hence $v_{p+1} \leq K_1 v_p + \beta_N(\xi) \, ||\tilde{x}_2 - \tilde{x}_1||$.
From here we get

$$v_p \leq K_1^{p-1} \xi \, ||\tilde{x}_2 - \tilde{x}_1|| + K_1^N \left(\omega \left(\frac{K_1^N - 1}{K_1 - 1} \xi \right) + \xi \right) \frac{K_1^{p-1}}{K_1 - 1} \, ||\tilde{x}_2 - \tilde{x}_1||$$

hence $v_p \leq \alpha_N(\xi) \, ||\tilde{x}_2 - \tilde{x}_1||$ for $0 \leq p \leq N$, $\lim_{\xi \to 0} \alpha_N(\xi) = 0$ and the theorem is proved.

Theorem 3. *Consider the system*

$$y_{n+1} = Y_n(y_n, \vartheta_n)$$
$$\vartheta_{n+1} - \vartheta_n = \Theta_n(y_n, \vartheta_n)$$

and suppose that:

a) Y_n, Θ_n *are defined for* $||y|| \leq H$, $\vartheta \in \mathfrak{C}$,

$$\left\| \frac{\partial Y_n}{\partial y}(y, \vartheta) \right\| \leq K_1, \quad \left\| \frac{\partial Y_n}{\partial \vartheta}(y, \vartheta) \right\| \leq K_1, \quad \left\| \frac{\partial \Theta_n}{\partial y}(y, \vartheta) \right\| \leq K_1,$$
$$\left\| \frac{\partial \Theta_n}{\partial \vartheta}(y, \vartheta) \right\| \leq K_1,$$
$$\left\| \frac{\partial Y_n}{\partial y}(y', \vartheta') - \frac{\partial Y_n}{\partial y}(y'', \vartheta'') \right\| \leq K_1(||y' - y''||^{\mu} + ||\vartheta' - \vartheta''||^{\mu}),$$
$$\left\| \frac{\partial Y_n}{\partial \vartheta}(y', \vartheta') - \frac{\partial Y_n}{\partial \vartheta}(y'', \vartheta'') \right\| \leq K_1(||y' - y''||^{\mu} + ||\vartheta' - \vartheta''||^{\mu}),$$
$$\left\| \frac{\partial \Theta_n}{\partial y}(y', \vartheta') - \frac{\partial \Theta_n}{\partial y}(y'', \vartheta'') \right\| \leq K_1(||y' - y''||^{\mu} + ||\vartheta' - \vartheta''||^{\mu}),$$
$$\left\| \frac{\partial \Theta_n}{\partial \vartheta}(y', \vartheta') - \frac{\partial \Theta_n}{\partial \vartheta}(y'', \vartheta'') \right\| \leq K_1(||y' - y''||^{\mu} + ||\vartheta' - \vartheta''||^{\mu});$$

b) $Y_n(0, \vartheta) \equiv 0$, $\Theta_n(0, \vartheta) \equiv \alpha_n$.

c) *Let* $A_n(\vartheta) = \dfrac{\partial Y_n}{\partial y}(0, \vartheta)$, $\delta_n = \tilde{\vartheta} + \sum_{k=\tilde{n}}^{n-1} \alpha_k$; *then there exists* $0 < q < 1$

and K such that $||z_n(\tilde{n}, \tilde{z})|| \leq Kq^{n-\tilde{n}}||\tilde{z}||$ *for all solutions of the system* $z_{n+1} = $ $= A_n(\delta_n) z_n$.

If all these conditions are fulfilled there exist q', K', l *such that*

1^0 $||\tilde{y}|| \leq l$ *implies* $||y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})|| \leq K' q'^{n-\tilde{n}} ||\tilde{y}||$ *for* $n \geq \tilde{n}$

2^0 $||\tilde{y}'|| \leq l$, $||\tilde{y}''|| \leq l$ *imply*

$$||y_n(\tilde{n}, \tilde{y}', \tilde{\vartheta}') - y_n(\tilde{n}, \tilde{y}'', \tilde{\vartheta}'')|| \leq K' q'^{n-\tilde{n}} (||\tilde{y}' - \tilde{y}''|| + \varrho ||\tilde{\vartheta}' - \tilde{\vartheta}''||)$$

$$||\vartheta_n(\tilde{n}, \tilde{y}', \tilde{\vartheta}') - \vartheta_n(\tilde{n}, \tilde{y}'', \tilde{\vartheta}'') - \tilde{\vartheta}' + \tilde{\vartheta}''|| \leq K'(||\tilde{y}' - \tilde{y}''|| + \varrho ||\tilde{\vartheta}' - \tilde{\vartheta}''||)$$

(l depends on ϱ).

Proof. A. Let $V_n(z) = \sup_{p \geq 0} ||z_{n+p}(n, z)|| \frac{1}{q^p}$; we have $||z|| \leq V_n(z) \leq K||z||$.

Let $V_n^* = V_n[z_n(\tilde{n}, \tilde{z})] = \sup_{p \geq 0} ||z_{n+p}(n, z_n(\tilde{n}, \tilde{z}))|| \frac{1}{q^p} = \sup_{p \geq 0} ||z_{n+p}(\tilde{n}, \tilde{z})|| \frac{1}{q^p}$;

it follows $V_{n+1}^* = \sup_{p \geq 0} ||z_{n+p+1}(\tilde{n}, \tilde{z})|| \frac{1}{q^p} = \sup_{p \geq 1} ||z_{n+p}(\tilde{n}, \tilde{z})|| \frac{1}{q^{p-1}} \leq$

$$\leq \sup_{p \geq 0} ||z_{n+p}(\tilde{n}, \tilde{z})|| \frac{1}{q^{p-1}}$$

hence

$$V_{n+1}^* - V_n^* \leq (q-1) \sup_{p \geq 0} ||z_{n+p}(\tilde{n}, \tilde{z})|| \frac{1}{q^p} = -(1-q) V_n^*.$$

We have further $V_n(z') - V_n(z'') = \sup_{p \geq 0} ||z_{n+p}(n, z')|| \frac{1}{q^p} - \sup_{p \geq 0} ||z_{n+p}(n, z'')|| \frac{1}{q^p} \leq$

$$\leq \sup_{p \geq 0} ||z_{n+p}(n, z') - z_{n+p}(n, z'')|| \frac{1}{q^p} = \sup_{p \geq 0} ||z_{n+p}(n, z' - z'')|| \frac{1}{q^p} =$$
$$= V_n(z' - z'') \leq K||z' - z''||$$

hence $|V_n(z') - V_n(z'')| \leq K||z' - z''||$.

B. We put the first equation of the system in the form

$$y_{n+1} = A_n(\delta_n) y_n + B_n(y_n, \vartheta_n);$$

$$B_n(y_n, \vartheta_n) = Y_n(y_n, \vartheta_n) - A_n(\delta_n) y_n = Y_n(y_n, \vartheta_n) - Y_n(0, \vartheta_n) - A_n(\delta_n) y_n =$$

$$= \int_0^1 \frac{\partial Y_n}{\partial y} (\lambda y_n, \vartheta_n) y_n \, d\lambda - \frac{\partial Y_n}{\partial y} (0, \delta_n) y_n =$$

$$= \int_0^1 \left[\frac{\partial Y_n}{\partial y} (\lambda y_n, \vartheta_n) - \frac{\partial Y_n}{\partial y} (0, \vartheta_n)\right] d\lambda y_n + \left(\frac{\partial Y_n}{\partial y} (0, \vartheta_n) - \frac{\partial Y_n}{\partial y} (0, \delta_n)\right) y_n;$$

hence $||B_n(y_n, \vartheta_n)|| \leq K_1 ||y_n||^{\mu+1} + K_1 ||y_n|| \, ||\vartheta_n - \delta_n||^{\mu}$.

Let $\beta_n = \vartheta_n - \delta_n$; we have $\beta_{n+1} - \beta_n = \vartheta_{n+1} - \vartheta_n - (\delta_{n+1} - \delta_n) =$ $= \Theta_n(y_n, \vartheta_n) - \alpha_n = \Theta_n(y_n, \vartheta_n) - \Theta_n(0, \vartheta_n)$, hence $||\beta_{n+1} - \beta_n|| \leq K_1 ||y_n||$.

If $\delta_{\tilde{n}} = \vartheta_{\tilde{n}}$ we get $||\beta_n|| \leq \sum_{k=\tilde{n}}^{n-1} ||y_k||$ for $n > \tilde{n}$, $\beta_{\tilde{n}} = 0$.

C. Let $y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})$, $\Theta_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})$ a solution of the system, $\tilde{V}_n^* = V_n[y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})]$. We have $\tilde{V}_{n+1}^* - \tilde{V}_n^* = \overset{\mu}{V}_{n+1}[z_{n+1}(n, y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}))] -$

$$- V_n[y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})] + V_{n+1}[y_{n+1}(n, y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}))] - V_{n+1}[z_{n+1}(n, y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}))] \leq$$

$$\leq -(1-q)\, \tilde{V}_n^* + K||y_{n+1}(n, y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})) - z_{n+1}(n, y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}))||.$$

But $y_{n+1}(n, y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})) = A_n(\delta_n)\, y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}) + B_n(y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}), \vartheta_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}))$

$$z_{n+1}(n, y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})) = A_n(\delta_n)\, y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})$$

hence

$$||y_{n+1}(n, y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})) - z_{n+1}(n, y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}))|| = ||B_n(y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}), \vartheta_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}))|| \leq$$

$$\leq K_1||y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})||^{\mu+1} + K_1^{1+\mu}||y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})||(\sum_{k=\tilde{n}}^{n-1} ||y_k(\tilde{n}, \tilde{y}, \tilde{\vartheta})||)^{\mu}.$$

It follows that

$$\tilde{V}_{n+1}^* - \tilde{V}_n^* \leq -(1-q)\, \tilde{V}_n^* + K_2||y_n(\tilde{n}, \tilde{y}, \vartheta)||^{\mu+1} +$$

$$+ K_3||y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})||(\sum_{k=\tilde{n}}^{n-1} ||y_k(\tilde{n}, \tilde{y}, \tilde{\vartheta})||)^{\mu} \leq$$

$$\leq -(1-q)\, \tilde{V}_n^* + K_2 \tilde{V}_n^{*1+\mu} + K_3 \tilde{V}_n^* (\sum_{k=\tilde{n}}^{n-1} \tilde{V}_k^*)^{\mu}.$$

Let $q < q' < 1$, $W_n = \frac{1}{q'^{n-\tilde{n}}}\, \tilde{V}_n^*$; we have

$$W_{n+1} - W_n = \frac{1}{q'^{n-\tilde{n}+1}}\, \tilde{V}_{n+1}^* - \frac{1}{q'^{n-\tilde{n}}}\, \tilde{V}_n^* = \frac{1}{q'^{n-\tilde{n}+1}}\, (\tilde{V}_{n+1}^* - \tilde{V}_n^*) +$$

$$+ \frac{1}{q'^{n-\tilde{n}}}\, \tilde{V}_n^* \left(\frac{1}{q'} - 1\right) \leq -\frac{1}{q'}\, W_n\left[1 - q - K_2 \tilde{V}_n^{*\mu} - K_3 (\sum_{k=\tilde{n}}^{n-1} \tilde{V}_n^*)^{\mu}\right] +$$

$$+ W_n \left(\frac{1}{q'} - 1\right) = -W_n \left[1 - \frac{q}{q'} - \frac{K_2}{q'}\, \tilde{V}_n^{*\mu} - \frac{K_3}{q'} (\sum_{k=\tilde{n}}^{n-1} \tilde{V}_k^*)^{\mu}\right].$$

We deduce that

$$W_{n+1} - W_n \leq -W_n \left[1 - \frac{q}{q'} - \frac{K_2}{q'}\, q'^{\mu(n-\tilde{n})} W_n^{\mu} - \frac{K_3}{q'} (\sum_{k=\tilde{n}}^{n-1} q'^{k-\tilde{n}} W_k)^{\mu}\right];$$

$$W_{\tilde{n}} = \tilde{V}_{\tilde{n}}^* = V_{\tilde{n}}(\tilde{y}) \leq K||\tilde{y}||.$$

Suppose $W_k \leq l'$ for $k \leq n$; then

$$\left(\sum_{k=\tilde{n}}^{n-1} q'^{k-\tilde{n}} W_k\right)^{\mu} \leq l'^{\mu} \left(\sum_{k=\tilde{n}}^{n-1} q'^{k-\tilde{n}}\right)^{\mu} < l'^{\mu} \frac{1}{(1-q')^{\mu}}, \qquad \text{and}$$

$$1-\frac{q}{q'}-\frac{K_2}{q'}q'^{\mu(n-\tilde{n})}W_n^{\mu}-\frac{K_3}{q'}\left(\sum_{k=\tilde{n}}^{n-1}q'^{k-\tilde{n}}W^k\right)^{\mu}\geq$$

$$\geq 1-\frac{q}{q'}-\frac{K_2}{q'}q'^{\mu(n-\tilde{n})}l'^{\mu}-\frac{K_3}{q'}\frac{l'^{\mu}}{(1-q')^{\mu}}>0$$

if $1-\frac{q}{q'}>l'^{\mu}\left(\frac{K_2}{q'}+\frac{K_3}{q'(1-q')^{\mu}}\right)$ hence if $l'<\frac{(q'-q)^{\frac{1}{\mu}}}{\left[K_2+\frac{K_3}{(1-q')^{\mu}}\right]^{\frac{1}{\mu}}}$.

For such l' and for $W_k \leq l'$, $k \leq n$ we get $W_{n+1}-W_n<0$ hence $W_{n+1}<$ $<W_n \leq l'$ and the inequality is proved by induction if it is true for $k=\tilde{n}$,

i.e. if $\|\tilde{y}\| \leq \frac{1}{K}\frac{(q'-q)^{\frac{1}{\mu}}}{\left[K_2+\frac{K_3}{(1-q')^{\mu}}\right]^{\frac{1}{\mu}}}$.

For such $\tilde{y}$ we have $W_n \leq l'$ for all $n \geq \tilde{n}$ hence

$W_{n+1}-W_n \leq -\alpha W_n$, $\quad W_{n+1} \leq (1-\alpha)\, W_n$,
$W_n \leq (1-\alpha)^{n-\tilde{n}} W_{\tilde{n}} = K(1-\alpha)^{n-\tilde{n}}\|\tilde{y}\|$;
it follows that $\tilde{V}_n^* \leq K[q'(1-\alpha)]^{n-\tilde{n}}\|\tilde{y}\|$ hence
$\|y_n(\tilde{n},\tilde{y},\tilde{\vartheta})\| \leq K[q'(1-\alpha)]^{n-\tilde{n}}\|\tilde{y}\|$
and the first assertion of the theorem is proved.

D. Let now $y_n'=y_n(\tilde{n},\tilde{y}',\tilde{\vartheta}')$, $y_n''=y_n(\tilde{n},\tilde{y}'',\tilde{\vartheta}_n'')$, $\vartheta_n'=\vartheta_n(\tilde{n},\tilde{y}',\tilde{\vartheta}')$, $\vartheta_n''=\vartheta_n(\tilde{n},\tilde{y}_n'',\tilde{\vartheta}_n'')$; suppose $\|\tilde{y}'\| \geq \|\tilde{y}''\|$, $\vartheta_{\tilde{n}}=\tilde{\vartheta}'$.

Denote $V_n^{**}=V_n(y_n'-y_n'')$; we have

$V_{n+1}^{**}-V_{n\mu}^{**}=V_{n+1}[z_{n+1}(n,y_n'-y_n'')]-V_n[z_n(n,y_n'-y_n'')]+$
$+V_{n+1}(y_{n+1}'-y_{n+1}'')-V_{n+1}[z_{n+1}(n,y_n'-y_n'')] \leq$
$\leq -(1-q)\,V_{n\mu}^{**}+K\|y_{n+1}'-y_{n+1}''-z_{n+1}(n,y_n'-y_n'')\|$.

But

$\|y_{n+1}'-y_{n+1}''-z_{n+1}(n,y_n'-y_n'')\|=$
$=\|Y_n(y_n',\vartheta_n')-Y_n(y_n'',\vartheta_n'')-A_n(\delta_n)\,(y_n'-y_n'')\| \leq$
$\leq \|Y_n(y_n'',\vartheta_n')-Y_n(y_n'',\vartheta_n'')\|+$
$+\|Y_n(y_n',\vartheta_n')-Y_n(y_n'',\vartheta_n')-A_n(\delta_n)\,(y_n'-y_n'')\| \leq$

$$\leq \left\|\int_0^1 \frac{\partial Y_n}{\partial\vartheta}(y_n'',\vartheta_n^{\lambda})\,(\vartheta_n'-\vartheta_n'')\,\mathrm{d}\lambda\right\|+$$

$$+\left\|\int_0^1\left(\frac{\partial Y_n}{\partial y}(y_n^{\lambda},\vartheta_n')-\frac{\partial Y_n}{\partial y}(0,\delta_n)\right)(y_n'-y_n'')\,\mathrm{d}\lambda\right\|=$$

$$= \left\| \int_0^1 \left(\frac{\partial Y_n}{\partial \vartheta} (y''_n, \vartheta^\lambda_n) - \frac{\partial Y_n}{\partial \vartheta} (0, \vartheta^\lambda_n) \right) \mathrm{d}\lambda (\vartheta'_n - \vartheta''_n) \right\| +$$

$$+ \left\| \int_0^1 \left(\frac{\partial Y_n}{\partial y} (y^\lambda_n, \vartheta'_n) - \frac{\partial Y_n}{\partial y} (0, \vartheta'_n) + \frac{\partial Y_n}{\partial y} (0, \delta'_n) - \right.\right.$$

$$\left.\left. - \frac{\partial Y_n}{\partial y} (0, \delta_n) \right) \mathrm{d}\lambda (y'_n - y''_n) \right\| \leq$$

$$\leq K_1 ||y''_n||^\mu ||\vartheta'_n - \vartheta''_n|| + K_1 \sup ||y^\lambda_n||^\mu ||y'_n - y''_n|| +$$
$$+ K_1 ||\vartheta'_n - \delta_n||^\mu ||y'_n - y''_n|| \leq$$
$$\leq K_1 K'^\mu [q'(1-\alpha)]^{\mu(n-\tilde{n})} ||\tilde{y}'|| \, (||y'_n - y''_n|| + ||\vartheta'_n - \vartheta''_n||) +$$
$$+ K_1 ||\vartheta'_n - \delta_n||^\mu ||y'_n - y''_n||.$$

We know that

$$||\beta_n|| = ||\vartheta'_n - \delta_n|| \leq K_1 \sum_{k=\tilde{n}}^{n-1} ||y'_k|| \leq K_1 K' ||\tilde{y}'|| \sum_{k=\tilde{n}}^{n-1} [q'(1-\alpha)]^{k-\tilde{n}},$$

hence $||\Theta'_n - \delta_n||^\mu \leq K_4 ||\tilde{y}'||^\mu$. We have further

$$\vartheta''_{n+1} - \vartheta'_{n+1} - (\vartheta''_n - \vartheta'_n) = \Theta_n(y''_n, \vartheta''_n) - \Theta_n(y'_n, \vartheta'_n) =$$

$$= \int_0^1 \left[\frac{\partial \Theta_n}{\partial y} (y^\lambda_n, \vartheta^\lambda_n)(y''_n - y'_n) + \frac{\partial \Theta_n}{\partial \vartheta} (y^\lambda_n, \vartheta^\lambda_n)(\vartheta''_n - \vartheta'_n) \right] \mathrm{d}\lambda =$$

$$= \int_0^1 \frac{\partial \Theta_n}{\partial y} (y^\lambda_n, \vartheta^\lambda_n)(y''_n - y'_n) \, \mathrm{d}\lambda +$$

$$+ \int_0^1 \left[\frac{\partial \Theta_n}{\partial \vartheta} (y^\lambda_n, \vartheta^\lambda_n) - \frac{\partial \Theta_n}{\partial \vartheta} (0, \vartheta^\lambda_n) \right] (\vartheta''_n - \vartheta'_n) \, \mathrm{d}\lambda$$

hence setting $\gamma_n = \vartheta''_n - \vartheta'_n$ we get

$$||\gamma_{n+1} - \gamma_n|| \leq K_1 ||y''_n - y'_n|| + K_1 K'^\mu [q'(1-\alpha)]^{\mu(n-\tilde{n})} ||\tilde{y}'||^\mu ||\gamma_n||.$$

It follows that

$$||\gamma_n|| \leq ||\gamma_{\tilde{n}}|| + \sum_{k=\tilde{n}}^{n-1} (K_1 ||y''_k - y'_k|| + K_1 K'^\mu [q'(1-\alpha)]^{\mu(k-\tilde{n})} ||\tilde{y}'||^\mu ||\gamma_k||)$$

hence

$$||\vartheta''_n - \vartheta'_n|| \leq ||\tilde{\vartheta}'' - \tilde{\vartheta}'|| + K_1 \sum_{k=\tilde{n}}^{n-1} ||y''_k - y'_k|| +$$

$$+ K_1 K'^\mu ||y'||^\mu \sum_{k=\tilde{n}}^{n-1} [q'(1-\alpha)]^{\mu(k-\tilde{n})} ||\vartheta''_k - \vartheta'_k||$$

for $n \geq \tilde{n} + 1$.

By a discrete analogue of the Gronwall lemma this inequality yields

$$||\vartheta''_n - \vartheta'_n|| \leq K_5||\tilde{\vartheta}'' - \tilde{\vartheta}'|| + \sum_{k=\tilde{n}}^{n-1} ||y''_k - y'_k||.$$

Let us estimate $||\vartheta''_n - \vartheta'_n - \tilde{\vartheta}'' + \tilde{\vartheta}'|| = ||\gamma_n - \gamma_{\tilde{n}}||$. We have

$$||\gamma_n - \gamma_{\tilde{n}}|| \leq \sum_{k=\tilde{n}}^{n-1} (K_1||y''_k - y'_k|| + K_1K'^{\mu}[q'(1-\alpha)]^{\mu(k-\tilde{n})}||\tilde{y}'||^{\mu}||\gamma_k||) \leq$$

$$\leq \sum_{k=\tilde{n}}^{n-1} (K_1||y''_k - y_k'|| + K_1K'^{\mu}[q'(1-\alpha)]^{\mu(k-\tilde{n})}||\tilde{y}'||^{\mu}||\gamma_{\tilde{n}}||) +$$

$$+ K_1K'^{\mu}||\tilde{y}'||^{\mu} \sum_{k=\tilde{n}}^{n-1} [q'(1-\alpha)]^{\mu(k-\tilde{n})}||\gamma_k - \gamma_{\tilde{n}}||$$

which yields the inequality

$$||\vartheta''_n - \vartheta'_n - \tilde{\vartheta}'' + \tilde{\vartheta}'|| \leq K_6\left(\sum_{k=\tilde{n}}^{n-1} ||y'' - y'_k|| + ||\tilde{y}'||^{\mu}||\vartheta'' - \vartheta'||\right).$$

Using these inequalities we have

$$V^{**}_{n+1} - V^{**}_n \leq -(1-q)\, V^{**}_n + K_7||\tilde{y}'||^{\mu}[q'(1-\alpha)]^{\mu(n-\tilde{n})}||y'_n - y''_n|| +$$

$$+ K_8||\tilde{y}'||^{\mu}||y'_n - y''_n|| + K_9[q'(1-\alpha)]^{\mu(n-\tilde{n})}||\tilde{y}'||^{\mu} \sum_{k=\tilde{n}}^{n-1} ||y''_k - y'_k|| +$$

$$+ K_9[q'(1-\alpha)]^{\mu(n-\tilde{n})}||\tilde{y}'||^{\mu}||\tilde{\vartheta}'' - \tilde{\vartheta}'||.$$

Let $q'' = q + K_8 l^{\mu}$ and choose l small enough in order that $q'' < q_1$, i.e. $l < \left(\frac{q_1 - q}{K_8}\right)^{\frac{1}{\mu}}$. Suppose $||y'|| < l$; it follows

$$V^{**}_{n+1} - V^{**}_n \leq -(1-q'')\, V^{**}_n + K_{10}[q'(1-\alpha)]^{\mu(n-\tilde{n})}||\tilde{y}'||^{\mu} \sum_{k=\tilde{n}}^{n-1} ||y''_k - y'_k|| +$$

$$+ K_9[q'(1-\alpha)]^{\mu(n-\tilde{n})}||\tilde{y}'||^{\mu}||\tilde{\vartheta}'' - \tilde{\vartheta}'||.$$

Let $W^*_n = \frac{1}{q''^{n-\tilde{n}}} V^{**}_n$; we have

$$W^*_{n+1} = \frac{1}{q''^{n+1-\tilde{n}}} V^{**}_{n+1} \leq$$

$$\leq \frac{1}{q''^{n+1-\tilde{n}}} (q'' V^{**}_n + K_{10}[q'(1-\alpha)]^{\mu(n-\tilde{n})}||\tilde{y}'|| \sum_{k=\tilde{n}}^{n-1} ||y''_k - y'_k|| +$$

$$+ K_9[q'(1-\alpha)]^{\mu(n-\tilde{n})}||y'||^{\mu}||\tilde{\vartheta}'' - \tilde{\vartheta}'||) =$$

$$= W^*_n + \frac{K_{10}}{q''} ||\tilde{y}'||^{\mu} \left(\frac{[q'(1-\alpha)]^{\mu}}{q''}\right)^{n-\tilde{n}} \sum_{k=\tilde{n}}^{n-1} ||y''_k - y'_k|| +$$

$$+ \frac{K_9}{q''} ||\tilde{y}'||^{\mu} \left(\frac{[q'(1-\alpha)]^{\mu}}{q''}\right)^{n-\tilde{n}} ||\tilde{\vartheta}'' - \tilde{\vartheta}'||,$$

hence

$$W_n^* \leq W_{\tilde{n}}^* + \sum_{k=\tilde{n}}^{n-1} \frac{K_{10}}{q''} ||\tilde{y}'||^\mu \left(\frac{[q'(1-\alpha)]^\mu}{q''}\right)^{k-\tilde{n}} \sum_{j=\tilde{n}}^{k} ||y_j'' - y_j'|| +$$

$$+ \frac{K_9}{q''} ||\tilde{y}'||^\mu ||\tilde{\vartheta}'' - \tilde{\vartheta}'|| \sum_{k=\tilde{n}}^{n-1} \left(\frac{[q'(1-\alpha)]^\mu}{q''}\right)^{k-\tilde{n}} \leq$$

$$\leq V_{\tilde{n}}^{**} + \frac{K_9}{q''} \cdot \frac{||\tilde{y}'||^\mu}{1 - \dfrac{[q'(1-\alpha)]^\mu}{q''}} ||\tilde{\vartheta}'' - \tilde{\vartheta}'|| +$$

$$+ \frac{K_{10}}{q''} ||\tilde{y}'||^\mu \sum_{j=\tilde{n}}^{n-1} \left(\sum_{k=j}^{n-1} \left(\frac{[q'(1-\alpha)]^\mu}{q''}\right)^{k-\tilde{n}} \right) ||y_j'' - y_j'|| \leq$$

$$\leq K\, ||\tilde{y}' - \tilde{y}''|| + \frac{K_9 ||\tilde{y}'||^\mu}{q'' - [q'(1-\alpha)]^\mu} ||\tilde{\vartheta}'' - \tilde{\vartheta}'|| +$$

$$+ \frac{K_{10}}{q''} ||\tilde{y}'^\mu|| \sum_{j=\tilde{n}}^{n-1} \left(\frac{[q'(1-\alpha)]^\mu}{q''}\right)^{j-\tilde{n}} \cdot \frac{||y_j'' - y_j'||}{1 - \dfrac{[q'(1-\alpha)]^\mu}{q''}}$$

hence

$$||y_n' - y_n''|| \leq V_n^{**} = q''^{n-\tilde{n}} W_n^* \leq$$

$$\leq K q''^{n-\tilde{n}} ||\tilde{y}' - \tilde{y}''|| + K_{11} ||\tilde{y}'||^\mu ||\vartheta'' - \vartheta'||\, q''^{n-\tilde{n}} +$$

$$+ K_{12} ||\tilde{y}'||^\mu q''^{n-\tilde{n}} \sum_{j=\tilde{n}}^{n-1} \left(\frac{[q'(1-\alpha)]^\mu}{q''}\right)^{j-\tilde{n}} ||y_j'' - y_j'||.$$

Let $u_n = \dfrac{1}{q_1^{n-\tilde{n}}} ||y_n' - y_n''||$; we have

$$q_1^{n-\tilde{n}} u_n \leq K q''^{n-\tilde{n}} ||\tilde{y}' - \tilde{y}''|| + K_{11} ||\tilde{y}'||^\mu ||\tilde{\vartheta}'' - \tilde{\vartheta}'||\, q''^{n-\tilde{n}} +$$

$$+ K_{12} ||\tilde{y}'||^\mu q''^{n-\tilde{n}} \sum_{j=\tilde{n}}^{n-1} \left(\frac{[q'(1-\alpha)]^\mu}{q''}\right)^{j-\tilde{n}} q_1^{j-\tilde{n}} u_j,$$

$$u_n \leq K\, ||\tilde{y}' - \tilde{y}''|| + K_{11} ||\tilde{y}'||^\mu ||\tilde{\vartheta}'' - \tilde{\vartheta}'|| +$$

$$+ K_{12} ||\tilde{y}'||^\mu \sum_{j=\tilde{n}}^{n-1} \left(\frac{[q'(1-\alpha)]^\mu q_1}{q''}\right)^{j-\tilde{n}} u_j$$

hence $u_n \leq K_{13}(||\tilde{y}' - \tilde{y}''|| + ||\tilde{y}'||^\mu ||\tilde{\vartheta}'' - \tilde{\vartheta}'||)$.

It follows that

$$||y_n' - y_n''|| \leq K_{13} q_1^{n-\tilde{n}} (||\tilde{y}' - \tilde{y}''|| + ||\tilde{y}'||^\mu ||\tilde{\vartheta}'' - \tilde{\vartheta}'||)$$

$$||\vartheta_n'' - \vartheta_n' - \tilde{\vartheta}'' + \tilde{\vartheta}'|| \leq K_{14}(||\tilde{y}' - \tilde{y}''|| + ||\tilde{y}'||^\mu ||\tilde{\vartheta}'' - \tilde{\vartheta}'||)$$

and the theorem is proved.

III. The theorem on invariant manifolds.

We may now prove the following theorem on the existence of exponentially stable invariant manifolds.

Theorem 4. *Consider the discrete system*

$$y_{n+1} = Y^{\circ}_n(y_n, \vartheta_n) + \varepsilon Y^1_n(y_n, \vartheta_n, \varepsilon)$$
$$\vartheta_{n+1} = \vartheta_n + \Theta^{\circ}_n(y_n, \vartheta_n) + \varepsilon\Theta^1_n(y_n, \vartheta_n, \varepsilon)$$

Suppose that Y°_n, Θ°_n *verify all the conditions of theorem* 3 *and* Y^1_n, Θ^1_n *verify the regularity conditions of theorem* 2. *Then for* $|\varepsilon|$ *small enough there exist* $p_n : \mathfrak{C} \to C$ *such that*

a) $||p_n(\vartheta)|| \leq l(\varepsilon)$,

b) $||p_n(\vartheta_1) - p_n(\vartheta_2)|| \leq L(\varepsilon)\, ||\vartheta_1 - \vartheta_2||$, $\lim\limits_{\varepsilon\to 0} l(\varepsilon) = \lim\limits_{\varepsilon\to 0} \mathrm{L}(\varepsilon) = 0$;

c) $||\tilde{y}|| \leq l$ *implies* $||y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}) - p_n(\vartheta_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}))|| \leq K'q'^{n-\tilde{n}}||\tilde{y} - p_{\tilde{n}}(\tilde{\vartheta})||$,

d) $\tilde{y} = p_{\tilde{n}}(\tilde{\vartheta})$ *implies* $y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}) = p_n(\vartheta_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}))$

and the solution is defined for all integers n.

e) p_n *is unique with the above properties,*

f) 1°. *If* $Y^{\circ}_{n+\nu}(y, \vartheta) = Y^{\circ}_n(y, \vartheta)$, $Y^1_{n+\nu}(y, \vartheta, \varepsilon) = Y^1_n(y, \vartheta, \varepsilon)$,
$\Theta^{\circ}_{n+\nu}(y, \vartheta) = \Theta^{\circ}_n(y, \vartheta)$, $\Theta^1_{n+\nu}(y, \vartheta, \varepsilon) = \Theta^1_n(y, \vartheta, \varepsilon)$
then $p_{n+\nu} \equiv p_n$.

2°. *If* $Y^{\circ}_n(y, \vartheta + \omega) = Y^{\circ}_n(y, \vartheta)$, $Y^1_n(y, \vartheta + \omega, \varepsilon) = Y^1_n(y, \vartheta, \varepsilon)$,
$\Theta^{\circ}_n(y, \vartheta + \omega) = \Theta^{\circ}_n(y, \vartheta)$, $\Theta^1_n(y, \vartheta + \omega, \varepsilon) = \Theta^1_n(y, \vartheta, \varepsilon)$,
then $p_n(\vartheta + \omega) = p_n(\vartheta)$.

g) *If* Y°_n, Y^1_n, Θ°_n, Θ^1_n *are almost periodic sequences (uniformly with respect to* y, ϑ, ε*) then* p_n *is an almost-periodic sequence.*

Proof. We have to verify that the discrete system considered verifies all conditions of theorem 1. Let y°_n, ϑ°_n be defined by the system for $\varepsilon = 0$. From theorem 3 we have $||y^{\circ}_n(\tilde{n}, \tilde{y}, \vartheta)|| \leq K'q'^{n-\tilde{n}}||\tilde{y}||$ for $n \geq \tilde{n}$, $||\tilde{y}|| \leq l$. Let N be such that $K'q'^N < \frac{1}{3}$; we have for $\tilde{n} \leq n \leq \tilde{n} + 2N$ using theorem 2 $||y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})|| \leq ||y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta}) - y^{\circ}_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})|| + ||y^{\circ}_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})|| \leq \beta_N|\varepsilon| + K'l \leq \leq \frac{3}{4} H$ for $|\varepsilon|$ and l small enough and the solution is defined for such n. Further, for $n \geq \tilde{n} + N$ we have $||y_n(\tilde{n}, \tilde{y}, \tilde{\vartheta})|| \leq \beta_N|\varepsilon| + K'q'^N l < \beta_N|\varepsilon| + + \frac{1}{3} l < l$, for $|\varepsilon|$ small enough. Condition 1° of theorem 1 is verified.

We have then by theorem 3

$$||y^{\circ}_n(\tilde{n}, \tilde{y}', \vartheta') - y^{\circ}_n(\tilde{n}, \tilde{y}'', \vartheta'')|| \leq K'q'^{n-\tilde{n}}(||\tilde{y}' - \tilde{y}''|| + \varrho||\vartheta' - \vartheta''||)$$
$$||\vartheta^{\circ}_n(\tilde{n}, \tilde{y}', \vartheta') - \vartheta^{\circ}_n(\tilde{n}, \tilde{y}'', \vartheta'') - \vartheta' + \vartheta''|| \leq K'(||\tilde{y}' - \tilde{y}''|| + \varrho||\vartheta' - \vartheta''||).$$

It follows using theorem 2 that

$$\begin{aligned}
&\|y_n(\tilde n, \tilde y') - y_n(\tilde n, \tilde y'', \vartheta)\| + L\,\|\vartheta_n(\tilde n, \tilde y', \vartheta) - \vartheta_n(\tilde n, \tilde y'', \vartheta)\| \le \\
&\quad \le \|y_n(\tilde n, \tilde y', \vartheta) - y_n(\tilde n, \tilde y'', \vartheta) - y_n^\circ(\tilde n, \tilde y', \vartheta) + y_n^\circ(\tilde n, \tilde y'', \vartheta)\| + \\
&\quad + L\,\|\vartheta_n(\tilde n, \tilde y', \vartheta) - \vartheta_n(\tilde n, \tilde y'', \vartheta) - \vartheta_n^\circ(\tilde n, \tilde y', \vartheta) + \vartheta_n^\circ(\tilde n, \tilde y'', \vartheta)\| + \\
&\quad + \|y_n^\circ(\tilde n, \tilde y', \vartheta) - y_n^\circ(\tilde n, \tilde y'', \vartheta)\| + L\,\|\vartheta_n^\circ(\tilde n, \tilde y', \vartheta) - \vartheta_n^\circ(\tilde n, \tilde y'', \vartheta)\| \le \\
&\quad \le \alpha_{2N}(\varepsilon)\,\|\tilde y' - \tilde y''\| + L\alpha_N(\varepsilon)\,\|\tilde y' - \tilde y''\| + K'q'^N\|\tilde y' - \tilde y''\| + \\
&\quad + LK'\|\tilde y' - \tilde y''\|
\end{aligned}$$

for $\tilde n + N \le n \le \tilde n + 2N$, hence for $|\varepsilon|$, L small enough we get

$$\begin{aligned}
&\|y_n(\tilde n, \tilde y', \vartheta) - y_n(\tilde n, \tilde y'', \vartheta)\| + L\,\|\vartheta_n(\tilde n, \tilde y', \vartheta) - \vartheta_n(\tilde n, \tilde y'', \vartheta)\| \le \\
&\quad \le \alpha_1\|y' - y''\|, \quad \alpha_1 < 1,
\end{aligned}$$

and condition 2^0 of theorem 1 is verified.

In order to verify condition 3^0 a) we see that for $\tilde n \le n \le \tilde n + 2N$,

$\|\tilde y' - \tilde y''\| \le L\,\|\vartheta' - \vartheta''\|$ we have

$$\begin{aligned}
&\|\vartheta_n(\tilde n, \tilde y', \vartheta') - \vartheta_n(\tilde n, \tilde y'', \vartheta'') - \vartheta' + \vartheta''\| \le \\
&\quad \le \|\vartheta_n(\tilde n, \tilde y', \vartheta') - \vartheta_n(\tilde n, \tilde y'', \vartheta'') - \vartheta_n^\circ(\tilde n, \tilde y', \vartheta') + \vartheta_n^\circ(\tilde n, \tilde y'', \vartheta'')\| + \\
&\quad + \|\vartheta_n^\circ(\tilde n, \tilde y', \vartheta') - \vartheta_n^\circ(\tilde n, \tilde y'', \vartheta'') - \vartheta' + \vartheta''\| \le \\
&\quad \le \alpha_{2N}(\varepsilon)\,(\|\tilde y' - \tilde y''\| + \|\vartheta' - \vartheta''\|) + K'(\|\tilde y' - \tilde y''\| + \varrho\|\vartheta' - \vartheta''\|) \le \\
&\quad \le \alpha_{2N}(\varepsilon)\,(1 + L)\,\|\vartheta' - \vartheta''\| + K'(L + \varrho)\,\|\vartheta' - \vartheta''\| \le \alpha_2\|\vartheta' - \vartheta''\|,
\end{aligned}$$

$\alpha_2 < \dfrac{1}{3}$ if $|\varepsilon|$, L and ϱ are small enough.

We have then for $\tilde n + N \le n \le \tilde n + 2N$, $\|\tilde y' - \tilde y''\| \le L\,\|\vartheta' - \vartheta''\|$ the estimation

$$\begin{aligned}
&\|y_n(\tilde n, \tilde y', \vartheta') - y_n(\tilde n, \tilde y'', \vartheta'')\| \le \|y_n^\circ(\tilde n, \tilde y', \vartheta') - y_n^\circ(\tilde n, \tilde y'', \vartheta'')\| + \\
&\quad + \|y_n(\tilde n, \tilde y', \vartheta') - y_n(\tilde n, \tilde y'', \vartheta'') - y_n^\circ(\tilde n, \tilde y', \vartheta') + y_n^\circ(\tilde n, \tilde y'', \vartheta'')\| \le \\
&\quad \le K'q'^N(\|\tilde y' - \tilde y''\| + \varrho\,\|\vartheta' - \vartheta''\|) + \alpha_{2N}(\varepsilon)\,(\|\tilde y' - \tilde y''\| + \|\vartheta' - \vartheta''\|) \le \\
&\quad \le \frac{1}{3}\,(L + \varrho)\,\|\vartheta' - \vartheta''\| + \alpha_{2N}(\varepsilon)\,(L + 1)\,\|\vartheta' - \vartheta''\| \le \\
&\quad \le (1 - \alpha_2)\,L\,\|\vartheta' - \vartheta''\|
\end{aligned}$$

if $|\varepsilon|$ and ϱ are small enough.

Condition 4^0 is obvious from the regularity conditions.

It is easy to see that conditions in f) and g) theorem 4 imply conditions in f) and g) theorem 1.

Theorem 4 is thus proved.

It is useful to consider the "autonomous" case

$$\begin{aligned}
y_{n+1} &= Y^\circ(y_n, \vartheta_n) + \varepsilon Y^1(y_n, \vartheta_n, \varepsilon) \\
\vartheta_{n+1} &= \vartheta_n + \Theta^\circ(y_n, \vartheta_n) + \varepsilon\Theta^1(y_n, \vartheta_n, \varepsilon)
\end{aligned}$$

An invariant manifold for such system will be a function $p\colon \mathfrak{C} \to C$ such that if $\bar{y} = p(\vartheta)$ then $Y^{\circ}(\bar{y}, \vartheta) + \varepsilon Y^{1}(\bar{y}, \vartheta) = p(\vartheta + \Theta^{\circ}(\bar{y}, \vartheta) + \varepsilon\Theta^{1}(y, \vartheta, \varepsilon))$, i.e. an invariant manifold for the mapping defined by the system.

BIBLIOGRAPHY

[1] J. KURZWEIL, *Exponentially Stable Integral oranifolds. Averaying Principle and Continuous dependence on a Parameter. Czech. Math. J. 16 (91), 1966.*